

# On cyclic $G$ -designs where $G$ is the one-point union of two cycles

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*Dedicated in honor of Roger B. Eggleton*

## Abstract

Let  $G$  be the one-point union of two cycles and suppose  $G$  has  $n$  edges. We show via various graph labelings that there exists a cyclic  $G$ -decomposition of  $K_{2nt+1}$  for every positive integer  $t$ .

## 1 Introduction

If  $a$  and  $b$  are integers we denote  $\{a, a+1, \dots, b\}$  by  $[a, b]$  (if  $a > b$ ,  $[a, b] = \emptyset$ ). Let  $\mathbb{N}$  denote the set of nonnegative integers and  $\mathbb{Z}_t$  the group of integers modulo  $t$ . For a graph  $G$ , let  $V(G)$  and  $E(G)$  denote the vertex set of  $G$  and the edge set of  $G$ , respectively. The *order* and the *size* of a graph  $G$  are  $|V(G)|$  and  $|E(G)|$ , respectively. If  $G_1$  and  $G_2$  are vertex-transitive graphs,  $G_1 \circ G_2$  denotes the one-point union of  $G_1$  and  $G_2$ .

Let  $V(K_t) = \{0, 1, \dots, t-1\}$ . The *length* of an edge  $\{i, j\}$  in  $K_t$  is  $\min\{|i-j|, t-|i-j|\}$ . Note that if  $t$  is odd, then  $K_t$  consists of  $t$  edges of length  $i$  for  $i = 1, 2, \dots, \frac{t-1}{2}$ .

Let  $V(K_t) = \mathbb{Z}_t$  and let  $G$  be a subgraph of  $K_t$ . By *clicking*  $G$ , we mean applying the permutation  $i \rightarrow i+1$  to  $V(G)$ . Note that clicking an edge does not change its length. Let  $H$  and  $G$  be graphs such that  $G$  is a

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\*Research supported by National Science Foundation Grant No. A0649210

subgraph of  $H$ . A  $G$ -decomposition of  $H$  is a set  $\Delta = \{G_1, G_2, \dots, G_r\}$  of pairwise edge-disjoint subgraphs of  $H$  each of which is isomorphic to  $G$  and such that  $E(H) = \bigcup_{i=1}^r E(G_i)$ . A  $G$ -decomposition of  $K_t$  is also known as a  $(K_t, G)$ -design. A  $(K_t, G)$ -design  $\Delta$  is *cyclic* if clicking is an automorphism of  $\Delta$ . The study of graph decompositions is generally known as the study of graph designs, or  $G$ -designs. For recent surveys on  $G$ -designs, see [1] and [6].

Let  $G$  be a graph of size  $n$ . A primary question in the study of graph designs is: *for what values of  $k$  does there exist a  $(K_k, G)$ -design?* For most studied graphs  $G$ , it is often the case that if  $k \equiv 1 \pmod{2n}$ , then there exists a  $(K_k, G)$ -design. A common approach to finding these designs is through the use of graph labelings.

In this paper, we will show via various graph labelings that if  $G$  of size  $n$  is the one-point union of two cycles, then there exists a cyclic  $G$ -decomposition of  $K_{2nt+1}$  for every positive integer  $t$ .

## 1.1 Graph Labelings

For any graph  $G$ , a one-to-one function  $f : V(G) \rightarrow \mathbb{N}$  is called a *labeling* (or a *valuation*) of  $G$ . In [12], Rosa introduced a hierarchy of labelings. Let  $G$  be a graph with  $n$  edges and no isolated vertices and let  $f$  be a labeling of  $G$ . Let  $f(V(G)) = \{f(u) : u \in V(G)\}$ . Define a function  $\bar{f} : E(G) \rightarrow \mathbb{Z}^+$  by  $\bar{f}(e) = |f(u) - f(v)|$ , where  $e = \{u, v\} \in E(G)$ . We will refer to  $\bar{f}(e)$  as the *label* of  $e$ . Let  $\bar{f}(E(G)) = \{\bar{f}(e) : e \in E(G)\}$ . Consider the following conditions:

$$(\ell 1) \quad f(V(G)) \subseteq [0, 2n],$$

$$(\ell 2) \quad f(V(G)) \subseteq [0, n],$$

$$(\ell 3) \quad \bar{f}(E(G)) = \{x_1, x_2, \dots, x_n\}, \text{ where for each } i \in [1, n] \text{ either } x_i = i \text{ or } x_i = 2n + 1 - i,$$

$$(\ell 4) \quad \bar{f}(E(G)) = [1, n].$$

If in addition  $G$  is bipartite with bipartition  $\{A, B\}$  of  $V(G)$  consider also

$$(\ell 5) \quad \text{for each } \{a, b\} \in E(G) \text{ with } a \in A \text{ and } b \in B, \text{ we have } f(a) < f(b),$$

$$(\ell 6) \quad \text{there exists an integer } \lambda \text{ such that } f(a) \leq \lambda \text{ for all } a \in A \text{ and } f(b) > \lambda \text{ for all } b \in B.$$

Then a labeling satisfying the conditions:

$$(\ell 1), (\ell 3) \quad \text{is called a } \rho\text{-labeling};$$

$$(\ell 1), (\ell 4) \quad \text{is called a } \sigma\text{-labeling};$$

(ℓ2), (ℓ4) is called a  $\beta$ -labeling.

A  $\beta$ -labeling is necessarily a  $\sigma$ -labeling which in turn is a  $\rho$ -labeling. Suppose  $G$  is bipartite. If a  $\rho$ ,  $\sigma$ , or  $\beta$ -labeling of  $G$  satisfies condition (ℓ5), then the labeling is *ordered* and is denoted by  $\rho^+$ ,  $\sigma^+$  or  $\beta^+$ , respectively. If in addition (ℓ6) is satisfied, the labeling is *uniformly-ordered* and is denoted by  $\rho^{++}$ ,  $\sigma^{++}$  or  $\beta^{++}$ , respectively.

A  $\beta$ -labeling is better known as a *graceful* labeling and a uniformly-ordered  $\beta$ -labeling is an  $\alpha$ -labeling as introduced in [12]. Labelings of the types above are called *Rosa-type* because of Rosa's original article [12] on the topic. (See [9] for a recent comprehensive survey of Rosa-type labelings). A dynamic survey on general graph labelings is maintained by Gallian [11].

Call a connected graph  $G$  *Eulerian* if every vertex of  $G$  has even degree. If a graph  $G$  with Eulerian components admits a  $\sigma$ -labeling, then we have the following well-known restriction on  $|E(G)|$ .

**Theorem 1** (Parity Condition in [12]) *If a graph  $G$  with Eulerian components and  $n$  edges has a  $\sigma$ -labeling, then  $n \equiv 0$  or  $3 \pmod{4}$ . If in addition  $G$  is bipartite, then  $n \equiv 0 \pmod{4}$ .*

Labelings are critical to the study of cyclic graph decompositions as seen in the following two results from [12].

**Theorem 2** *Let  $G$  be a graph with  $n$  edges. There exists a cyclic  $G$ -decomposition of  $K_{2n+1}$  if and only if  $G$  admits a  $\rho$ -labeling.*

**Theorem 3** *Let  $G$  be a bipartite graph with  $n$  edges that admits an  $\alpha$ -labeling. Then there exists a cyclic  $G$ -decomposition of  $K_{2nt+1}$  for all positive integers  $t$ .*

It is easy to see how Theorem 3 works. Let  $G$  have bipartition  $\{A, B\}$  and let  $h$  be an  $\alpha$ -labeling for  $G$  with  $h(A) < h(B)$ . Let  $B_1, B_2, \dots, B_t$  be  $t$  vertex-disjoint copies of  $B$ . The vertex in  $B_i$  corresponding to  $b \in B$  will be called  $b_i$ . Let  $B^* = \bigcup_{i=1}^t B_i$ . We define a new graph  $G^*$  with vertex set  $A \cup B^*$  and edges  $\{a, b_i\}$ ,  $1 \leq i \leq t$ , whenever  $a \in A$  and  $\{a, b\}$  is an edge of  $G$ . Clearly  $G^*$  has  $nt$  edges and  $G$  divides  $G^*$ . Define a labeling  $h^*$  on  $G^*$  by

$$h^*(v) = \begin{cases} h(v) & v \in A, \\ h(b) + (i-1)n & v = b_i \in B_i. \end{cases}$$

The labeling  $h^*$  is an  $\alpha$ -labeling of  $G^*$  (which is also a  $\rho$ -labeling) and thus the result follows by Theorem 2.

From a graph decompositions perspective, Theorem 3 offers a great advantage over Theorem 2. However, there are many classes of bipartite graphs (see [9]) that do not admit  $\alpha$ -labelings. Theorem 3 was extended to cover graphs that admit  $\rho^+$ -labelings in [8].

**Theorem 4** *Let  $G$  be a bipartite graph with  $n$  edges that admits an  $\rho^+$ -labeling. Then there exists a cyclic  $G$ -decomposition of  $K_{2nt+1}$  for all positive integers  $t$ .*

Theorem 4 is set up in exactly the same way as Theorem 3 except that now we let  $h^*(v) = h(b) + (i - 1)2n$  for  $v = b_i \in B_i$ . It is conjectured by El-Zanati and Vanden Eynden (see [9]) that every bipartite graph admits a  $\rho^+$ -labeling.

Labelings that lead to results similar those of Theorem 4 have now been introduced for almost-bipartite graphs [4] and for tripartite graphs in general [7].

A non-bipartite graph  $G$  is said to be *almost-bipartite* if  $G - e$  is bipartite for some  $e \in E(G)$ . Note that if  $G$  is almost-bipartite with  $e = \{\hat{b}, c\}$ , then  $G$  is necessarily tripartite and  $V(G)$  can be partitioned into three sets  $A$ ,  $B$  and  $C = \{c\}$  such that  $\hat{b} \in B$  and  $e$  is the only edge joining an element of  $B$  to  $c$ .

Let  $G$  be an almost-bipartite graph with  $n$  edges with vertex tripartition  $A, B, C$  as above. A labeling  $h$  of the vertices of  $G$  is called a  $\gamma$ -labeling of  $G$  if the following conditions hold:

- (g1) The function  $h$  is a  $\rho$ -labeling of  $G$ .
- (g2) If  $\{a, v\}$  is an edge of  $G$  with  $a \in A$ , then  $h(a) < h(v)$ .
- (g3) We have  $h(c) - h(\hat{b}) = n$ .

Several classes of almost-bipartite graphs have been shown to have  $\gamma$ -labelings (see [9]). It was shown in [4] that  $\gamma$ -labelings yield results similar to  $\rho^+$ -labelings.

**Theorem 5** *Let  $G$  be an almost-bipartite graph with  $n$  edges that admits a  $\gamma$ -labeling. Then there exists a cyclic  $G$ -decomposition of  $K_{2nt+1}$  for all positive integers  $t$ .*

We illustrate how Theorem 5 works. Let  $G$  have  $n$  edges and let  $h$  be a  $\gamma$ -labeling for  $G$ , with  $A, B, C, c$ , and  $\hat{b}$  as in the above definition. Let  $B_1, B_2, \dots, B_t$  be  $t$  vertex-disjoint copies of  $B$ , and let  $c_1, c_2, \dots, c_t$  be  $t$  new vertices. The vertex in  $B_i$  corresponding to  $b \in B$  will be called  $b_i$ . Let  $B^* = \bigcup_{i=1}^t B_i$  and  $C^* = \{c_1, c_2, \dots, c_t\}$ . We define a new graph  $G^*$  with vertex set  $A \cup B^* \cup C^*$  and edges  $\{a, v_i\}$ ,  $1 \leq i \leq t$ , whenever  $a \in A$  and  $\{a, v\}$  is an edge of  $G$ , and  $\{\hat{b}_i, c_i\}$ ,  $1 \leq i \leq t$ . Clearly  $G^*$  has  $nt$  edges and  $G$  divides  $G^*$ . Define a labeling  $h^*$  on  $G^*$  by

$$h^*(v) = \begin{cases} h(v) & v \in A, \\ h(b) + (i - 1)2n & v = b_i \in B_i, \\ h(c) + (t - i)2n & v = c_i. \end{cases}$$

The labeling  $h^*$  is a  $\rho$ -labeling of  $G^*$  and thus the result follows by Theorem 2.

The concept of a  $\gamma$ -labeling was generalized in [7] to cover tripartite graphs that are not necessarily almost-bipartite. Let  $G$  be a tripartite graph with  $n$  edges having the vertex tripartition  $\{A, B, C\}$ . A  $\sigma$ -*tripartite* labeling of  $G$  is a one-to-one function  $h : V(G) \rightarrow [0, 2n]$  that satisfies

(s1)  $h$  is a  $\sigma$ -labeling of  $G$ .

(s2) If  $\{a, v\} \in E(G)$  with  $a \in A$ , then  $h(a) < h(v)$ .

(s3) If  $e = \{b, c\} \in E(G)$  with  $b \in B$  and  $c \in C$ , then there exists an edge  $e' = \{b', c'\} \in E(G)$  with  $b' \in B$  and  $c' \in C$  such that

$$|h(c') - h(b')| + |h(c) - h(b)| = n.$$

(s4) If  $a \in A$  and  $v \in B \cup C$ , then  $h(a) - h(v) \neq n$ .

(s5) If  $b \in B$  and  $c \in C$ , then  $|h(b) - h(c)| \notin \{n, 2n\}$ .

A  $\sigma$ -tripartite labeling of  $G$  is necessarily a  $\sigma$ -labeling of  $G$ . Thus the parity condition must be satisfied in order for  $G$  to admit a  $\sigma$ -tripartite labeling.

Also, a  $\rho$ -*tripartite* labeling of  $G$  is a one-to-one function  $h : V(G) \rightarrow [0, 2n]$  that satisfies

(r1)  $h$  is a  $\rho$ -labeling of  $G$ .

(r2) If  $\{a, v\} \in E(G)$  with  $a \in A$ , then  $h(a) < h(v)$ .

(r3) If  $e = \{b, c\} \in E(G)$  with  $b \in B$  and  $c \in C$ , then there exists an edge  $e' = \{b', c'\} \in E(G)$  with  $b' \in B$  and  $c' \in C$  such that

$$|h(c') - h(b')| + |h(c) - h(b)| = 2n.$$

(r4) If  $b \in B$  and  $c \in C$ , then  $|h(b) - h(c)| \neq 2n$ .

We note that a  $\gamma$ -labeling of a graph  $G$  is necessarily a  $\rho$ -tripartite labeling of  $G$ . For the purposes of this manuscript, we will consider almost-bipartite graphs separately from general tripartite graphs.

The following theorem shows that the above tripartite labelings yield results similar to  $\gamma$ -labelings:

**Theorem 6** *If a tripartite graph  $G$  with  $n$  edges has a  $\sigma$ -tripartite or a  $\rho$ -tripartite labeling, then there exists a cyclic  $G$ -decomposition of  $K_{2nt+1}$  for all positive integers  $t$ .*

Theorem 6 works in a similar way to Theorem 5. Let  $G$  have  $n$  edges and let  $h$  be a  $\sigma$ -tripartite or a  $\rho$ -tripartite labeling for  $G$ , with  $A$ ,  $B$ , and  $C$  as in the above definitions. Let  $B_1, B_2, \dots, B_t$  be  $t$  vertex-disjoint copies of  $B$ , and let  $C_1, C_2, \dots, C_t$  be  $t$  vertex-disjoint copies of  $C$ . The vertex in  $B_i$  corresponding to  $b \in B$  will be called  $b_i$ . Similarly, the vertex in  $C_i$  corresponding to  $c \in C$  will be called  $c_i$ . Let  $B^* = \bigcup_{i=1}^t B_i$  and  $C^* = \bigcup_{i=1}^t C_i$ . We define a new graph  $G^*$  with vertex set  $A \cup B^* \cup C^*$  and edges  $\{a, v_i\}$ ,  $1 \leq i \leq t$ , whenever  $a \in A$  and  $\{a, v\}$  is an edge of  $G$ , and  $\{b_i, c_i\}$ ,  $1 \leq i \leq t$ , whenever  $\{b, c\}$  is an edge of  $G$  with  $b \in B$  and  $c \in C$ . Clearly  $G^*$  has  $nt$  edges and  $G$  divides  $G^*$ . Define a labeling  $h^*$  on  $G^*$  by

$$h^*(v) = \begin{cases} h(v) & v \in A, \\ h(b) + (i-1)\delta n & v = b_i \in B_i, \\ h(c) + (t-i)\delta n & v = c_i \in C_i. \end{cases}$$

We use  $\delta = 1$  if  $h$  is a  $\sigma$ -tripartite labeling of  $G$  and  $\delta = 2$  if  $h$  is a  $\rho$ -tripartite labeling. Either way, the labeling  $h^*$  is either a  $\sigma$ -labeling or a  $\rho$ -labeling of  $G^*$  and the result follows by Theorem 2.

Let  $r \geq 3$  and  $s \geq 4$  be integers and let  $G = C_r \cup C_s$ . We will show that  $G$  admits: an  $\alpha$ -labeling if  $r \equiv s \equiv 0 \pmod{4}$ , a  $\rho^+$ -labeling if  $r \equiv s \equiv 2 \pmod{4}$ , a  $\gamma$ -labeling if  $r + s \equiv 1$  or  $3 \pmod{4}$  and  $G \neq C_3 \cup C_4$ , a  $\sigma$ -tripartite labeling if  $r$  and  $s$  are both odd and  $r + s \equiv 0 \pmod{4}$ , and a  $\rho$ -tripartite labeling if  $r$  and  $s$  are both odd and  $r + s \equiv 2 \pmod{4}$  or  $G = C_3 \cup C_4$ .

## 1.2 Some Known Results

Several authors have investigated labelings of the one-point union of various graphs. We direct the interested reader to Gallian's graph labelings survey [11] for a detailed list of results. We will only cite the most relevant results for our problem. Most of the previous investigations have focused on graceful and  $\alpha$ -labelings of one-point union graphs.

Bodendiek, Schumacher, and Wegner [5] proved that the one-point union of any two cycles is graceful when the number of edges is congruent to 0 or 3 modulo 4. Figueroa-Centeno, Ichishima, and Muntaner-Batle [10] have shown that if  $m \equiv 0 \pmod{4}$  then the one-point union of 2, 3, or 4 copies of  $C_m$  admits an  $\alpha$ -labeling, and if  $m \equiv 2 \pmod{4}$  then the one-point union of 2 or 4 copies of  $C_m$  admits an  $\alpha$ -labeling. They conjecture that the one-point union of  $n$  copies of  $C_m$  admits an  $\alpha$ -labeling if and only if  $mn \equiv 0 \pmod{4}$ . Let  $C_n^{(t)}$  denote the one-point union of  $t$  cycles of length  $n$ . Bermond, Brouwer, and Germa [2] and Bermond, Kotzig, and Turgeon [3] proved that  $C_3^{(t)}$  is graceful if and only if  $t \equiv 0$  or  $1 \pmod{4}$ .

### 1.3 Additional Notation and Definitions

For ease of notation, we will henceforth consider graphs whose vertices are (distinct) nonnegative integers. Each vertex will be its own label, so the label of the edge  $\{x, y\}$  in such a graph will be simply  $|x - y|$ .

We denote the directed path with vertices  $x_0, x_1, \dots, x_k$ , where  $x_i$  is adjacent to  $x_{i+1}$ ,  $0 \leq i \leq k - 1$ , by  $(x_0, x_1, \dots, x_k)$ . The *first vertex* of this path is  $x_0$ , the *second vertex* is  $x_1$ , and the *last vertex* is  $x_k$ . If  $G_1 = (x_0, x_1, \dots, x_j)$  and  $G_2 = (y_0, y_1, \dots, y_k)$  are directed paths with  $V(G_1) \cap V(G_2) = \{x_j = y_0\}$ , then by  $G_1 + G_2$  we mean the directed path  $(x_0, x_1, \dots, x_j, y_1, y_2, \dots, y_k)$ .

Let  $P(k)$  be the path with  $k$  edges and  $k + 1$  vertices  $0, 1, \dots, k$  given by  $(0, k, 1, k - 1, 2, k - 2, \dots, \lfloor k/2 \rfloor)$ . Note that the set of vertices of this graph is  $A \cup B$ , where  $A = [0, \lfloor k/2 \rfloor]$ ,  $B = [\lfloor k/2 \rfloor + 1, k]$ , and every edge joins a vertex of  $A$  to one of  $B$ . Furthermore the set of labels of the edges of  $P(k)$  is  $[1, k]$ .

Now let  $a$  and  $b$  be nonnegative integers with  $a \leq b$  and let us add  $a$  to all the vertices of  $A$  and  $b$  to all the vertices of  $B$ . We will denote the resulting graph by  $P(a, b, k)$ . Note that this graph has the following properties.

- P1**  $P(a, b, k)$  is a path with first vertex  $a$  and second vertex  $b + k$ . If  $k$  is even, its last vertex is  $a + k/2$ .
- P2** Each edge of  $P(a, b, k)$  joins a vertex of  $A' = [a, \lfloor k/2 \rfloor + a]$  to a larger vertex of  $B' = [\lfloor k/2 \rfloor + 1 + b, k + b]$ .
- P3** The set of edge labels of  $P(a, b, k)$  is  $[b - a + 1, b - a + k]$ .

Figure 1 shows examples of this path notation.

Figure 1: The paths  $P(6)$  and  $P(4, 7, 6)$ .

## 2 Main Results

**Lemma 7** *If a graph  $G$  is the one-point union of even cycles  $C_r$  and  $C_s$  where  $r \equiv s \pmod{4}$ , then  $G$  has an  $\alpha$ -labeling.*

*Proof.* We will consider two cases.

**Case 1**  $r \equiv s \equiv 0 \pmod{4}$ .

Let  $G = C_{4x} \cup C_{4y}$  where  $x, y \geq 1$ . Let  $C_{4x} = G_1 + G_2 + (2x - 1, b_1, 0)$  and

$C_{4y} = G_3 + G_4 + (2x + 2y - 2, b_2, 2x - 1)$  where  $b_1 = 2x + 4y$ ,  $b_2 = 2x + 4y - 1$ , and

$$\begin{aligned} G_1 &= P(0, 2x + 4y, 2x), \\ G_2 &= P(x, x + 4y + 1, 2x - 2), \\ G_3 &= P(2x - 1, 2x + 2y, 2y - 2), \\ G_4 &= P(2x + y - 2, 2x + y - 2, 2y). \end{aligned}$$

(Note: In the case when  $x = 1$ , the path  $G_2$  is empty, and when  $y = 1$ , the path  $G_3$  is empty. However, this does not change the proof in any way.)

Figure 2: An  $\alpha$ -labeling of  $C_8 \circlearrowleft C_8$ .

First, we show that  $G_1 + G_2 + (2x - 1, b_1, 0)$  is a cycle of length  $4x$ , and  $G_3 + G_4 + (2x + 2y - 2, b_2, 2x - 1)$  is a cycle of length  $4y$ . Note that by **P1**, the first vertex of  $G_1$  is 0, and the last is  $x$ ; the first vertex of  $G_2$  is  $x$ , and the last is  $2x - 1$ ; the first vertex of  $G_3$  is  $2x - 1$ , and the last is  $2x + y - 2$ ; and the first vertex of  $G_4$  is  $2x + y - 2$ , and the last is  $2x + 2y - 2$ . For  $1 \leq i \leq 4$ , let  $A_i$  and  $B_i$  denote the sets labeled  $A'$  and  $B'$  in **P2** corresponding to the path  $G_i$ . Then using **P2**, we compute

$$\begin{aligned} A_1 &= [0, x], & B_1 &= [3x + 4y + 1, 4x + 4y], \\ A_2 &= [x, 2x - 1], & B_2 &= [2x + 4y + 1, 3x + 4y - 1], \\ A_3 &= [2x - 1, 2x + y - 2], & B_3 &= [2x + 3y, 2x + 4y - 2], \\ A_4 &= [2x + y - 2, 2x + 2y - 2], & B_4 &= [2x + 2y - 1, 2x + 3y - 2]. \end{aligned}$$

Thus,

$$A_1 \leq A_2 \leq A_3 \leq A_4 < B_4 < B_3 < b_2 < b_1 < B_2 < B_1. \quad (1)$$

Note that  $V(G_1) \cap V(G_2) = \{x\}$ ,  $V(G_2) \cap V(G_3) = \{2x - 1\}$ , and  $V(G_3) \cap V(G_4) = \{2x + y - 2\}$ ; otherwise,  $G_i$  and  $G_j$  are vertex-disjoint for  $i \neq j$ . Therefore,  $G_1 + G_2 + (2x - 1, b_1, 0)$  is a cycle of length  $4x$ , and  $G_3 + G_4 + (2x + 2y - 2, b_2, 2x - 1)$  is a cycle of length  $4y$ . Furthermore,  $V(C_{4x}) \cap V(C_{4y}) = \{2x - 1\}$ ; therefore,  $G$  is a graph composed of two cycles that share a single vertex.

Next, let  $E_i$  denote the set of edge labels in  $G_i$  for  $1 \leq i \leq 4$ . By **P3**, we have edge labels

$$\begin{aligned} E_1 &= [2x + 4y + 1, 4x + 4y], \\ E_2 &= [4y + 2, 2x + 4y - 1], \\ E_3 &= [2y + 2, 4y - 1], \\ E_4 &= [1, 2y]. \end{aligned}$$



Moreover, the path  $(2x - 1, b_1, 0)$  consists of edges with labels  $4y + 1$  and  $2x + 4y$ , and the path  $(2x + 2y - 2, b_2, 2x - 1)$  consists of edges with labels  $2y + 1$  and  $4y$ . Thus, the edge set of  $G$  has one edge of each label  $i$  where  $1 \leq i \leq 4x + 4y$ , and condition  $(\ell 4)$  for an  $\alpha$ -labeling is satisfied.

Finally, let  $A = \bigcup_{i=1}^4 A_i$  and  $B = \bigcup_{i=1}^4 B_i \cup \{b_1, b_2\}$ . Then,  $\{A, B\}$  is a bipartition of  $V(G)$ . Conditions  $(\ell 2)$  and  $(\ell 6)$  for an  $\alpha$ -labeling are clear from (1). Thus, we have an  $\alpha$ -labeling of  $G$ .

**Case 2**  $r \equiv s \equiv 2 \pmod{4}$ .

Let  $G = C_{4x+2} \circlearrowleft C_{4y+2}$  where  $x, y \geq 1$ . Let  $C_{4x+2} = G_1 + G_2 + (2x, b_1, 0)$  and  $C_{4y+2} = G_3 + G_4 + (2x + 2y, b_2, 2x)$  where  $b_1 = 2x + 4y + 2$ ,  $b_2 = 2x + 4y + 3$ , and

$$\begin{aligned} G_1 &= P(0, 2x + 4y + 2, 2x + 2), \\ G_2 &= P(x + 1, x + 4y + 4, 2x - 2), \\ G_3 &= P(2x, 2x + 2y + 3, 2y - 2), \\ G_4 &= P(2x + y - 1, 2x + y - 1, 2y + 2). \end{aligned}$$

(Note: In the case when  $x = 1$ , the path  $G_2$  is empty, and when  $y = 1$ , the path  $G_3$  is empty. However, this does not change the proof in any way.) If we proceed as in Case 1, it is easy to verify that we have an  $\alpha$ -labeling of  $G$ . ■

Figure 3: An  $\alpha$ -labeling of  $C_{10} \circlearrowleft C_{10}$ .

It is necessary here to define a new operator on the edge label values and sets. Let  $G$  be a graph with  $n$  edges. If  $m$  is the label of an edge, let  $m^* = \min\{m, 2n + 1 - m\}$  be the length of the edge, and let  $S^* = \{m^* : m \in S\}$  be the corresponding set of edge lengths. Thus if the set of vertices of  $G$  is a subset of  $[0, 2n]$  and the set  $E$  of edge labels of  $G$  satisfies  $E^* = [1, n]$ , then  $G$  has a  $\rho$ -labeling.

**Lemma 8** *If a graph  $G$  is the one-point union of even cycles  $C_r$  and  $C_s$  where  $r \not\equiv s \pmod{4}$ , then  $G$  has a uniformly-ordered  $\rho$ -labeling.*

*Proof.* Let  $G = C_{4x+2} \circlearrowleft C_{4y}$  where  $x, y \geq 1$ . Let  $C_{4x+2} = G_1 + G_2 + (2x, b_1, 0)$  and  $C_{4y} = G_3 + G_4 + (2x + 2y - 1, b_2, 2x)$  where  $b_1 = 4x + 4y + 3$ ,  $b_2 = 2x + 2y$ , and

$$\begin{aligned} G_1 &= P(0, 2x + 4y + 3, 2x - 2), \\ G_2 &= P(x - 1, x + 4y - 1, 2x + 2), \\ G_3 &= P(2x, 2x + 2y, 2y), \\ G_4 &= P(2x + y, 2x + y + 1, 2y - 2). \end{aligned}$$

(Note: In the case when  $x = 1$ , the path  $G_1$  is empty, and when  $y = 1$ , the path  $G_4$  is empty. However, this does not change the proof in any way.)

Figure 4: A uniformly-ordered  $\rho$ -labeling of  $C_{10} \cup C_8$ .

First, we show that  $G_1 + G_2 + (2x, b_1, 0)$  is a cycle of length  $4x + 2$ , and  $G_3 + G_4 + (2x + 2y - 1, b_2, 2x)$  is a cycle of length  $4y$ . Note that by **P1**, the first vertex of  $G_1$  is 0, and the last is  $x - 1$ ; the first vertex of  $G_2$  is  $x - 1$ , and the last is  $2x$ ; the first vertex of  $G_3$  is  $2x$ , and the last is  $2x + y$ ; and the first vertex of  $G_4$  is  $2x + y$ , and the last is  $2x + 2y - 1$ . For  $1 \leq i \leq 4$ , let  $A_i$  and  $B_i$  denote the sets labeled  $A'$  and  $B'$  in **P2** corresponding to the path  $G_i$ . Then using **P2**, we compute

$$\begin{aligned} A_1 &= [0, x - 1], & B_1 &= [3x + 4y + 3, 4x + 4y + 1], \\ A_2 &= [x - 1, 2x], & B_2 &= [2x + 4y + 1, 3x + 4y + 1], \\ A_3 &= [2x, 2x + y], & B_3 &= [2x + 3y + 1, 2x + 4y], \\ A_4 &= [2x + y, 2x + 2y - 1], & B_4 &= [2x + 2y + 1, 2x + 3y - 1]. \end{aligned}$$

Thus,

$$A_1 \leq A_2 \leq A_3 \leq A_4 < b_2 < B_4 < B_3 < B_2 < B_1 < b_1. \quad (2)$$

Note that  $V(G_1) \cap V(G_2) = \{x - 1\}$ ,  $V(G_2) \cap V(G_3) = \{2x\}$ , and  $V(G_3) \cap V(G_4) = \{2x + y\}$ ; otherwise,  $G_i$  and  $G_j$  are vertex-disjoint for  $i \neq j$ . Therefore,  $G_1 + G_2 + (2x, b_1, 0)$  is a cycle of length  $4x + 2$ , and  $G_3 + G_4 + (2x + 2y - 1, b_2, 2x)$  is a cycle of length  $4y$ . Furthermore,  $V(C_{4x+2}) \cap V(C_{4y}) = \{2x\}$ ; therefore,  $G$  is a graph composed of two cycles that share a single vertex.

Next, let  $E_i$  denote the set of edge labels in  $G_i$  for  $1 \leq i \leq 4$ . By **P3**, we have edge labels

$$\begin{aligned} E_1 &= [2x + 4y + 4, 4x + 4y + 1], \\ E_2 &= [4y + 1, 2x + 4y + 2], \\ E_3 &= [2y + 1, 4y], \\ E_4 &= [2, 2y - 1] \end{aligned}$$

yielding edge lengths of the same values. Moreover, the path  $(2x, b_1, 0)$  consists of edges with lengths  $2x + 4y + 3$  and  $(4x + 4y + 3)^* = 4x + 4y + 2$ , and the path  $(2x + 2y - 1, b_2, 2x)$  consists of edges with lengths 1 and  $2y$ . Thus, the edge set of  $G$  has one edge of each length  $i$  where  $1 \leq i \leq 4x + 4y + 2$ , and condition ( $\ell 3$ ) for a  $\rho^{++}$ -labeling is satisfied.

Finally, let  $A = \bigcup_{i=1}^4 A_i$ ,  $B = \bigcup_{i=1}^4 B_i \cup \{b_1, b_2\}$ . Conditions ( $\ell 1$ ) and ( $\ell 6$ ) of a  $\rho^{++}$ -labeling are clear from (2). Thus, we have a uniformly-ordered  $\rho$ -labeling of  $G$ . ■

**Lemma 9** *If a graph  $G$  is the one-point union of  $C_r$  and  $C_s$  where  $r \not\equiv s \pmod{2}$  and  $\{r, s\} \neq \{3, 4\}$ , then  $G$  has a  $\gamma$ -labeling.*

*Proof.* We will consider four cases.

**Case 1**  $r \equiv 0 \pmod{4}$  and  $s \equiv 1 \pmod{4}$ .

Let  $G = C_{4x} \cup C_{4y+1}$  where  $x, y \geq 1$ . We will consider two subcases.

**Case 1.1**  $x = 1$ .

Let  $C_4 = (a_1, b_1, a_2, b_2, a_1)$  and  $C_{4y+1} = G_1 + G_2 + (2y - 1, \hat{b}, c, a_1, b_3, 1)$  where  $a_1 = 0$ ,  $b_1 = 2y + 4$ ,  $a_2 = 2y$ ,  $b_2 = 2y + 5$ ,  $\hat{b} = 4y + 5$ ,  $c = 8y + 10$ ,  $b_3 = 8y + 9$ , and

$$G_1 = P(1, 6y + 8, 2y - 2),$$

$$G_2 = P(y, 5y + 6, 2y - 2).$$

(Note: In the case when  $y = 1$ , the paths  $G_1$  and  $G_2$  are empty. However, this does not change the proof in any way.)

Figure 5: A  $\gamma$ -labeling of  $C_4 \cup C_9$ .

First, we show that  $G_1 + G_2 + (2y - 1, \hat{b}, c, a_1, b_3, 1)$  is a cycle of length  $4y + 1$ . Note that by **P1**, the first vertex of  $G_1$  is 1, and the last is  $y$ ; and the first vertex of  $G_2$  is  $y$ , and the last is  $2y - 1$ . For  $1 \leq i \leq 2$ , let  $A_i$  and  $B_i$  denote the sets labeled  $A'$  and  $B'$  in **P2** corresponding to the path  $G_i$ . Then using **P2**, we compute

$$A_1 = [1, y],$$

$$B_1 = [7y + 8, 8y + 6],$$

$$A_2 = [y, 2y - 1],$$

$$B_2 = [6y + 6, 7y + 4].$$

Thus,

$$a_1 < A_1 \leq A_2 < a_2 < b_1 < b_2 < \hat{b} < B_2 < B_1 < b_3 < c. \quad (3)$$

Note that  $V(G_1) \cap V(G_2) = \{y\}$ ; otherwise,  $G_1$  and  $G_2$  are vertex-disjoint. Therefore,  $G_1 + G_2 + (2y - 1, \hat{b}, c, a_1, b_3, 1)$  is a cycle of length  $4y + 1$ . Furthermore,  $V(C_4) \cap V(C_{4y+1}) = \{a_1 = 0\}$ ; therefore,  $G$  is a graph composed of two cycles that share a single vertex.

Next, let  $E_i$  denote the set of edge labels in  $G_i$  for  $1 \leq i \leq 2$ . By **P3**, we have edge labels

$$E_1 = [6y + 8, 8y + 5],$$

$$E_2 = [4y + 7, 6y + 4],$$

yielding edge lengths

$$E_1^* = \{8y + 11 - \ell : \ell \in E_1\} = [6, 2y + 3],$$

$$E_2^* = \{8y + 11 - \ell : \ell \in E_2\} = [2y + 7, 4y + 4].$$

Moreover, the path  $(a_1, b_1, a_2, b_2, a_1)$  consists of edges with lengths  $2y + 4$ ,  $4$ ,  $5$ , and  $2y + 5$ ; and the path  $(2y - 1, \hat{b}, c, a_1, b_3, 1)$  consists of edges with lengths  $2y + 6$ ,  $4y + 5$ ,  $(8y + 10)^* = 1$ ,  $(8y + 9)^* = 2$ , and  $(8y + 8)^* = 3$ . Thus, the edge set of  $G$  has one edge of each length  $i$  where  $1 \leq i \leq 4y + 5$ . It is clear from (3) that  $V(G) \subseteq [0, 8y + 10]$ . Hence, the defined labeling is a  $\rho$ -labeling, and condition (g1) for a  $\gamma$ -labeling is satisfied.

Finally, let  $A = A_1 \cup A_2 \cup \{a_1, a_2\}$ ,  $B = B_1 \cup B_2 \cup \{\hat{b}, b_1, b_2, b_3\}$ , and  $C = \{c\}$ . Then,  $\{A, B, C\}$  is a tripartition of  $V(G)$ . Condition (g2) of a  $\gamma$ -labeling is clear from (3). Note that  $c - \hat{b} = (8y + 10) - (4y + 5) = 4y + 5$ , the number of edges of  $G$ . Thus, condition (g3) is satisfied, and we have a  $\gamma$ -labeling of  $G$ .

**Case 1.2**  $x > 1$ .

Let  $C_{4x} = G_1 + G_2 + (2x + 2y - 2, b_1, a_1, b_2, 2y)$  and  $C_{4y+1} = G_3 + G_4 + (2y - 1, \hat{b}, c, a_1, b_3, 1)$  where  $b_1 = 4x + 2y$ ,  $a_1 = 0$ ,  $b_2 = 4x + 2y + 1$ ,  $\hat{b} = 4x + 4y + 1$ ,  $c = 8x + 8y + 2$ ,  $b_3 = 4x + 8y + 4$ , and

$$G_1 = P(2y, 2x + 2y + 2, 2x - 4),$$

$$G_2 = P(x + 2y - 2, x + 2y - 1, 2x),$$

$$G_3 = P(1, 4x + 6y + 4, 2y - 2),$$

$$G_4 = P(y, 4x + 5y + 2, 2y - 2).$$

(Note: In the case when  $x = 2$ , the path  $G_1$  is empty, and when  $y = 1$ , the paths  $G_3$  and  $G_4$  are empty. However, this does not change the proof in any way.)

Figure 6: A  $\gamma$ -labeling of  $C_{12} \cup C_9$ .

First, we show that  $G_1 + G_2 + (2x + 2y - 2, b_1, a_1, b_2, 2y)$  is a cycle of length  $4x$ , and  $G_3 + G_4 + (2y - 1, \hat{b}, c, a_1, b_3, 1)$  is a cycle of length  $4y + 1$ . Note that by **P1**, the first vertex of  $G_1$  is  $2y$ , and the last is  $x + 2y - 2$ ; the first vertex of  $G_2$  is  $x + 2y - 2$ , and the last is  $2x + 2y - 2$ ; the first vertex of  $G_3$  is  $1$ , and the last is  $y$ ; and the first vertex of  $G_4$  is  $y$ , and the last is  $2y - 1$ . For  $1 \leq i \leq 4$ , let  $A_i$  and  $B_i$  denote the sets labeled  $A'$  and  $B'$  in

**P2** corresponding to the path  $G_i$ . Then using **P2**, we compute

$$\begin{aligned} A_1 &= [2y, x + 2y - 2], & B_1 &= [3x + 2y + 1, 4x + 2y - 2], \\ A_2 &= [x + 2y - 2, 2x + 2y - 2], & B_2 &= [2x + 2y, 3x + 2y - 1], \\ A_3 &= [1, y], & B_3 &= [4x + 7y + 4, 4x + 8y + 2], \\ A_4 &= [y, 2y - 1], & B_4 &= [4x + 6y + 2, 4x + 7y]. \end{aligned}$$

Thus,

$$a_1 < A_3 \leq A_4 < A_1 \leq A_2 < B_2 < B_1 < b_1 < b_2 < \hat{b} < B_4 < B_3 < b_3 < c. \quad (4)$$

Note that  $V(G_1) \cap V(G_2) = \{x + 2y - 2\}$  and  $V(G_3) \cap V(G_4) = \{y\}$ ; otherwise,  $G_i$  and  $G_j$  are vertex-disjoint for  $i \neq j$ . Therefore,  $G_1 + G_2 + (2x + 2y - 2, b_1, a_1, b_2, 2y)$  is a cycle of length  $4x$ , and  $G_3 + G_4 + (2y - 1, \hat{b}, c, a_1, b_3, 1)$  is a cycle of length  $4y + 1$ . Furthermore,  $V(C_{4x}) \cap V(C_{4y+1}) = \{a_1 = 0\}$ ; therefore,  $G$  is a graph composed of two cycles that share a single vertex.

Next, let  $E_i$  denote the set of edge labels in  $G_i$  for  $1 \leq i \leq 4$ . By **P3**, we have edge labels

$$\begin{aligned} E_1 &= [2x + 3, 4x - 2], \\ E_2 &= [2, 2x + 1], \\ E_3 &= [4x + 6y + 4, 4x + 8y + 1], \\ E_4 &= [4x + 4y + 3, 4x + 6y] \end{aligned}$$

yielding edge lengths

$$\begin{aligned} E_1^* &= E_1 = [2x + 3, 4x - 2], \\ E_2^* &= E_2 = [2, 2x + 1], \\ E_3^* &= \{8x + 8y + 3 - \ell : \ell \in E_3\} = [4x + 2, 4x + 2y - 1], \\ E_4^* &= \{8x + 8y + 3 - \ell : \ell \in E_4\} = [4x + 2y + 3, 4x + 4y]. \end{aligned}$$

Moreover, the path  $(2x + 2y - 2, b_1, a_1, b_2, 2y)$  consists of edges with lengths  $2x + 2$ ,  $4x + 2y$ ,  $4x + 2y + 1$ , and  $4x + 1$ ; and the path  $(2y - 1, \hat{b}, c, a_1, b_3, 1)$  consists of edges with lengths  $4x + 2y + 2$ ,  $4x + 4y + 1$ ,  $(8x + 8y + 2)^* = 1$ ,  $(4x + 8y + 4)^* = 4x - 1$ , and  $(4x + 8y + 3)^* = 4x$ . Thus, the edge set of  $G$  has one edge of each length  $i$  where  $1 \leq i \leq 4x + 4y + 1$ . It is clear from (4) that  $V(G) \subseteq [0, 8x + 8y + 2]$ . Hence, the defined labeling is a  $\rho$ -labeling, and condition (g1) for a  $\gamma$ -labeling is satisfied.

Finally, let  $A = \bigcup_{i=1}^4 A_i \cup \{a_1\}$ ,  $B = \bigcup_{i=1}^4 B_i \cup \{\hat{b}, b_1, b_2, b_3\}$ , and  $C = \{c\}$ . Then,  $\{A, B, C\}$  is a tripartition of  $V(G)$ . Condition (g2) of a  $\gamma$ -labeling is clear from (4). Note that  $c - \hat{b} = (8x + 8y + 2) - (4x + 4y + 1) = 4x + 4y + 1$ , the number of edges of  $G$ . Thus, condition (g3) is satisfied, and we have a  $\gamma$ -labeling of  $G$ .

**Case 2**  $r \equiv 0 \pmod{4}$  and  $s \equiv 3 \pmod{4}$ .

Let  $G = C_{4x+3} \cup C_{4y}$  where  $x \geq 0$  and  $y \geq 1$  but  $(x, y) \neq (0, 1)$ . We will consider two subcases.

**Case 2.1**  $x = 0$  and  $y > 1$ .

If  $y = 2$ , let  $C_3 = (0, 3, 14, 0)$  and  $C_8 = (1, 8, 4, 10, 5, 7, 6, 14, 1)$ . It is easily checked that this is a  $\gamma$ -labeling with  $A = \{0, 1, 4, 5, 6\}$ ,  $B = \{\hat{b} = 3\} \cup \{7, 8, 10\}$ , and  $C = \{14\}$ .

If  $y \geq 3$ , let  $C_3 = (0, \hat{b}, c, 0)$  and  $C_{4y} = G_1 + G_2 + (2y - 2, b_1, a_1, b_2, 0)$  where  $\hat{b} = 4y + 1$ ,  $c = 8y + 4$ ,  $b_1 = 2y$ ,  $a_1 = 2y - 1$ ,  $b_2 = 4y + 5$  and

$$G_1 = P(0, 2y + 6, 2y - 6),$$

$$G_2 = P(y - 3, y, 2y + 2).$$

(Note: In the case when  $y = 3$ , the path  $G_1$  is empty. However, this does not change the proof in any way.) If we proceed as in Case 1, it is easy to verify that we have a  $\gamma$ -labeling of  $G$ .

Figure 7: A  $\gamma$ -labeling of  $C_3 \cup C_{16}$ .

**Case 2.2**  $x > 0$ .

Let  $C_{4x+3} = G_1 + G_2 + (2x + 1, b_1, a_1, \hat{b}, c, 2)$  and  $C_{4y} = G_3 + G_4 + (2x + 2y, b_2, 2x + 1)$  where  $b_1 = 2x + 4y + 3$ ,  $a_1 = 0$ ,  $\hat{b} = 1$ ,  $c = 4x + 4y + 4$ ,  $b_2 = 2x + 4y + 2$ , and

$$G_1 = P(2, 2x + 4y + 5, 2x - 2),$$

$$G_2 = P(x + 1, x + 4y + 3, 2x),$$

$$G_3 = P(2x + 1, 2x + 2y + 3, 2y - 2),$$

$$G_4 = P(2x + y, 2x + y + 1, 2y).$$

(Note: In the case when  $x = 1$ , the path  $G_1$  is empty, and when  $y = 1$ , the path  $G_3$  is empty. However, this does not change the proof in any way.) If we proceed as in Case 1, it is easy to verify that we have a  $\gamma$ -labeling of  $G$ .

Figure 8: A  $\gamma$ -labeling of  $C_{11} \cup C_8$ .

**Case 3**  $r \equiv 2 \pmod{4}$  and  $s \equiv 1 \pmod{4}$ .

Let  $G = C_{4x+1} \cup C_{4y+2}$  where  $x, y \geq 1$ . Let  $C_{4x+1} = G_1 + G_2 + (2x, c, \hat{b}, a_1, b_1, 2)$  and  $C_{4y+2} = G_3 + G_4 + (2x + 2y, b_2, 2x)$  where  $c = 4x + 4y + 4$ ,  $\hat{b} = 1$ ,  $a_1 = 0$ ,  $b_1 = 8x + 4y + 4$ ,  $b_2 = 2x + 4y + 4$ , and

$$\begin{aligned}
G_1 &= P(2, 6x + 4y + 5, 2x - 2), \\
G_2 &= P(x + 1, 5x + 4y + 5, 2x - 2), \\
G_3 &= P(2x, 2x + 2y + 4, 2y - 2), \\
G_4 &= P(2x + y - 1, 2x + y, 2y + 2).
\end{aligned}$$

(Note: In the case when  $x = 1$ , the paths  $G_1$  and  $G_2$  are empty, and when  $y = 1$  the path  $G_3$  is empty. However, this does not change the proof in any way.) If we proceed as in Case 1, it is easy to verify that we have a  $\gamma$ -labeling of  $G$ .

Figure 9: A  $\gamma$ -labeling of  $C_9 \cup C_{10}$ .

**Case 4**  $r \equiv 2 \pmod{4}$  and  $s \equiv 3 \pmod{4}$ .

Let  $G = C_{4x+2} \cup C_{4y+3}$  where  $x \geq 1$  and  $y \geq 0$ . We will consider three subcases.

**Case 4.1**  $y = 0$ .

Let  $C_{4x+2} = G_1 + G_2 + (2x, b_1, a_1, b_2, 1)$  and  $C_3 = (2x, \hat{b}, c, 2x)$  where  $b_1 = 6x + 7$ ,  $a_1 = 0$ ,  $b_2 = 8x + 10$ ,  $\hat{b} = 2x + 3$ ,  $c = 6x + 8$ , and

$$\begin{aligned}
G_1 &= P(1, 6x + 8, 2x), \\
G_2 &= P(x + 1, 5x + 9, 2x - 2).
\end{aligned}$$

(Note: In the case when  $x = 1$ , the path  $G_2$  is empty. However, this does not change the proof in any way.) If we proceed as in Case 1, it is easy to verify that we have a  $\gamma$ -labeling of  $G$ .

Figure 10: A  $\gamma$ -labeling of  $C_{10} \cup C_3$ .

**Case 4.2**  $y > 0$  and  $x \leq y + 2$ .

Let  $C_{4x+2} = G_1 + (2x, b_1, 0)$  and  $C_{4y+3} = G_2 + G_3 + G_4 + (2x + 2y, \hat{b}, c, 2x)$  where  $b_1 = 6x + 6y + 5$ ,  $\hat{b} = 2x + 2y + 1$ ,  $c = 6x + 6y + 6$ , and

$$\begin{aligned}
G_1 &= P(0, 4x + 8y + 9, 4x), \\
G_2 &= P(2x, 6x + 2y + 6, 2y - 2), \\
G_3 &= P(2x + y - 1, 4x + 3y + 5, 2x - 2), \\
G_4 &= P(3x + y - 2, 7x + y - 1, -2x + 2y + 4).
\end{aligned}$$

(Note: In the case when  $x = 1$ , the path  $G_3$  is empty; when  $y = 1$ , the path  $G_2$  is empty; and when  $x = y + 2$ , the path  $G_4$  is empty. However,

Figure 11: A  $\gamma$ -labeling of  $C_{10} \cup C_{11}$ .

this does not change the proof in any way.) If we proceed as in Case 1, it is easy to verify that we have a  $\gamma$ -labeling of  $G$ .

**Case 4.3**  $y > 0$  and  $x > y + 2$ .

Let  $C_{4x+2} = G_1 + G_2 + (2x, b_1, 0)$  and  $C_{4y+3} = G_3 + G_4 + (2x + 2y, \hat{b}, c, 2x)$  where  $b_1 = 6x + 6y + 5$ ,  $\hat{b} = 2x + 2y + 1$ ,  $c = 6x + 6y + 6$ , and

$$\begin{aligned} G_1 &= P(0, 6x + 6y + 5, 2x + 2y + 4), \\ G_2 &= P(x + y + 2, 5x + 9y + 10, 2x - 2y - 4), \\ G_3 &= P(2x, 6x + 2y + 6, 2y - 2), \\ G_4 &= P(2x + y - 1, 6x + y + 1, 2y + 2). \end{aligned}$$

Figure 12: A  $\gamma$ -labeling of  $C_{22} \cup C_{11}$ .

(Note: In the case when  $y = 1$ , the path  $G_3$  is empty. However, this does not change the proof in any way.) If we proceed as in Case 1, it is easy to verify that we have a  $\gamma$ -labeling of  $G$ . ■

**Lemma 10** *If a graph  $G$  is the one-point union of odd cycles  $C_r$  and  $C_s$  where  $r \not\equiv s \pmod{4}$ , then  $G$  has a  $\sigma$ -tripartite labeling.*

*Proof.* Let  $G = C_{4x+1} \cup C_{4y+3}$  where  $x \geq 1$  and  $y \geq 0$ . We will consider four cases.

**Case 1**  $y = 0$ .

Let  $C_{4x+1} = G_1 + G_2 + (2x - 1, b_1, c_1, 0)$  and  $C_3 = (a_1, b_2, c_2, a_1)$  where  $b_1 = 2x + 1$ ,  $c_1 = 4x + 4$ ,  $a_1 = 3x + 3$ ,  $b_2 = 3x + 4$ ,  $c_2 = 5x + 5$ , and

$$\begin{aligned} G_1 &= P(0, 2x + 3, 2x), \\ G_2 &= P(x, x + 2, 2x - 2). \end{aligned}$$

(Note: In the case when  $x = 1$ , the path  $G_2$  is empty. However, this does not change the proof in any way.)

Figure 13: A  $\sigma$ -tripartite labeling of  $C_9 \cup C_3$ .

First, we show that  $G_1 + G_2 + (2x - 1, b_1, c_1, 0)$  is a cycle of length  $4x + 1$ . Note that by P1, the first vertex of  $G_1$  is 0, and the last is  $x$ ; and the first vertex of  $G_2$  is  $x$ , and the last is  $2x - 1$ . For  $1 \leq i \leq 2$ , let  $A_i$  and  $B_i$  denote



the sets labeled  $A'$  and  $B'$  in **P2** corresponding to the path  $G_i$ . Then using **P2**, we compute

$$\begin{aligned} A_1 &= [0, x], & B_1 &= [3x + 4, 4x + 3], \\ A_2 &= [x, 2x - 1], & B_2 &= [2x + 2, 3x]. \end{aligned}$$

Thus,

$$A_1 \leq A_2 < b_1 < B_2 < a_1 < b_2 \leq B_1 < c_1 < c_2. \quad (5)$$

Note that  $V(G_1) \cap V(G_2) = \{x\}$ ; otherwise,  $G_1$  and  $G_2$  are vertex-disjoint. Therefore,  $G_1 + G_2 + (2x - 1, b_1, c_1, 0)$  is a cycle of length  $4x + 1$ . Furthermore,  $V(C_{4x+1}) \cap V(C_3) = \{b_2 = 3x + 4\}$ ; therefore,  $G$  is a graph composed of two cycles that share a single vertex.

Next, let  $E_i$  denote the set of edge labels in  $G_i$  for  $1 \leq i \leq 2$ . By **P3**, we have edge labels

$$\begin{aligned} E_1 &= [2x + 4, 4x + 3], \\ E_2 &= [3, 2x]. \end{aligned}$$

Moreover, the path  $(2x - 1, b_1, c_1, 0)$  consists of edges with labels  $2, 2x + 3,$  and  $4x + 4$ ; and the path  $(a_1, b_2, c_2, a_1)$  consists of edges with labels  $1, 2x + 1,$  and  $2x + 2$ . Thus, the edge set of  $G$  has one edge of each label  $i$  where  $1 \leq i \leq 4x + 4$ . It is clear from (5) that  $V(G) \subseteq [0, 8x + 8]$ . Hence, the defined labeling is a  $\sigma$ -labeling, and condition (s1) for a  $\sigma$ -tripartite labeling is satisfied.

Now, let  $A = A_1 \cup A_2 \cup \{a_1\}$ ,  $B = B_1 \cup B_2 \cup \{b_1, b_2\}$ , and  $C = \{c_1, c_2\}$ . Then,  $\{A, B, C\}$  is a tripartition of  $V(G)$ . Since  $a_1$  is only adjacent to  $b_2$  and  $c_2$ , condition (s2) of a  $\sigma$ -tripartite labeling is clear from (5). Note that  $|b_1 - c_1| + |b_2 - c_2| = (2x + 3) + (2x + 1) = 4x + 4$ , the number of edges of  $G$ . Thus, condition (s3) is satisfied. Also,  $a = v + (4x + 4)$ , where  $a \in A$  and  $v \in B \cup C$ , is impossible, since by (5) we have

$$v + (4x + 4) \geq b_1 + 4x + 4 = 6x + 5 > 3x + 3 = \max A.$$

Thus, condition (s4) holds.

Finally, suppose  $b \in B$  and  $c \in C$ . The equation  $|b - c| = 8x + 8$  is impossible since all vertices are in  $[0, 8x + 8]$  and  $\{0\} \in A$ . Likewise,  $|b - c_1| = 4x + 4$  is impossible since  $c_1 = 4x + 4$  and  $0 < B < 8x + 4$ . The case remains that  $c_2 - b = 4x + 4$ , which gives  $b = x + 1$ . This contradicts (5), since  $x + 1 < b_1$ . Thus, condition (s5) holds, and we have a  $\sigma$ -tripartite labeling of  $G$ .

**Case 2**  $1 \leq y < x$ .

Let  $C_{4x+1} = G_1 + G_2 + G_3 + (4x + 4y + 2, c, b_1, 2x + 4y + 3)$  and  $C_{4y+3} =$

$G_4 + G_5 + (2y, b_2, c, a_1, b_3, 1)$  where  $c = 4x + 4y + 4$ ,  $b_1 = 4x + 6y + 5$ ,  $b_2 = 2y + 1$ ,  $a_1 = 0$ ,  $b_3 = 4x + 4y + 3$ , and

$$\begin{aligned} G_1 &= P(2x + 4y + 3, 6x + 6y + 6, 2y - 2), \\ G_2 &= P(2x + 5y + 2, 4x + 7y + 4, 2x), \\ G_3 &= P(3x + 5y + 2, 3x + 9y + 3, 2x - 2y), \\ G_4 &= P(1, 2y + 2, 2y), \\ G_5 &= P(y + 1, y + 3, 2y - 2). \end{aligned}$$

(Note: In the case when  $y = 1$ , the paths  $G_1$  and  $G_5$  are empty. However, this does not change the proof in any way.)

Figure 14: A  $\sigma$ -tripartite labeling of  $C_{13} \cup C_{11}$ .

First, we show that  $G_1 + G_2 + G_3 + (4x + 4y + 2, c, b_1, 2x + 4y + 3)$  is a cycle of length  $4x + 1$ , and  $G_4 + G_5 + (2y, b_2, c, a_1, b_3, 1)$  is a cycle of length  $4y + 3$ . Note that by **P1**, the first vertex of  $G_1$  is  $2x + 4y + 3$ , and the last is  $2x + 5y + 2$ ; the first vertex of  $G_2$  is  $2x + 5y + 2$ , and the last is  $3x + 5y + 2$ ; the first vertex of  $G_3$  is  $3x + 5y + 2$ , and the last is  $4x + 4y + 2$ ; the first vertex of  $G_4$  is 1, and the last is  $y + 1$ ; and the first vertex of  $G_5$  is  $y + 1$ , and the last is  $2y$ . For  $1 \leq i \leq 5$ , let  $A_i$  and  $B_i$  denote the sets labeled  $A'$  and  $B'$  in **P2** corresponding to the path  $G_i$ . Then using **P2**, we compute

$$\begin{aligned} A_1 &= [2x + 4y + 3, 2x + 5y + 2], & B_1 &= [6x + 7y + 6, 6x + 8y + 4], \\ A_2 &= [2x + 5y + 2, 3x + 5y + 2], & B_2 &= [5x + 7y + 5, 6x + 7y + 4], \\ A_3 &= [3x + 5y + 2, 4x + 4y + 2], & B_3 &= [4x + 8y + 4, 5x + 7y + 3], \\ A_4 &= [1, y + 1], & B_4 &= [3y + 3, 4y + 2], \\ A_5 &= [y + 1, 2y], & B_5 &= [2y + 3, 3y + 1]. \end{aligned}$$

Thus,

$$a_1 < A_4 \leq A_5 < b_2 < B_5 < B_4 < A_1 \leq A_2 \leq A_3 < b_3 < c < b_1 < B_3 < B_2 < B_1. \quad (6)$$

Note that  $V(G_1) \cap V(G_2) = \{2x + 5y + 2\}$ ,  $V(G_2) \cap V(G_3) = \{3x + 5y + 2\}$ , and  $V(G_4) \cap V(G_5) = \{y + 1\}$ ; otherwise,  $G_i$  and  $G_j$  are vertex-disjoint for  $i \neq j$ . Therefore,  $G_1 + G_2 + G_3 + (4x + 4y + 2, c, b_1, 2x + 4y + 3)$  is a cycle of length  $4x + 1$ , and  $G_4 + G_5 + (2y, b_2, c, a_1, b_3, 1)$  is a cycle of length  $4y + 3$ . Furthermore,  $V(C_{4x+1}) \cap V(C_{4y+3}) = \{c = 4x + 4y + 4\}$ ; therefore,  $G$  is a graph composed of two cycles that share a single vertex.

Next, let  $E_i$  denote the set of edge labels in  $G_i$  for  $1 \leq i \leq 5$ . By **P3**,

we have edge labels

$$\begin{aligned} E_1 &= [4x + 2y + 4, 4x + 4y + 1], \\ E_2 &= [2x + 2y + 3, 4x + 2y + 2], \\ E_3 &= [4y + 2, 2x + 2y + 1], \\ E_4 &= [2y + 2, 4y + 1], \\ E_5 &= [3, 2y]. \end{aligned}$$

Moreover, the path  $(4x+4y+2, c, b_1, 2x+4y+3)$  consists of edges with edge labels 2,  $2y + 1$ , and  $2x + 2y + 2$ ; and the path  $(2y, b_2, c, a_1, b_3, 1)$  consists of edges with edge labels 1,  $4x + 2y + 3$ ,  $4x + 4y + 4$ ,  $4x + 4y + 3$ , and  $4x + 4y + 3$ . Thus, the edge set of  $G$  has one edge of each length  $i$  where  $1 \leq i \leq 4x+4y+4$ . It is clear from (6) that  $V(G) \subseteq [0, 8x+8y+8]$ . Hence, the defined labeling is a  $\sigma$ -labeling, and condition (s1) for a  $\sigma$ -tripartite labeling is satisfied.

Now, let  $A = \bigcup_{i=1}^5 A_i \cup \{a_1\}$ ,  $B = \bigcup_{i=1}^5 B_i \cup \{b_1, b_2, b_3\}$ , and  $C = \{c\}$ . Then,  $\{A, B, C\}$  is a tripartition of  $V(G)$ . Since all vertices in  $A_1 \cup A_2 \cup A_3$  are only adjacent to vertices in  $\{c, b_1\} \cup B_3 \cup B_2 \cup B_1$ , condition (s2) of a  $\sigma$ -tripartite labeling is clear from (6). Note that  $|b_1 - c| + |b_2 - c| = (2y + 1) + (4x + 2y + 3) = 4x + 4y + 4$ , the number of edges of  $G$ . Thus, condition (s3) is satisfied. Also,  $a = v + (4x + 4y + 4)$ , where  $a \in A$  and  $v \in B \cup C$ , is impossible, since by (6)

$$v + (4x + 4y + 4) \geq b_2 + 4x + 4y + 4 = 4x + 6y + 5 > 4x + 4y + 2 = \max A.$$

Thus, condition (s4) holds.

Finally, suppose  $b \in B$  and  $c \in C$ . The equation  $|b - c| = 8x + 8y + 8$  is impossible since all vertices are in  $[0, 8x + 8y + 8]$  and  $0 \in A$ . Likewise,  $|b - c| = 4x + 4y + 4$  is impossible since  $c = 4x + 4y + 4$  and  $0 < B < 8x + 8y + 8$ . Thus, condition (s5) holds, and we have a  $\sigma$ -tripartite labeling of  $G$ .

**Case 3**  $x \leq y < 2x$ .

Let  $C_{4x+1} = G_1 + G_2 + (4x + 4y + 2, c, b_1, 2x + 4y + 3)$  and  $C_{4y+3} = G_3 + G_4 + G_5 + (2y, b_2, c, a_1, b_3, 1)$  where  $c = 4x + 4y + 4$ ,  $b_1 = 4x + 6y + 5$ ,  $b_2 = 2y + 1$ ,  $a_1 = 0$ ,  $b_3 = 4x + 4y + 3$ , and

$$\begin{aligned} G_1 &= P(2x + 4y + 3, 6x + 6y + 6, 2y - 2), \\ G_2 &= P(2x + 5y + 2, 2x + 9y + 4, 4x - 2y), \\ G_3 &= P(1, 2x + 2y + 3, -2x + 2y), \\ G_4 &= P(-x + y + 1, -x + 3y + 2, 2x), \\ G_5 &= P(y + 1, y + 3, 2y - 2). \end{aligned}$$

(Note: In the case when  $y = 1$ , the paths  $G_1$ ,  $G_3$ , and  $G_5$  are empty, and when  $y = x$ , the path  $G_3$  is empty. However, this does not change the proof in any way.) If we proceed as in Case 2, it is easy to verify that we have a  $\sigma$ -tripartite labeling of  $G$ .

Figure 15: A  $\sigma$ -tripartite labeling of  $C_9 \circlearrowleft C_{15}$ .

**Case 4**  $y \geq 2x$ .

Let  $C_{4x+1} = G_1 + (4x + 4y + 2, c, b_1, 2x + 4y + 3)$  and  $C_{4y+3} = G_2 + G_3 + G_4 + G_5 + (2y, b_2, c, a_1, b_3, 1)$  where  $c = 4x + 4y + 4$ ,  $b_1 = 4x + 6y + 5$ ,  $b_2 = 2y + 1$ ,  $a_1 = 0$ ,  $b_3 = 4x + 4y + 3$ , and

$$\begin{aligned} G_1 &= P(2x + 4y + 3, 2x + 8y + 6, 4x - 2), \\ G_2 &= P(1, 4x + 2y + 4, -4x + 2y), \\ G_3 &= P(-2x + y + 1, 3y + 3, 2x), \\ G_4 &= P(-x + y + 1, -x + 3y + 2, 2x), \\ G_5 &= P(y + 1, y + 3, 2y - 2). \end{aligned}$$

If we proceed as in Case 2, it is easy to verify that we have a  $\sigma$ -tripartite labeling of  $G$ . ■

Figure 16: A  $\sigma$ -tripartite labeling of  $C_5 \circlearrowleft C_{15}$ .

**Lemma 11** *If a graph  $G$  is the one-point union of  $C_3$  and  $C_4$  or if  $G$  is the one-point union of odd cycles  $C_r$  and  $C_s$  where  $r \equiv s \pmod{4}$  and  $(r, s) \neq (3, 3)$ , then  $G$  has a  $\rho$ -tripartite labeling.*

*Proof.* It is easy to verify that  $C_3 \circlearrowleft C_3$  does not admit a  $\rho$ -tripartite labeling. If  $G = C_3 \circlearrowleft C_4$ , let  $C_3 = (0, 4, 13, 0)$  and  $C_4 = (0, 1, 8, 3, 0)$ . It is easily checked that this is a  $\rho$ -tripartite labeling with  $A = \{0\}$ ,  $B = \{1, 3, 4\}$ , and  $C = \{8, 13\}$ .

For the remainder of this proof, assume that  $G$  is the one-point union of odd cycles  $C_r$  and  $C_s$  where  $r \equiv s \pmod{4}$  and  $(r, s) \neq (3, 3)$ . We will consider two cases.

**Case 1**  $r \equiv s \equiv 1 \pmod{4}$ .

Let  $G = C_{4x+1} \circlearrowleft C_{4y+1}$  where  $x \geq y \geq 1$ . We will consider two subcases.

**Case 1.1**  $y = 1$ .

Let  $C_{4x+1} = G_1 + G_2 + (2x - 1, b_1, c_1, 0)$  and  $C_5 = (a_1, b_2, a_2, b_3, c_2, a_1)$

where  $b_1 = 2x$ ,  $c_1 = 6x + 7$ ,  $a_1 = 0$ ,  $b_2 = 2x + 4$ ,  $a_2 = 2x + 1$ ,  $b_3 = 2x + 3$ ,  $c_2 = 6x + 8$ , and

$$\begin{aligned} G_1 &= P(0, 6x + 9, 2x), \\ G_2 &= P(x, 3x + 6, 2x - 2). \end{aligned}$$

(Note: In the case when  $x = 1$ , the path  $G_2$  is empty. However, this does not change the proof in any way.)

Figure 17: A  $\rho$ -tripartite labeling of  $C_9 \cup C_5$ .

First, we show that  $G_1 + G_2 + (2x - 1, b_1, c_1, 0)$  is a cycle of length  $4x + 1$ . Note that by **P1**, the first vertex of  $G_1$  is 0, and the last is  $x$ ; and the first vertex of  $G_2$  is  $x$ , and the last is  $2x - 1$ . For  $1 \leq i \leq 2$ , let  $A_i$  and  $B_i$  denote the sets labeled  $A'$  and  $B'$  in **P2** corresponding to the path  $G_i$ . Then using **P2**, we compute

$$\begin{aligned} A_1 &= [0, x], & B_1 &= [7x + 10, 8x + 9], \\ A_2 &= [x, 2x - 1], & B_2 &= [4x + 6, 5x + 4]. \end{aligned}$$

Thus,

$$a_1 \leq A_1 \leq A_2 < b_1 < a_2 < b_3 < b_2 < B_2 < c_1 < c_2 < B_1. \quad (7)$$

Note that  $V(G_1) \cap V(G_2) = \{x\}$ ; otherwise,  $G_1$  and  $G_2$  are vertex-disjoint. Therefore,  $G_1 + G_2 + (2x - 1, b_1, c_1, 0)$  is a cycle of length  $4x + 1$ . Furthermore,  $V(C_{4x+1}) \cap V(C_5) = \{a_1 = 0\}$ ; therefore,  $G$  is a graph composed of two cycles that share a single vertex.

Next, let  $E_i$  denote the set of edge labels in  $G_i$  for  $1 \leq i \leq 2$ . By **P3**, we have edge labels

$$\begin{aligned} E_1 &= [6x + 10, 8x + 9], \\ E_2 &= [2x + 7, 4x + 4] \end{aligned}$$

yielding edge lengths

$$\begin{aligned} E_1^* &= \{8x + 13 - e : e \in E_1\} = [4, 2x + 3], \\ E_2^* &= \{e : e \in E_2\} = [2x + 7, 4x + 4]. \end{aligned}$$

Moreover, the path  $(2x - 1, b_1, c_1, 0)$  consists of edges with lengths 1,  $(4x + 7)^* = 4x + 6$ , and  $(6x + 7)^* = 2x + 6$ , and the path  $(a_1, c_2, b_3, a_2, b_2, a_1)$  consists of edges with lengths  $(6x + 8)^* = 2x + 5$ ,  $4x + 5$ , 2, 3, and  $2x + 4$ . Thus, the edge set of  $G$  has one edge of each length  $i$  where  $1 \leq i \leq 4x + 6$ .

It is clear from (7) that  $V(G) \subseteq [0, 8x + 12]$ . Hence, the defined labeling is a  $\rho$ -labeling, and condition (r1) for a  $\rho$ -labeling is satisfied.

Finally, let  $A = A_1 \cup A_2 \cup \{a_1, a_2\}$ ,  $B = B_1 \cup B_2 \cup \{b_1, b_2, b_3\}$ , and  $C = \{c_1, c_2\}$ . Then,  $\{A, B, C\}$  is a tripartition of  $V(G)$ . Condition (r2) of a  $\rho$ -tripartite labeling is clear from (7). Note that  $|b_1 - c_1| + |b_3 - c_2| = (4x + 7) + (4x + 5) = 8x + 12$ , twice the number of edges of  $G$ . Thus condition (r3) is satisfied. Also,  $|b - c| = 8x + 12$ , where  $b \in B$  and  $c \in C$ , is impossible since all vertices are in  $[0, 8x + 12]$  and  $0 \in A$ . Thus, condition (r4) holds, and we have a  $\rho$ -tripartite labeling of  $G$ .

**Case 1.2**  $y > 1$ .

Let  $C_{4x+1} = G_1 + G_2 + G_3 + (2x - 1, b_1, c_1, 0)$  and  $C_{4y+1} = G_4 + (2x + 4y - 3, b_2, a_1, c_2, b_3, a_2, b_4, 2x + 2y)$  where  $b_1 = 2x$ ,  $c_1 = 6x + 4y + 3$ ,  $b_2 = 2x + 4y$ ,  $a_1 = 0$ ,  $c_2 = 6x + 4y + 4$ ,  $b_3 = 2x + 3$ ,  $a_2 = 2x + 1$ ,  $b_4 = 2x + 6y - 2$ , and

$$\begin{aligned} G_1 &= P(0, 8x + 2y + 8, 2y - 2), \\ G_2 &= P(y - 1, 6x + 5y + 4, 2x - 2y + 2), \\ G_3 &= P(x, 3x + 4y + 2, 2x - 2), \\ G_4 &= P(2x + 2y, 2x + 2y + 3, 4y - 6). \end{aligned}$$

Figure 18: A  $\rho$ -tripartite labeling of  $C_9 \cup C_9$ .

First, we show that  $G_1 + G_2 + G_3 + (2x - 1, b_1, c_1, 0)$  is a cycle of length  $4x + 1$ , and  $G_4 + (2x + 4y - 3, b_2, a_1, c_2, b_3, a_2, b_4, 2x + 2y)$  is a cycle of length  $4y + 1$ . Note that by P1, the first vertex of  $G_1$  is 0, and the last is  $y - 1$ ; the first vertex of  $G_2$  is  $y - 1$ , and the last is  $x$ ; the first vertex of  $G_3$  is  $x$ , and the last is  $2x - 1$ ; and the first vertex of  $G_4$  is  $2x + 2y$ , and the last is  $2x + 4y - 3$ . For  $1 \leq i \leq 4$ , let  $A_i$  and  $B_i$  denote the sets labeled  $A'$  and  $B'$  in P2 corresponding to the path  $G_i$ . Then using P2, we compute

$$\begin{aligned} A_1 &= [0, y - 1], & B_1 &= [8x + 3y + 8, 8x + 4y + 6], \\ A_2 &= [y - 1, x], & B_2 &= [7x + 4y + 6, 8x + 3y + 6], \\ A_3 &= [x, 2x - 1], & B_3 &= [4x + 4y + 2, 5x + 4y], \\ A_4 &= [2x + 2y, 2x + 4y - 3], & B_4 &= [2x + 4y + 1, 2x + 6y - 3]. \end{aligned}$$

Thus,

$$a_1 \leq A_1 \leq A_2 \leq A_3 < b_1 < a_2 < b_3 < A_4 < b_2 < B_4 < b_4 < B_3 < c_1 < c_2 < B_2 < B_1. \quad (8)$$

Note that  $V(G_1) \cap V(G_2) = \{y - 1\}$  and  $V(G_2) \cap V(G_3) = \{x\}$ ; otherwise,  $G_i$  and  $G_j$  are vertex-disjoint for  $i \neq j$ . Therefore,  $G_1 + G_2 +$

$G_3 + (2x - 1, b_1, c_1, 0)$  is a cycle of length  $4x + 1$ , and  $G_4 + (2x + 4y - 3, b_2, a_1, c_2, b_3, a_2, b_4, 2x + 2y)$  is a cycle of length  $4y + 1$ . Furthermore,  $V(C_{4x+1}) \cap V(C_{4y+1}) = \{a_1 = 0\}$ ; therefore,  $G$  is a graph composed of two cycles that share a single vertex.

Next, let  $E_i$  denote the set of edge labels in  $G_i$  for  $1 \leq i \leq 4$ . By **P3**, we have edge labels

$$E_1 = [8x + 2y + 9, 8x + 4y + 6],$$

$$E_2 = [6x + 4y + 6, 8x + 2y + 7],$$

$$E_3 = [2x + 4y + 3, 4x + 4y],$$

$$E_4 = [4, 4y - 3]$$

yielding edge lengths

$$E_1^* = \{8x + 8y + 5 - e : e \in E_1\} = [4y - 1, 6y - 4],$$

$$E_2^* = \{8x + 8y + 5 - e : e \in E_2\} = [6y - 2, 2x + 4y - 1],$$

$$E_3^* = \{e : e \in E_3\} = [2x + 4y + 3, 4x + 4y],$$

$$E_4^* = \{e : e \in E_4\} = [4, 4y - 3].$$

Moreover, the path  $(2x - 1, b_1, c_1, 0)$  consists of edges with lengths 1,  $(4x + 4y + 3)^* = 4x + 4y + 2$ , and  $(6x + 4y + 3)^* = 2x + 4y + 2$ , and the path  $(2x + 4y - 3, b_2, a_1, c_2, b_3, a_2, b_4, 2x + 2y)$  consists of edges with lengths 3,  $2x + 4y$ ,  $(6x + 4y + 4)^* = 2x + 4y + 1$ ,  $4x + 4y + 1$ , 2,  $6y - 3$ , and  $4y - 2$ . Thus, the edge set of  $G$  has one edge of each length  $i$  where  $1 \leq i \leq 4x + 4y + 2$ . It is clear from (8) that  $V(G) \subseteq [0, 8x + 4y + 4]$ . Hence, the defined labeling is a  $\rho$ -labeling, and condition (r1) for a  $\rho$ -tripartite labeling is satisfied.

Finally, let  $A = \bigcup_{i=1}^4 A_i \cup \{a_1, a_2\}$ ,  $B = \bigcup_{i=1}^4 B_i \cup \{b_1, b_2, b_3, b_4\}$ , and  $C = \{c_1, c_2\}$ . Then,  $\{A, B, C\}$  is a tripartition of  $V(G)$ . Condition (r2) of a  $\rho$ -tripartite labeling is clear from (8). Note that  $|b_1 - c_1| + |b_3 - c_2| = (4x + 4y + 3) + (4x + 4y + 1) = 8x + 8y + 4$ , twice the number of edges of  $G$ . Thus, condition (r3) is satisfied. Also,  $|b - c| = 8x + 8y + 4$ , where  $b \in B$  and  $c \in C$ , is impossible since all vertices are in  $[0, 8x + 8y + 4]$  and  $0 \in A$ . Thus, condition (r4) holds, and we have a  $\rho$ -tripartite labeling of  $G$ .

**Case 2**  $r \equiv s \equiv 3 \pmod{4}$ .

Let  $G = C_{4x+3} \uplus C_{4y+3}$  where  $y \geq x \geq 0$  but  $(x, y) \neq (0, 0)$ . We will consider three subcases.

**Case 2.1**  $x = 0$  and  $y \geq 1$ .

Let  $C_3 = (a_1, b_1, c_1, a_1)$  and  $C_{4y+3} = G_1 + G_2 + (2y - 1, b_2, a_2, c_2, b_3, 0)$  where  $a_1 = 0$ ,  $b_1 = 4y + 3$ ,  $c_1 = 8y + 10$ ,  $b_2 = 2y + 1$ ,  $a_2 = 2y$ ,  $c_2 = 8y + 9$ ,  $b_3 = 4y + 4$ , and

$$G_1 = P(0, 2y + 4, 2y - 2),$$

$$G_2 = P(y - 1, y + 2, 2y).$$

(Note: In the case when  $y = 1$ , the path  $G_1$  is empty. However, this does not change the proof in any way.) If we proceed as in Case 1, it is easy to verify that we have a  $\rho$ -tripartite labeling of  $G$ .

Figure 19: A  $\rho$ -tripartite labeling of  $C_3 \cup C_{11}$ .

**Case 2.2**  $1 \leq x \leq y < x + 2$ .

Let  $C_{4x+3} = G_1 + G_2 + (2x, c_1, b_1, 0)$  and  $C_{4y+3} = G_3 + G_4 + (2x + 2y, b_2, c_2, 2x)$  where  $c_1 = 8x + 8y + 11$ ,  $b_1 = 4x + 4y + 4$ ,  $b_2 = 4x + 4y + 7$ ,  $c_2 = 8x + 8y + 12$ , and

$$G_1 = P(0, 2x + 2y + 7, 2x + 2y - 4),$$

$$G_2 = P(x + y - 2, x + 5y, 2x - 2y + 4),$$

$$G_3 = P(2x, 4x + 2, 4y - 2x),$$

$$G_4 = P(x + 2y, x + 2y, 2x).$$

(Note: In the case when  $x = y = 1$ , the path  $G_1$  are empty. However, this does not change the proof in any way.) If we proceed as in Case 1, it is easy to verify that we have a  $\rho$ -tripartite labeling of  $G$ .

Figure 20: A  $\rho$ -tripartite labeling of  $C_{11} \cup C_{11}$ .

**Case 2.3**  $x \geq 1$  and  $y \geq x + 2$ .

Let  $C_{4x+3} = G_1 + (2x, c_1, b_1, 0)$  and  $C_{4y+3} = G_2 + G_3 + G_4 + (2x + 2y, b_2, c_2, 2x)$  where  $c_1 = 8x + 8y + 11$ ,  $b_1 = 4x + 4y + 4$ ,  $b_2 = 4x + 4y + 7$ ,  $c_2 = 8x + 8y + 12$ , and

$$G_1 = P(0, 4y + 3, 4x),$$

$$G_2 = P(2x, 4x + 2y + 7, 2y - 2x - 4),$$

$$G_3 = P(x + y - 2, 3x + y, 2y + 4),$$

$$G_4 = P(x + 2y, x + 2y, 2x).$$

(Note: In the case when  $y = x + 2$ , the path  $G_2$  is empty. However, this does not change the proof in any way.) If we proceed as in Case 1, it is easy to verify that we have a  $\rho$ -tripartite labeling of  $G$ . ■



Figure 21: A  $\rho$ -tripartite labeling of  $C_7 \cup C_{19}$ .

Although the graph  $G = C_3 \cup C_3$  does not admit a labeling that yields cyclic  $G$ -decompositions of  $K_{12t+1}$ , it is easy to verify that such a cyclic decomposition does exist. It is well known that there exists a cyclic  $C_3$ -decomposition of  $K_{6t+1}$  for all positive integers  $t$ . (Such a decomposition is better known as a *cyclic Steiner Triple System of order  $6t + 1$* .) The  $t$  starters in such a decomposition are usually obtained from the partition of  $[1, 3t]$  into  $t$  triples  $\{a, b, c\}$  such that  $a + b = c$  or  $a + b + c \equiv 0 \pmod{6t + 1}$ . (This is known as the *First Heffter difference problem*.) Then the set of all triples  $\{0, a, a + b\}$  gives the  $t$  starter blocks in the triple system. In other words, the triples  $\{0, a, a + b\}$  give a  $\rho$ -labeling of the graph  $C_3^{(t)}$ . Thus there exists a cyclic  $C_3^{(t)}$ -decomposition of  $K_{6nt+1}$  for all positive integers  $t$ . Since  $C_3 \cup C_3$  divides  $C_3^{(2t)}$ , there exists a cyclic  $(C_3 \cup C_3)$ -decomposition of  $K_{12t+1}$ . We can now state our main theorem.

**Theorem 12** *If  $G$  with  $n$  edges is the one-point union of any two cycles, then there exists a cyclic  $G$ -decomposition of  $K_{2nt+1}$  for all positive integers  $t$ .*

### 3 Acknowledgment

This research is supported by grant number A0649210 from the Division of Mathematical Sciences at the National Science Foundation. This work was done under the supervision of the second and fourth authors as part of: *REU Site: Mathematics Research Experience for Pre-service and for In-service Teachers* at Illinois State University.

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