Purely heterogeneous spanning tree decompositions

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Abstract

Decompositions of complete or near-complete graphs into spanning trees have been widely studied, but usually in the homogeneous case, where all component trees are isomorphic. A spanning tree decomposition $\mathcal{T} = (T_1, \ldots, T_n)$ of such a graph is purely heterogeneous if no two trees T_i are isomorphic. We show existence of such decompositions with the maximum degree condition $\Delta(T_i) = i+1$ for each $i \in [1..n]$, for every largest possible graph of odd order, and every even order graph which is the complement of a spanning tree satisfying a necessary maximum degree condition.

1. Introduction

We discuss the possibility that a graph G of order n and size m has a *purely heterogeneous spanning tree decomposition*, that is, a partition of the edges of G into spanning trees, no two of which are isomorphic. This contrasts with the more familiar problem of finding a *homogeneous* spanning tree decomposition, where all the trees are isomorphic. To admit any spanning tree decomposition, it is clear that G must satisfy the *size constraint*: $m \equiv 0 \pmod{n-1}$. When m is as large as possible, consistent with the size constraint, we shall show that G has a purely heterogeneous spanning tree decomposition — indeed, it has such a decomposition in which no two trees have the same maximum degree.

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A key result for purely heterogeneous decompositions is Wallis's theorem [4] for even order complete graphs (the parity forced by the size constraint):

Theorem 1 (Wallis) *If $n \geq 3$, the complete graph K_{2n} has a purely heterogeneous spanning tree decomposition.*

The odd order complete graph K_{2n+1} has size $n(2n+1)$, and its spanning trees have size $2n$, so its largest subgraphs which meet the size constraint have size $2n^2$. Hence we shall consider heterogeneous spanning tree decompositions of $G = K_{2n+1} - E(H)$, formed from K_{2n+1} by deleting the edge set $E(H)$ of an arbitrary subgraph H of size n. It will be shown in all cases that G does have a purely heterogeneous spanning tree decomposition.

The even order complete graph K_{2n} has size $n(2n-1)$, and its spanning trees have size $2n-1$, so it satisfies the size constraint. In this case we shall choose a spanning tree T in advance, and consider spanning tree decompositions of K_{2n} constrained to include T in the decomposition. Thus, for even order we shall actually study decompositions of $G = K_{2n} - E(T)$, formed from K_{2n} by deleting the edge set of the prescribed tree T. With mild constraints on the choice of T, it will be shown that G has a purely heterogeneous spanning tree decomposition.

The odd order case is treated first. The theorem and an outline of its proof are given in Section 2, a relevant generalization of Hall's theorem is proved in Section 3, and applied to complete proof details in Sections 4 and 5, in two parts which depend on particular features of the subgraph H. The even order case is then treated in similarly paced stages. In Section 6 necessary constraints on T are established, the theorem is then stated and its proof outlined, a relevant lemma is proved in Section 7, and proof details are completed in Section 8.

Let $[a..b]$ denote the integer set $\{i \in \mathbf{Z}: a \leq i \leq b\}$, for any a, $b \in \mathbf{Z}$. Spanning tree decompositions are a sequence

$\mathcal{T} = (T_1, \ldots, T_n)$ of n spanning trees; typically we impose further conditions on T_i for each $i \in [1..n]$.

2. Decompositions for odd order

First consider the odd order case. With $n \geq 1$, let G be any graph of order $2n+1$ and size $2n^2$. Thus $G = K_{2n+1} - E(H)$, where H is a size n subgraph of K_{2n+1}. We show that G can be decomposed into n spanning trees, no two of which have the same maximum degree:

Theorem 2. *For $n \geq 1$, let $G = K_{2n+1} - E(H)$, where H is any subgraph of size n in K_{2n+1}. Then G has a purely heterogeneous spanning tree decomposition $\mathcal{T} = (T_1, \ldots, T_n)$ which satisfies the maximum degree condition $\Delta(T_i) = i+1$ for each $i \in [1..n]$.*

Proof. The theorem is immediate when $n = 1$. Fix an integer $k \geq 1$ and suppose the theorem holds when $n = k$. Let $n = k+1$, and consider $G = K_{2n+1} - E(H)$, where H is any spanning (for convenience) subgraph of size n in K_{2n+1}. We consider two cases, simply outlining here the construction for each case, and reserving full details till later, after obtaining a relevant extension of Hall's Theorem.

Case 1: H has a degree 1 vertex, x. Since H has order $2n+1 = 2k+3$ and size $k+1$, it has at least one degree 0 vertex, y. Deleting vertices x, y from H yields the graph $H' = H - \{x, y\}$, of order $2k+1$ and size k. Thus $G' = G - \{x, y\} = K_{2k+1} - E(H')$ satisfies the induction hypothesis, and so it has a decomposition $\mathcal{T}' = (T_1', \ldots, T_k')$ into k spanning trees with the maximum degree property $\Delta(T_i') = i+1$ for each $i \in [1..k]$. From this we build an appropriate decomposition $\mathcal{T} = (T_1, \ldots, T_{k+1})$ of G. From the spanning path T_1' in G' we form a spanning path $T_1 = T_1' \cup \{wx, xy\}$ in G, where w is a suitable endpoint of T_1'. Similarly, for each $i \in [2..k]$ we form a spanning tree T_i by adding two edges to T_i', one incident with x and the other incident with y, taking care not to change the maximum degree.

The edges used are chosen in such a way that the remaining $2k+2$ edges of G form a spanning tree T_{k+1} of maximum degree $k+2$. The construction details are fully described and verified in Section 4.

Case 2: No vertex of H has degree 1. In this case no H component of order greater than 1 is a tree, so each such component has order at most equal to its size. Hence H has at least $n+1 = k+2$ degree 0 vertices. Select vertices x and y of degree 0 in H, and select any edge $e = uv$ in H. Note that u and v each have degree at least 2 in H. Deleting the edge uv and vertices x and y from H yields the graph $H' = H - \{uv\} - \{x, y\}$, of order $2k+1$ and size k, so $G' = G \cup \{uv\} - \{x, y\} = K_{2k+1} - E(H')$ satisfies the induction hypothesis. Thus G' has a spanning tree decomposition $\mathcal{T}' = (T_1', \ldots, T_k')$ which satisfies the maximum degree condition. In particular, T_k' has order $2k+1$ and maximum degree $k+1$, so it has a unique vertex w of maximum degree. Note w must have degree 1 in each T_i' with $i \in [1..k-1]$, since its degree in G' cannot exceed $2k$. Thus w has degree $2k$ in G', so is distinct from u and v, each of which has degree at most $2k-1$ in G'.

We extend $\mathcal{T}'$ to a suitable decomposition $\mathcal{T} = (T_1, \ldots, T_{k+1})$ of G, the details of our construction depending on where the selected edge $e = uv$ occurs in $\mathcal{T}'$. If e is in T_1' then $T_1 = T_1' - uv + ux + vx + wy$, so T_1 does not contain e, and is a Hamilton path in G since w is an endpoint of T_1'. Otherwise e is in T_r' for some $r \in [2..k]$. Then $T_1 = T_1' \cup \{wy, xy\}$ is a spanning path in G, and $T_r = T_r' \cup \{ux, vx, yz\} - \{uv\}$, for a suitable vertex z not of maximum degree in T_r', so T_r is a spanning tree that does not contain e, and $\Delta(T_r) = \Delta(T_r') = r+1$. As in Case 1, for every other $i \in [2..k]$ we form a tree T_i by adding two edges to T_i' without changing the maximum degree, in such a way that the remaining $2k+2$ edges form a tree T_{k+1} with maximum degree $k+2$. The details are given in Section 5.

Thus, subject to details pending in Sections 4 and 5, in every case G has a suitable decomposition, and the theorem follows by induction on n. □

3. An extension of Hall's Theorem

Let $\mathcal{A} = (A_1, \ldots, A_n)$ be a sequence of n sets and $\boldsymbol{m} = (m_1, \ldots, m_n)$ be a sequence of n positive integers. A sequence $\mathcal{B} = (B_1, \ldots, B_n)$ of n pairwise disjoint sets is a *system of distinct representatives* of $\mathcal{A}$ with *multiplicity sequence* $\boldsymbol{m}$ if $B_i \subseteq A_i$ and $|B_i| = m_i$ for each $i \in [1..n]$. The sequence $\mathcal{B}$ is a standard system of distinct representatives when $\boldsymbol{m} = (1, \ldots, 1)$. We shall apply the following modest generalization of Philip Hall's Theorem [3]:

Theorem 3. *A sequence $\mathcal{A} = (A_1, \ldots, A_n)$ of n sets has a system of distinct represent-atives with multiplicity sequence $\boldsymbol{m} = (m_1, \ldots, m_n)$ if and only if*

$$|\cup_{i \in I} A_i| \geq \Sigma_{i \in I}\, m_i \qquad \textit{for every } I \subseteq [1..n].$$

Proof. Necessity of the inequalities is clear. Now to see the sufficiency, suppose all the inequalities hold. For each $i \in [1..n]$, let $\mathcal{A}_i$ be a constant sequence of m_i sets, each equal to A_i, and let $\mathcal{X} = (X_1, \ldots, X_M)$ be the sequence of $M = \Sigma_{1 \leq i \leq n}\, m_i$ sets, formed by concatenating the sequences $\mathcal{A}_i$. Note that we are seeking a standard system of distinct representatives for the sequence $\mathcal{X}$. With any subset $J \subseteq [1..M]$, associate the subset $\Omega(J) = \{i \in [1..n]\text{: } X_j \in \mathcal{A}_i \text{ for some } j \in J\}$. Since all m_i sets in $\mathcal{A}_i$ are equal to A_i, we have

$$|\cup_{j \in J} X_j| = |\cup_{i \in \Omega(J)} A_i| \geq \Sigma_{i \in \Omega(J)}\, m_i \geq \Sigma_{j \in J}\, 1 = |J|.$$

Now Hall's Theorem ensures that $\mathcal{X}$ has a system of distinct representatives. □

Our construction for Theorem 2 uses a *twofold system of distinct representatives,* corresponding to the constant multiplicity sequence $\boldsymbol{m} = (2, \ldots, 2)$.

4. Case 1 construction for Theorem 2

We now describe in detail and verify the construction for Case 1 of the proof of Theorem 2, when H has vertices x and y of degree 1 and 0, respectively. Then $H' = H - \{x, y\}$ and $G' = G - \{x, y\} = K_{2k+1} - E(H')$ has a spanning tree decomposition $\mathcal{T}' = (T_1', \ldots, T_k')$ with $\Delta(T_i') = i+1$ for each $i \in [1..k]$. We build the decomposition $\mathcal{T} = (T_1, \ldots, T_{k+1})$ of G. Let z be the neighbor of x in H. Since x is adjacent in G to all vertices except z, at least one endpoint w of the path T_1' is adjacent in G to x, so w and z are distinct, $T_1 = T_1' \cup \{wx, xy\}$ is a spanning path in G, and $\Delta(T_1) = 2$.

For each $i \in [2..k]$, let A_i be the set of vertices of degree less than $i+1$ in T_i', and put $A_i' = A_i \setminus \{w, z\}$. Suppose $\mathcal{A}' = (A_2', \ldots, A_k')$ has a twofold system of distinct representatives $\mathcal{B} = (B_2, \ldots, B_k)$, say $B_i = \{u_i, v_i\}$ for each $i \in [2..k]$. Then $T_i = T_i' \cup \{u_ix, v_iy\}$ is a spanning tree in G with $\Delta(T_i) = \Delta(T_i') = i+1$. In G the k edges $\{wx\} \cup \{u_ix : i \in [2..k]\}$ incident with x but not y are distinct from the k edges $\{xy\} \cup \{v_iy : i \in [2..k]\}$ incident with y, so the k trees $\{T_i : i \in [1..k]\}$ are edge-disjoint. Since G' has order $2k+1$, and the set $\{u_i, v_i : i \in [2..k]\} \cup \{w, z\}$ comprises $2k$ of its vertices, there is a unique remaining vertex v in G'. The remaining edges in G form two stars: the set $\{vx\} \cup \{v_ix : i \in [2..k]\}$ forms S_{k+1}, of order $k+1$ with center x, and the set $\{vy, wy, yz\} \cup \{u_iy : i \in [2..k]\}$ forms S_{k+3}, of order $k+3$ with center y. These two stars share the single vertex v, so together they form a diameter 4 spanning tree T_{k+1} of G with maximum degree $\Delta(T_{k+1}) = k+2$ at y. The details of the construction will now be verified.

When $k = 1$ the graph H must have a vertex x of degree 1, so is necessarily in Case 1. Then $\mathcal{T}' = (T_1')$, so $\mathcal{A}'$ is empty and T_2 has edge set $\{vx\} \cup \{vy, wy, yz\}$. When $k \geq 2$ we must confirm that $\mathcal{A}' = (A_2', \ldots, A_k')$ does have a twofold system of distinct representatives. By Theorem 3, it suffices to show $|\cup_{i \in I} A_i'| \geq 2|I|$ for every $I \subseteq [2..k]$. Any tree T of order $2k+1$ has degree sum $4k$, so if T has maximum degree $\Delta \geq 3$ then it has at least $k+2$ vertices of degree less than Δ, otherwise its degree sum would be at least $k\Delta+k+1 \geq 4k+1$, a contradiction. Hence $|A_i'|$

$= |A_i \setminus \{w, z\}| \geq |A_i| - 2 \geq k$ for every $i \in [2..k]$. Thus $|\bigcup_{i \in I} A_i'| \geq k$ holds for every nonempty index set $I \subseteq [2..k]$, so the inequality $|\bigcup_{i \in I} A_i'| \geq 2|I|$ certainly holds whenever $|I| \leq \lfloor k/2 \rfloor$. In particular, it follows that $\mathcal{A}'$ has a twofold system of distinct representatives when $k = 2$, so we may suppose $k \geq 3$.

We now show that the required inequality also holds when $|I| \geq \lfloor (k+2)/2 \rfloor$. Now $|I| \geq 2$, and the two largest members s, t of I are at least as large as the two largest members of $[2..\lfloor (k+4)/2 \rfloor]$, so $s+t \geq \lfloor (k+2)/2 \rfloor + \lfloor (k+4)/2 \rfloor \geq k+2$. We claim that $|A_s' \cup A_t'| = 2k-1$. Then $|\bigcup_{i \in I} A_i'| \geq 2k-1 > 2(k-1) \geq 2|I|$, so the required inequality certainly holds. To verify the claim, suppose a vertex u has maximum degree in two different trees in $\mathcal{T}'$, say T_i' and T_j'. Trees in $\mathcal{T}'$ are edge-disjoint, and u has degree at most $2k$ in G', so its degree sum over all members of $\mathcal{T}'$ yields $2k \geq (i+1) + (j+1) + (k-2)$, whence $i+j \leq k$. Therefore no vertex of G' has maximum degree in two different trees T_i' and T_j' with index sum $i+j \geq k+1$, so in that case $A_i \cup A_j$ contains all vertices of G'. Then $|A_i' \cup A_j'| = |(A_i \cup A_j) \setminus \{w, z\}| = |A_i \cup A_j| - 2 = 2k-1$ whenever $i+j \geq k+1$. The claim follows with $\{i, j\} = \{s, t\}$. Thus $\mathcal{A}'$ always has a twofold system of distinct representatives, as required. □

5. Case 2 construction for Theorem 2

To complete the proof of Theorem 2 we now describe and verify the details of the construction for Case 2, when H has no vertices of degree 1. In this case H has order $2k+3$ and size $k+1$, where $k \geq 2$. Select vertices x and y of degree 0 and any edge $e = uv$ in H. Then $H' = H - \{uv\} - \{x, y\}$ and $G' = G \cup \{uv\} - \{x, y\} = K_{2k+1} - E(H')$ has a spanning tree decomposition $\mathcal{T}' = (T_1', \ldots, T_k')$ with $\Delta(T_i') = i+1$ for each $i \in [1..k]$. As noted earlier, T_k' has a unique vertex w of degree n, and w has degree 1 in T_i' for each $i \in [1..k-1]$. We build the decomposition $\mathcal{T} = (T_1, \ldots, T_{k+1})$ of G. Since $e = uv$ is an edge of G', it belongs to a unique tree in $\mathcal{T}'$, say T_r' for some $r \in [1..k]$. But e is not in G, so it must be deleted to produce $\mathcal{T}$. This results in two variants to the construction, depending upon the index r.

For each $i \in [2..k]$, let A_i be the set of all vertices of degree less than i+1 in T_i', and put $A_i' = A_i \setminus \{u, v, w\}$. Suppose that $\mathcal{A}' = (A_2', \ldots, A_k')$ has a twofold system of distinct representatives $\mathcal{B} = (B_2, \ldots, B_k)$, say $B_i = \{u_i, v_i\}$ for each $i \in [2..k]$. If $r = 1$, put $T_1 = (T_1' - \{uv\}) \cup \{ux, vx, wy\}$, and $T_i = T_i' \cup \{u_i x, v_i y\}$ for each $i \in [2..k]$. If $r > 1$, put $T_1 = T_1' \cup \{wx, xy\}$, $T_r = (T_r' - \{uv\}) \cup \{ux, vx, v_r y\}$ and $T_i = T_i' \cup \{u_i x, v_i y\}$ for each $i \in [2..k] \setminus \{r\}$. Since we have deleted $e = uv$, and added edges incident with x and y, in every instance we have produced a spanning tree of G. In both cases T_1 is a path, since w is an endpoint of T_1', so $\Delta(T_1) = 2$. Also $\Delta(T_i) = \Delta(T_i') = i$+1 for each $i \in [2..k]$. Since $\mathcal{T}'$ is a decomposition of G', clearly the trees in $\{T_i : i \in [1..k]\}$ are edge-disjoint. Let E be the set of $2k$+2 edges remaining in G. When $r = 1$, the set E comprises $\{wx\} \cup \{v_i x : i \in [2..k]\}$, forming a star S_{k+1} with center x, and the set $\{uy, vy, xy\} \cup \{u_i y : i \in [2..k]\}$, forming a star S_{k+3} with center y. These stars are edge-disjoint, and x is the single vertex they have in common, so they form a diameter 4 spanning tree T_{k+1} of G with maximum degree $\Delta(T_{k+1}) = k+2$ at y. When $r > 1$, the set E comprises $\{u_r x\} \cup \{v_i x : i \in [2..k]\}$, forming a star S_{k+1} with center x, and the set $\{uy, vy, wy\} \cup \{u_i y : i \in [2..k]\}$, forming a star S_{k+3} with center y. Again these stars are edge-disjoint, but now they have the single vertex u_r in common, so they form a diameter 4 spanning tree T_{k+1} of G with maximum degree $\Delta(T_{k+1}) = k+2$ at y.

Again, it remains to confirm that $\mathcal{A}' = (A_2', \ldots, A_k')$ has a twofold system of distinct representatives. It suffices to show $|\bigcup_{i \in I} A_i'| \geq 2|I|$ for every $I \subseteq [2..k]$. As w is the unique vertex of maximum degree in T_k', so $|A_k'| = |A_k \setminus \{u, v, w\}| = |A_k \setminus \{u, v\}| = |A_k| - 2 = 2k-2$. In particular, if $k = 2$ then $\mathcal{A}' = (A_2')$ and $|A_2'| = 2$, so $\mathcal{A}'$ certainly has a twofold system of distinct representatives. Now take $k \geq 3$. As noted in Case 1, a tree of order $2k$+1 and maximum degree $\Delta \geq 3$ has at least k+2 vertices of degree less than Δ, so $|A_i'| = |A_i \setminus \{u, v, w\}| \geq |A_i| - 3 \geq k-1$ for every $i \in [2..k]$. Therefore $|\bigcup_{i \in I} A_i'| \geq k-1$ holds for every nonempty set $I \subseteq [2..k]$, so $|\bigcup_{i \in I} A_i'| \geq 2|I|$ certainly holds whenever $|I| \leq \lfloor (k-1)/2 \rfloor$.

We must show $|\bigcup_{i \in I} A_i'| \geq 2|I|$ also holds if $|I| \geq \lfloor (k+1)/2 \rfloor$. Then $|I| \geq 2$, and the two largest members s, t of I are at least as large as those of $[2..\lfloor (k+3)/2 \rfloor]$, so $s+t \geq \lfloor (k+1)/2 \rfloor + \lfloor (k+3)/2 \rfloor \geq k+1$. As noted in Case 1, no vertex can attain maximum degree in two trees T_i' and T_j' with index sum $i+j \geq k+1$, so in that case $A_i \cup A_j$ contains all vertices of G'. With $\{i, j\} = \{s, t\}$ in particular, $|\bigcup_{i \in I} A_i'| \geq |A_s' \cup A_t'| = |(A_s \cup A_t) \setminus \{u, v, w\}| = |A_s \cup A_t| - 3 = 2k-2 = 2(k-1) \geq 2|I|$, so the desired inequality certainly holds. Hence $\mathcal{A}'$ always has a twofold system of distinct representatives, as required. □

The proof of Theorem 2 is now complete. It is worth remarking that, with greater attention to the structural details of the decomposition $\mathcal{T}'$ of G', in both Case 1 and Case 2 it can be shown that $|A_i'| \geq 2(i-1)$ for every $i \in [2..k]$. Thus the sequence of sets $\mathcal{A}' = (A_2', \ldots, A_k')$ has a triangular structure which ensures that when building $\mathcal{T}$ from $\mathcal{T}'$ we can simply proceed in order, choosing in each T_i' with $i \in [2..k]$ any two unused vertices not of maximum degree in T_i' and not in the set S, where $S = \{w, z\}$ in Case 1, and $S = \{u, v, w\}$ in Case 2.

6. Decompositions for even order

With $n \geq 2$, now let T be a given tree of order $2n$, and consider spanning tree decompositions of the graph $G = K_{2n} - E(T)$, of order $2n$ and size $(n-1)(2n-1)$. Thus G meets the size constraint for decomposition into spanning trees. However, if such a decomposition is to be possible we must further restrict T; in particular, its maximum degree $\Delta(T)$ cannot be too large. For instance, G has no spanning tree decomposition when T is a star, since G then has a degree 0 vertex.

Lemma 1. *Let T be a spanning tree of K_{2n} and let $G = K_{2n} - E(T)$. If $n \geq 2$ and G has a spanning tree decomposition $\mathcal{T} = (T_1, \ldots, T_{n-1})$ of any kind, then $\Delta(T) \leq n$. Moreover, if $n \geq 3$ and G has a spanning tree decomposition $\mathcal{T}$ with $\Delta(T_1) = 2$ and $\Delta(T_{n-1}) = n$, then T has at most one vertex of degree n.*

Proof. Suppose $G = K_{2n} - E(T)$ has a spanning tree decomposition $\mathcal{T} = (T_1, \ldots, T_{n-1})$ of any kind. Each vertex of G must have degree at least $n-1$ since it contributes at least one edge to each of $n-1$ edge-disjoint spanning trees, so $\Delta(T) \leq n$. Suppose $\Delta(T) = n$. Then some vertex v has degree n in T, and degree at least 1 in T_i for each $i \in [1..n-1]$; but its degree is $2n-1$ in K_{2n}, so its degree in each T_i is exactly 1. Now suppose further that $\Delta(T_1) = 2$ and $\Delta(T_{n-1}) = n$. There is a vertex w of degree n in T_{n-1} and, by the previous argument, w has degree 1 in T_i for each $i \in [1..n-2]$ and also in T. Hence v and w are distinct, and both have degree 1 in T_1. But T_1 is a path because $\Delta(T_1) = 2$, and T_1 is distinct from T_{n-1} when $n \geq 3$, so in that case v and w are the two endpoints of T_1. But v and w are any vertices of degree n in T and T_{n-1}, respectively, so neither tree can have more than one vertex of degree n. The lemma follows. □

When $n \geq 3$ and $G = K_{2n} - E(T)$ has a spanning tree decomposition $\mathcal{T}$ with $\Delta(T) = n$, $\Delta(T_1) = 2$ and $\Delta(T_{n-1}) = n$, we have just shown that T and T_{n-1} each have exactly one vertex of degree n. But if $\Delta(T) < n$, then T_{n-1} can have two vertices of degree n. For example, K_6 can easily be decomposed into two Hamilton paths and a double star $S_{3,3}$ with two vertices of degree 3; taking T to be one of the Hamilton paths yields an instance.

The degree sum immediately shows that any tree of order $2n$ has at most two vertices of degree n. If two vertices do have degree n then all others have degree 1, so the tree is the double star $S_{n,n}$ with adjacent vertices of degree n. Thus Lemma 1 allows T to be any tree of maximum degree $\Delta(T) \leq n$ except $S_{n,n}$.

Theorem 4. *Let T be a spanning tree of K_{2n} and let $G = K_{2n} - E(T)$, with $n \geq 2$. Then G has a purely heterogeneous spanning tree decomposition $\mathcal{T} = (T_1, \ldots, T_{n-1})$ which satisfies the maximum degree condition $\Delta(T_i) = i+1$ for each $i \in [1..n-1]$ if and only if $\Delta(T) \leq n$, and T is not the double star $S_{n,n}$ when $n \geq 3$.*

Proof. The stated conditions are necessary, by Lemma 1. When $n = 2$ their sufficiency is shown by the decomposition of K_4 into two complementary Hamilton paths. We now outline the construction showing their sufficiency when $n \geq 3$, but another structural lemma will be required to complete the proof.

We proceed by induction on n. For a fixed integer $k \geq 2$, assume the theorem holds when $n = k$; now let $n = k+1$. The construction begins with two vertices x and y of degree 1 in T, chosen so that the tree $T' = T - \{x, y\}$ of order $2k$ has maximum degree $\Delta(T') \leq k$ and at most one vertex of degree k. The induction hypothesis applies to the graph $G' = K_{2k} - E(T')$, so G' has a spanning tree decomposition $\mathcal{T}' = (T_1', \ldots, T_{k-1}')$ which satisfies the maximum degree condition $\Delta(T_i') = i+1$ for each $i \in [1..k-1]$. From $\mathcal{T}'$ we build an appropriate decomposition $\mathcal{T} = (T_1, \ldots, T_k)$ of G, beginning with the path $T_1 = T_1' \cup \{xy, yz\}$, where z is a suitable endpoint of T_1'. Next, the trees T_i with $i \in [2..k-1]$ are constructed from the corresponding trees T_i' by adding two edges to each, one incident with x and the other incident with y, in such a way that the maximum degree of each tree is preserved, and all vertices of attachment are distinct. Moreover, the added edges are chosen so that the remaining $2k+1$ edges incident with x or y form a tree T_k with a unique vertex of maximum degree $k+1$. Subject to verification of details pending in the next two sections, it follows that G does always have a suitable decomposition, so the theorem follows by induction. □

7. A pruning property of trees

We now establish the pruning property of T used in the proof of Theorem 4.

Lemma 2. *For $n \geq 2$, let T be any tree of order $2n$ with maximum degree $\Delta(T) \leq n$. Then T has two leaves x, y with no common neighbor, such that $T' = T - \{x, y\}$ has maximum degree $\Delta(T') \leq n-1$. Further, if $n \geq 4$ and T is not the double star $S_{n,n}$ then x, y can be chosen so that T' has at most one vertex of degree $n-1$.*

Proof. When T is a path the claim is immediate, so suppose $n \geq 3$ and $3 \leq \Delta(T) \leq n$. If T has a vertex of degree n, that vertex is incident with more than half the edges in T, so it is adjacent to at least one leaf. The degree sum of T shows that it has at most two vertices of degree n, and this maximum occurs only when all other vertices are leaves, so when $T = S_{n,n}$. Thus in all cases we can choose two leaves x, y with no common neighbor such that $T' = T - \{x, y\}$ satisfies $\Delta(T') \leq n-1$.

Now suppose $n \geq 4$ and T is not $S_{n,n}$. Then T satisfies the lemma if T' is not the double star $S_{n-1,n-1}$. So consider the case $T' = S_{n-1,n-1}$ and let u, v be the two vertices of degree $n-1$ in T'. Since T is not $S_{n,n}$, it follows that at least one of x, y is adjacent to neither u nor v in T: without loss of generality, suppose x has this property, and is at distance 2 from u in T. If y is adjacent to neither u nor v in T, choose a leaf w adjacent to v in T, and replace T' by $T^* = T - \{w, y\}$. Then the lemma is satisfied, since u has degree $n-1$ and v has degree $n-2$ in T^*. If y is adjacent to either u or v in T, there is a leaf w at distance 3 from y in T. In that case, replace T' by $T^* = T - \{w, y\}$ and the lemma is satisfied, since in T^* one of u, v has degree $n-1$, the other $n-2$. □

8. Construction details for Theorem 4

For $n \geq 3$ we now show that a spanning tree decomposition $\mathcal{T}' = (T_1', \ldots, T_{n-1}')$ which satisfies the maximum degree condition $\Delta(T_i') = i+1$ for each $i \in [1..n-1]$ of $G' = K_{2n} - E(T')$ extends to a corresponding decomposition of $G = K_{2n+2} - E(T)$, provided the tree T of order $2n+2$ satisfies $\Delta(T) \leq n+1$ and is not $S_{n+1,n+1}$. But in fact there is a potential technical hitch that could prevent the final set of $2n+1$ edges from forming a tree. To overcome this difficulty we shall prove a stronger, more technical result which always lets us select a spanning tree decomposition that does extend as required (and has two extensions in which the Hamilton paths do not have the same pair of endpoints).

Claim. *For $n \geq 3$, if T is any tree of order $2n$ satisfying $\Delta(T) \leq n$, excluding $S_{n,n}$, there are always two spanning tree decompositions $\mathcal{T}$ and $\mathcal{T}^*$ of $G = K_{2n} - E(T)$ that satisfy the maximum degree condition and include Hamilton paths T_1 and $T_1{}^*$ which do not have the same pair of endpoints.*

Proof of claim. When $n = 3$ choose any tree T of order 6 with $\Delta(T) \leq 3$, except $S_{3,3}$. Lemma 2 ensures that T has two leaves x, y with no common neighbor, such that $T' = T - \{x, y\}$ has maximum degree $\Delta(T') \leq 2$. Thus $T' = P_4 = abcd$. Since T is not $S_{3,3}$, without loss of generality x is adjacent to a in T. Then y is adjacent to exactly one of b, c, d; in each case take $T_1 = P_6$: ac, ad, bd, xy, xz with $z = c, b, b$ respectively, and let T_2 comprise the remaining edges of $G = K_6 - E(T)$. Then $\mathcal{T} = \{T_1, T_2\}$ is a spanning tree decomposition of G in which x is an interior point and y is an endpoint of the Hamilton path T_1, and y is the unique vertex of degree 3 in T_2. If xz is replaced by yz in this construction, yielding trees and $T_1{}^*$ and $T_2{}^*$, then $\mathcal{T}^* = \{T_1{}^*, T_2{}^*\}$ is also a spanning tree decomposition of G, where x is an endpoint and y is an interior point of the Hamilton path $T_1{}^*$, and x is the unique vertex of degree 3 in $T_2{}^*$. Thus the claim holds when $n = 3$.

For a fixed integer $k \geq 3$, assume the claim holds when $n = k$; now let $n = k+1$. Fix any spanning tree T of K_{2k+2} with $\Delta(T) \leq k+1$, excluding $S_{k+1,k+1}$. Choose vertices x, y of K_{2k+2} which are leaves of T satisfying the full $n \geq 4$ conditions in Lemma 2, and put $T' = T - \{x, y\}$. Let $\mathcal{T}' = (T_1', \ldots, T_{k-1}')$ and $\mathcal{T}'' = (T_1'', \ldots, T_{k-1}'')$ be two spanning tree decompositions of $G' = K_{2k} - E(T')$ satisfying the maximum degree condition and such that T_1' and T_1'' do not have the same pair of endpoints. At least one of $\mathcal{T}'$ and $\mathcal{T}''$ is such that one endpoint of its Hamilton path is adjacent in T to neither x nor y. Assume $\mathcal{T}'$ has this property, and z is an endpoint of T_1' adjacent to neither x nor y in T. Let $T_1 = T_1' \cup \{xy, yz\}$ and $T_1^* = T_1' \cup \{xy, xz\}$: these are two Hamilton paths in K_{2k+2} and they do not have the same pair of endpoints, since each has a different vertex of xy as an endpoint.

We now build a spanning tree decomposition $\mathcal{T} = (T_1, \ldots, T_k)$ of $G = K_{2k+2} - E(T)$ which includes the Hamilton path T_1 just constructed; a second decomposition, $\mathcal{T}^* = (T_1^*, \ldots, T_k^*)$, which includes the Hamilton path T_1^*, arises in essentially the same way. Let u, v be the neighbors of x, y in T. Then $S = \{u, v, x, y, z\}$ is a set of five distinct vertices in G, exactly three of which are in G'.

For each $i \in [2..k-1]$, let A_i be the set of vertices of degree less than $i+1$ in T_i', and put $A_i' = A_i \backslash S$. Suppose that $\mathcal{A}' = (A_2', \ldots, A_{k-1}')$ has a twofold system of distinct representatives $\mathcal{B} = (B_2, \ldots, B_{k-1})$, say $B_i = \{u_i, v_i\}$ for each $i \in [2..k-1]$. Put $T_i = T_i' \cup \{u_i x, v_i y\}$. Then T_i is a spanning tree of G with maximum degree $\Delta(T_i) = \Delta(T_i') = i+1$. Now let T_k be the subgraph of G induced by the edges not included in any T_i with $i \in [1..k-1]$. Then T_k has $2k+1$ edges. It does not include any edge in G', nor does it include xy or yz, so T_k is bipartite with $\{x, y\}$ as one independent set and the other $2k$ vertices as the other independent set. The degree of x and y in G is degree $2k$, so in T_k they have degree $k+1$ and k respectively. Again, z has degree 2 in T_k, while all other vertices have degree 1. Thus T_k is acyclic, so is a spanning tree with maximum degree $\Delta(T_k) = k+1$ achieved by a unique vertex, x. Hence $\mathcal{T} = (T_1, \ldots, T_k)$ is an appropriate decomposition of G.

It remains to show that $\mathcal{A}' = (A_2', \ldots, A_{k-1}')$ has a twofold system of distinct representatives. For $k = 3$ we have $\mathcal{A}' = (A_2')$ and T_2' is a tree of order 6 with maximum degree 3. If $|A_2'| \geq 2$ then trivially $\mathcal{A}'$ has a twofold system of distinct representatives: this certainly occurs when T_2' has exactly one vertex of degree 3. However, it is possible that T_2' has exactly two vertices of degree 3: this occurs precisely when T' is a

Hamilton path and $\mathcal{T}' = \{P_6, S_{3,3}\}$. But in that case z is an endpoint of $T_1' = P_6$ and has degree at most 2 in T', so z must have degree 3 in $T_2' = S_{3,3}$. Then z is not in A_2, so $|A_2'| \geq 2$ as before.

When $k \geq 4$, the degree sum guarantees that any tree of order $2k$ and maximum degree $\Delta \geq 3$ has at least $k+1$ vertices of degree less than Δ, so $|A_i'| \geq k-2$ for each $i \in [2..k-1]$. Suppose $|\bigcup_{i\in I} A_i'| < 2|I|$ for some subset $I \subseteq [2..k-1]$. Then we have $2|I| > k-2$; but $k \geq 4$, so $|I| \geq \lceil (k-1)/2 \rceil \geq 2$. Let the two highest indices in I be s, t. Then $s+t \geq \lceil (k-1)/2 \rceil + \lceil (k+1)/2 \rceil \geq k$. Suppose has a vertex w of K_{2k} is not in $A_s' \cup A_t' \cup \{u, v, z\}$. Then w has degree $s+1$ in T_s' and degree $t+1$ in T_t', and has positive degree in T' and in the $k-3$ trees T_i' with $i \in [1..k-1] \setminus \{s, t\}$, so the total degree of w would be at least $(s+1) + (t+1) + (k-2) \geq 2k$. But this is too large for any vertex in K_{2k} so, by contradiction, $A_s' \cup A_t' \cup \{u, v, z\}$ contains every vertex of K_{2k}. It follows that $|A_s' \cup A_t'| = 2k-3 > 2|I|$ for every subset $I \subseteq [2..k-1]$. Therefore $|\bigcup_{i\in I} A_i'| \geq 2|I|$ for every subset $I \subseteq [2..k-1]$, so $\mathcal{A}'$ always has a twofold system of distinct representatives, by Theorem 3. Thus the construction of $\mathcal{T}$ and $\mathcal{T}^*$ from $\mathcal{T}'$ is always possible, and the claim follows by induction on n. Clearly theorem follows. □

9. Conjecture

A much stronger constraint than that specified in Theorem 4 for spanning tree decompositions of K_{2n} is proposed in the following conjecture.

Conjecture. *For $n \geq 2$, let $\mathcal{T} = (T_1, \ldots, T_{n-1})$ be any sequence of $n-1$ trees of order $2n$ which satisfies the maximum degree condition $\Delta(T_i) = i+1$ for each $i \in [1..n-1]$. Then there exists a tree T of order $2n$ such that, with a suitable labeling of vertices, $\mathcal{T}$ is a purely heterogeneous spanning tree decomposition of $G = K_{2n} - E(T)$.*

Note that all but one of the n trees are specified (up to isomorphism). The conjecture is easily checked for $n = 2$ and 3. We thank Angela McCombs for verifying the far more complicated case $n = 4$. Other than the freedom to choose one tree, this conjecture is similar in several respects to the well-known elegant conjecture of Gyárfás & Lehel [**2**]:

Tree Packing Conjecture (Gyárfás & Lehel). *For any given positive integer n, and any sequence* $\mathcal{T} = (T_1, \ldots, T_n)$ *of n trees in which* T_i *has order i for each* $i \in [1..n]$*, there is a suitable labeling of vertices so that* $\mathcal{T}$ *forms a decomposition of* K_n.

In recent joint work, as yet unpublished, Abueida, Blinco, Clark, Daven and Eggleton have obtained results on heterogeneous spanning tree decompositions of uniform complete multigraphs. Close in spirit to the present paper is their theorem that $K_n^{(2)}$ has a purely heterogeneous spanning tree decomposition exactly when $n \geq 6$. This generalizes a result of Eggleton [**1**], showing that $K_6^{(2)}$ has a spanning tree decomposition into one copy of every tree of order 6.

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