Domination Value in $P_2 \Box P_n$ and $P_2 \Box C_n$

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Abstract

A set $D \subseteq V(G)$ is a *dominating set* of a graph G if every vertex of G not in D is adjacent to at least one vertex in D. A *minimum dominating set* of G, also called a $\gamma(G)$-set, is a dominating set of G of minimum cardinality. For each vertex $v \in V(G)$, we define the *domination value* of v to be the number of $\gamma(G)$-sets to which v belongs. In this paper, we find the total number of minimum dominating sets and characterize the domination values for $P_2 \Box P_n$ and $P_2 \Box C_n$.

1 Introduction

Let $G = (V(G), E(G))$ be a simple, undirected, and nontrivial graph. For $S \subseteq V(G)$, we denote by $\langle S \rangle$ the subgraph of G induced by S. For a vertex $v \in V(G)$, the *open neighborhood of* v is the set $N(v) = \{u \mid uv \in E(G)\}$, and the *closed neighborhood of* v is the set $N[v] = N(v) \cup \{v\}$. For $S \subseteq V(G)$, the *open neighborhood of* S is the set $N(S) = \cup_{v \in S} N(v)$ and the *closed neighborhood of* S is the set $N[S] = N(S) \cup S$.

A set $D \subseteq V(G)$ is a *dominating set* if $N[D] = V(G)$, and is a *total dominating set* if $N(D) = V(G)$. The *domination number of a graph* G, denoted by $\gamma(G)$, is the minimum of the cardinalities of all dominating sets of G. A *minimum dominating set* of G, also called a $\gamma(G)$-set, is a dominating set of G of minimum cardinality. For discussions on domination (resp. total domination) in graphs, see [1, 2, 6, 9, 10, 17] (resp. see [5, 9, 12]). Slater [18] introduced the notion of the number of dominating sets of G, which he denoted by HED(G) in honor of Steve Hedetniemi on the occasion of his 60th birthday; further, Slater used $\#\gamma(G)$ to denote the number of $\gamma(G)$-sets. Following [14, 19], we denote by $\tau(G)$ the total

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number of $\gamma(G)$-sets. For each vertex $v \in V(G)$, we define the *domination value* of v in G, denoted by $DV_G(v)$, to be the number of $\gamma(G)$-sets to which v belongs; we often drop G when ambiguity is not a concern. Clearly, $0 \leq DV_G(v) \leq \tau(G)$ for any graph G and for any vertex $v \in V(G)$. See [19] for an introductory discussion on domination value in graphs and [14] for an introductory discussion on total domination value in graphs.

The *Cartesian product* of two graphs G and H, denoted by $G \Box H$, is the graph with the vertex set $V(G) \times V(H)$ such that (u, v) is adjacent to (u', v') if and only if (i) $u = u'$ and $vv' \in E(H)$ or (ii) $v = v'$ and $uu' \in E(G)$. For other graph theory terminology, refer to [4].

We denote by P_n and C_n the path and the cycle on n vertices, respectively. In [13], Jacobson and Kinch obtained the results on $\gamma(P_m \Box P_n)$ for $m = 2, 3, 4$. Later, Hare developed an algorithm to compute $\gamma(P_m \Box P_n)$ and was able to find expressions for $\gamma(P_m \Box P_n)$ for a number of different values of m and n (see [8]). Chang and Clark proved the formulas found by Hare for $\gamma(P_5 \Box P_n)$ and $\gamma(P_6 \Box P_n)$ in [3]. The complexity of determining $\gamma(P_m \Box P_n)$ is open as of [11]. In [15], Klavžar and Seifter obtained results on $\gamma(C_m \Box C_n)$ for $m = 3, 4, 5$.

In section 2, we present relevant results from [19]. In sections 3 and 4, noting $\gamma(P_2 \Box P_n) \neq \gamma(P_2 \Box C_n)$ for $n \equiv 0 \pmod 4$, we investigate the total number of minimum dominating sets and the domination value for two classes of graphs, $P_2 \Box P_n$ and $P_2 \Box C_n$.

2 Preliminaries and domination value in paths and cycles

We first recall the following observations.

Observation 2.1. *[19]* $\sum_{v \in V(G)} DV_G(v) = \tau(G) \cdot \gamma(G)$

Observation 2.2. *[19] If there is an isomorphism of graphs carrying a vertex v in G to a vertex v' in G', then $DV_G(v) = DV_{G'}(v')$.*

It is well known that $\gamma(P_n) = \gamma(C_n) = \lceil \frac{n}{3} \rceil$. If we let the vertices of the path P_n be labeled 1 through n consecutively, then we have the following

Theorem 2.3. *[19] For $n \geq 2$,*

$$\tau(P_n) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod 3 \\ n + \frac{1}{2}\lfloor \frac{n}{3} \rfloor (\lfloor \frac{n}{3} \rfloor - 1) & \text{if } n \equiv 1 \pmod 3 \\ 2 + \lfloor \frac{n}{3} \rfloor & \text{if } n \equiv 2 \pmod 3). \end{cases}$$

For the domination value of a vertex v on P_n, by Observation 2.2, $DV(v) = DV(n+1-v)$ for $1 \le v \le n$. More precisely, we have the classification results which follow.

Corollary 2.4. *[19] Let $v \in V(P_{3k})$, where $k \ge 1$. Then*

$$DV(v) = \begin{cases} 0 & \text{if } v \equiv 0, 1 \pmod 3 \\ 1 & \text{if } v \equiv 2 \pmod 3 \end{cases} .$$

Proposition 2.5. *[19] Let $v \in V(P_{3k+1})$, where $k \ge 1$. Write $v = 3q + r$, where q and r are non-negative integers such that $0 \le r < 3$. Then, noting $\tau(P_{3k+1}) = \frac{1}{2}(k^2 + 5k + 2)$, we have*

$$DV(v) = \begin{cases} \frac{1}{2}q(q+3) & \text{if } v \equiv 0 \pmod 3 \\ (q+1)(k-q+1) & \text{if } v \equiv 1 \pmod 3 \\ \frac{1}{2}(k-q)(k-q+3) & \text{if } v \equiv 2 \pmod 3 \end{cases} .$$

Proposition 2.6. *[19] Let $v \in V(P_{3k+2})$, where $k \ge 0$. Write $v = 3q + r$, where q and r are non-negative integers such that $0 \le r < 3$. Then, noting $\tau(P_{3k+2}) = k + 2$, we have*

$$DV(v) = \begin{cases} 0 & \text{if } v \equiv 0 \pmod 3 \\ 1 + q & \text{if } v \equiv 1 \pmod 3 \\ k + 1 - q & \text{if } v \equiv 2 \pmod 3 \end{cases} .$$

If we let the vertices of the cycle C_n be labeled 1 though n cyclically, then we have the following

Theorem 2.7. *[19] For $n \ge 3$,*

$$\tau(C_n) = \begin{cases} 3 & \text{if } n \equiv 0 \pmod 3 \\ n(1 + \frac{1}{2}\lfloor \frac{n}{3} \rfloor) & \text{if } n \equiv 1 \pmod 3 \\ n & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

By Theorem 2.7, Observation 2.1, Observation 2.2, and the vertex-transitivity of C_n, we have the following

Corollary 2.8. *[19] Let $v \in V(C_n)$, where $n \ge 3$. Then*

$$DV(v) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod 3 \\ \frac{1}{2}\lceil \frac{n}{3} \rceil (1 + \lceil \frac{n}{3} \rceil) & \text{if } n \equiv 1 \pmod 3 \\ \lceil \frac{n}{3} \rceil & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

3 Total number of minimum dominating sets and domination value in $P_2\Box P_n$

We consider $P_2\Box P_n$ ($n \geq 2$) as two copies of P_n with vertices labeled $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ with only the edges x_iy_i, for each i ($1 \leq i \leq n$), between two paths (see Figure 1).

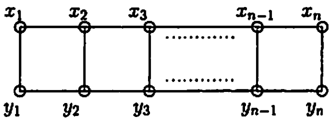

Figure 1: Labeling of vertices of $P_2\Box P_n$

We first recall the following.

Theorem 3.1. *[13] For $n \geq 2$, $\gamma(P_2\Box P_n) = \lceil \frac{n+1}{2} \rceil$.*

Lemma 3.2. *Let $G = P_2\Box P_n$, where $n \geq 2$. If neither x_1 nor y_1 belongs to a $\gamma(G)$-set D, then $\{x_2, y_2\} \subseteq D$. (Likewise, if neither x_n nor y_n belongs to D, then $\{x_{n-1}, y_{n-1}\} \subseteq D$.)*

Proof. By definition of a dominating set, either x_1 or a vertex in $N(x_1) = \{x_2, y_1\}$ belongs to D. If $x_1 \notin D$ and $y_1 \notin D$, then $x_2 \in D$. Similarly, either $y_1 \in D$ or a vertex in $N(y_1) = \{x_1, y_2\}$ belongs to D. If $x_1 \notin D$ and $y_1 \notin D$, then $y_2 \in D$ as well. Thus $x_1 \notin D$ and $y_1 \notin D$ implies $\{x_2, y_2\} \subseteq D$. □

Lemma 3.3. *Let $G = P_2\Box P_n$, where $n \geq 3$. If there exists a $\gamma(G)$-set containing no vertex of degree two, then $n = 3$ or $n = 6$.*

Proof. Suppose that D is a $\gamma(G)$-set such that $\{x_1, y_1, x_n, y_n\} \cap D = \emptyset$. Let $S_0 = \{x_2, y_2, x_{n-1}, y_{n-1}\}$. Then, by Lemma 3.2, $S_0 \subseteq D$. Note that $|S_0| = 2$ if and only if $n = 3$: in this case, $\gamma(P_2\Box P_3) = 2$ and $S_0 = \{x_2, y_2\}$ is a $\gamma(P_2\Box P_3)$-set. If $4 \leq n \leq 5$, then $|S_0| = 4$ and $\gamma(P_2\Box P_n) = 3$, and thus $S_0 \not\subseteq D$. If $n = 6$, then $|S_0| = 4$ and $\gamma(P_2\Box P_6) = 4$: in fact, $S_0 = \{x_2, y_2, x_5, y_5\}$ is a $\gamma(P_2\Box P_6)$-set. Now, we need to consider $n \geq 7$. Suppose that $S_0 \subseteq D$; we consider two cases.

Case 1. $n = 2k$, *where* $k \geq 4$*:* Here, $\gamma(P_2\Box P_{2k}) = k + 1$. Since $N[S_0] = \{x_i, y_i \mid 1 \leq i \leq 3\} \cup \{x_j, y_j \mid 2k-2 \leq j \leq 2k\}$, the part of $P_2\Box P_{2k}$ not dominated by S_0 is a $P_2\Box P_{2k-6}$. So, $k - 3$ vertices of $D - S_0$ must dominate $P_2\Box P_{2k-6}$. But $\gamma(P_2\Box P_{2k-6}) = k - 2$ by Theorem 3.1, and we reach a contradiction.

Case 2. $n = 2k+1$, where $k \geq 3$: Here, $\gamma(P_2 \square P_{2k+1}) = k+1$. Since $N[S_0] = \{x_i, y_i \mid 1 \leq i \leq 3\} \cup \{x_j, y_j \mid 2k-1 \leq j \leq 2k+1\}$, the part of $P_2 \square P_{2k+1}$ not dominated by S_0 is a $P_2 \square P_{2k-5}$. So, $k-3$ vertices of $D - S_0$ must dominate $P_2 \square P_{2k-5}$. But $\gamma(P_2 \square P_{2k-5}) = k-2$ by Theorem 3.1, and we reach a contradiction.

Thus, we have shown that if $S_0 \subseteq D$, then $n = 3$ or $n = 6$. □

Next we compute the total number of $\gamma(P_2 \square P_n)$-sets for $n \geq 2$.

Theorem 3.4. *For $n \geq 2$,*

$$\tau(P_2 \square P_n) = \begin{cases} 6 & \text{if } n = 2 \\ 3 & \text{if } n = 3 \\ 17 & \text{if } n = 6 \\ 2 & \text{if } n \text{ is odd and } n \neq 3 \\ 2n+4 & \text{if } n \text{ is even and } n \neq 2, 6\ . \end{cases}$$

Proof. Let D be a $\gamma(P_2 \square P_n)$-set for $n \geq 2$. Notice that no D contains both x_1 and y_1, or both x_n and y_n, unless $n = 2$. We consider two cases.

Case 1. $n \geq 3$ is odd: Here, $\gamma(P_2 \square P_n) = \frac{n+1}{2}$. By Lemma 3.3, if there is a D containing no vertex of degree two then $n = 3$. Moreover, we note that $\{x_2, y_2\} \subseteq D$ if and only if $n = 3$: If $\{x_2, y_2\} \subseteq D$ and $n > 3$, then the part of $P_2 \square P_n$ not dominated by $\{x_2, y_2\}$ is a $P_2 \square P_{n-3}$, and $\frac{n-3}{2}$ vertices of $D - \{x_2, y_2\}$ must dominate $P_2 \square P_{n-3}$. But $\gamma(P_2 \square P_{n-3}) = \frac{n-1}{2}$ by Theorem 3.1, and we reach a contradiction. So, if $n > 3$, by Lemma 3.2, either $x_1 \in D$ or $y_1 \in D$. One can easily check that $x_1 \in D$ uniquely determines a γ-set $D = \{x_i, y_j \mid i \equiv 1, j \equiv 3 \pmod 4\}$. Similarly, $y_1 \in D$ uniquely determines a γ-set $D = \{x_i, y_j \mid i \equiv 3, j \equiv 1 \pmod 4\}$. Thus, $\tau(P_2 \square P_n) = 2$ for $n \neq 3$, and $\tau(P_2 \square P_3) = 3$ by Lemma 3.3. (See Figure 2 for the three $\gamma(P_2 \square P_3)$-sets, where the solid black vertices in each $P_2 \square P_3$ form a $\gamma(P_2 \square P_3)$-set.)

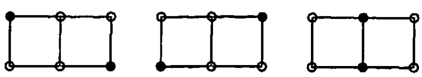

Figure 2: γ-sets for $P_2 \square P_3$

Case 2. $n \geq 2$ is even: Here, $\gamma(P_2 \square P_n) = \frac{n}{2} + 1$. If $n = 2$, then $\gamma(P_2 \square P_2) = 2$ and $\tau(P_2 \square P_2) = \tau(C_4) = \binom{4}{2} = 6$. We consider $n \geq 4$. By Lemma 3.3, if there is a D containing no vertex of degree two (i.e., $\{x_2, y_2, x_{n-1}, y_{n-1}\} \subseteq D$), then $n = 6$. We consider three subcases.

Subcase 2.1. $\{x_2, y_2\} \subseteq D$ *and* $\{x_{n-1}, y_{n-1}\} \cap D = \emptyset$: Let τ_1 be the number of such $\gamma(P_2 \Box P_n)$-sets for $n \geq 4$. Note that the part of $P_2 \Box P_n$ not dominated by $\{x_2, y_2\}$ is a $P_2 \Box P_{n-3}$. So, τ_1 equals the number of $\gamma(P_2 \Box P_{n-3})$-sets with $\gamma(P_2 \Box P_{n-3}) = \frac{n}{2} - 1$. One can easily see that $\tau_1 = 2$ when $n = 4, 6$. Since $\tau_1(P_2 \Box P_{n-3}) = 2$ for $n \geq 8$ by Case 1, we have $\tau_1 = 2$ for $n \geq 4$.

Subcase 2.2. $\{x_2, y_2\} \cap D = \emptyset$ *and* $\{x_{n-1}, y_{n-1}\} \subseteq D$: Let τ_2 be the number of such $\gamma(P_2 \Box P_n)$-sets for $n \geq 4$. By Observation 2.2 and Subcase 2.1, we have $\tau_2 = 2$ for $n \geq 4$.

Subcase 2.3. $\{x_2, y_2\} \not\subseteq D$ *and* $\{x_{n-1}, y_{n-1}\} \not\subseteq D$: By Lemma 3.2, $|\{x_1, y_1\} \cap D| = 1$ and $|\{x_n, y_n\} \cap D| = 1$. Let D (resp. D') be such a γ-set of $G = P_2 \Box P_n$ (resp. $G' = P_2 \Box P_{n+2}$), where $n \geq 4$. And let τ_3 (resp. τ_3') be the number of such γ-sets of G (resp. G'). We will show that $\tau_3 = 2n$, for $n \geq 4$, using induction. The base case, $n = 4$, is easily verified (see Figure 3). Assume that $\tau_3 = 2n$ for $n \geq 4$. If $x_1 \in D$, then each D extends to D' such that $D' = D \cup \{x_{n+2}\}$ if $y_n \in D$ and $D' = D \cup \{y_{n+2}\}$ if $x_n \in D$; in addition, there are two additional $\gamma(G')$-sets which do not come from any $\gamma(G)$-sets, i.e., $\{x_i, y_j \mid i \equiv 1, j \equiv 3 \pmod 4 \text{ and } 1 \leq i, j \leq n+1\} \cup \{x_{n+2}\}$ and $\{x_i, y_j \mid i \equiv 1, j \equiv 3 \pmod 4 \text{ and } 1 \leq i, j \leq n+1\} \cup \{y_{n+2}\}$. Similarly, if $y_1 \in D$, then each D extends to D' and there are two additional $\gamma(G')$-sets which do not come from $\gamma(G)$-sets. So, $\tau_3' = \tau_3 + 4 = 2n + 4 = 2(n+2)$.

Now, noting that $\{x_2, y_2, x_{n-1}, y_{n-1}\} \subseteq D$ implies $n = 6$, combine the three disjoint cases to get $\tau = \tau_1 + \tau_2 + \tau_3 = 2 + 2 + 2n = 2n + 4$ if $n \neq 2, 6$ and $\tau(P_2 \Box P_6) = (2 \cdot 6 + 4) + 1 = 17$. □

See Figure 3 for the collection of $\gamma(P_2 \Box P_4)$-sets, where the solid black vertices in each $P_2 \Box P_4$ form a $\gamma(P_2 \Box P_4)$-set.

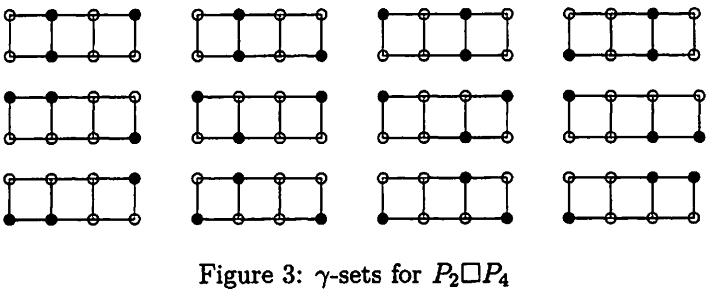

Figure 3: γ-sets for $P_2 \Box P_4$

As an immediate consequence of Theorem 3.4 for an odd $n \geq 3$, we have the following

Corollary 3.5. *Let* $n \geq 3$ *be an odd number.*

(i) For each $v \in V(P_2 \Box P_3)$, $DV(v) = 1$.

(ii) For $x_i, y_i \in V(P_2 \Box P_n)$, where $n \geq 5$,

$$DV(x_i) = DV(y_i) = \begin{cases} 1 & \textit{if } i \textit{ is odd} \\ 0 & \textit{if } i \textit{ is even} . \end{cases}$$

Proposition 3.6. *Let $n \geq 2$ be an even number.*

(i) For each $v \in V(P_2 \Box P_2)$, $DV(v) = 3$.

(ii) For $x_i, y_i \in V(P_2 \Box P_n)$, where $n \geq 4$ and $n \neq 6$,

$$DV(x_i) = DV(y_i) = \begin{cases} n+2-i & \textit{if } i \textit{ is odd and } 1 \leq i \leq n-3 \\ 4 & \textit{if } i = 2 \textit{ or } i = n-1 \\ i+1 & \textit{if } i \textit{ is even and } 4 \leq i \leq n . \end{cases} \quad (1)$$

(iii) For $x_i, y_i \in V(P_2 \Box P_6)$,

$$DV(x_i) = DV(y_i) = \begin{cases} 7 & \textit{if } i = 1 \textit{ or } i = 6 \\ 5 & \textit{if } 2 \leq i \leq 5 . \end{cases} \quad (2)$$

Proof. Let $n \geq 2$ be an even number.

(i) Note that $P_2 \Box P_2 \cong C_4$, $\gamma(C_4) = 2$, and $\tau(C_4) = 6$. By Observation 2.1, Observation 2.2, and the vertex-transitivity, $DV(v) = 3$ for each $v \in V(P_2 \Box P_2)$.

(ii) For an even $n \geq 4$, let D (resp. D') be a γ-set of $G = P_2 \Box P_n$ (resp. $G' = P_2 \Box P_{n+2}$). Since $DV_G(x_i) = DV_G(y_i)$ for each i $(1 \leq i \leq n)$, it suffices to compute $DV_G(x_i)$ for $1 \leq i \leq n$. We consider two cases.

Case 1. $\{x_1, y_1\} \cap D = \emptyset$*:* By Lemma 3.2, $\{x_2, y_2\} \subseteq D$. Denote by $DV^1(v)$ the number of such D's containing v. Notice that there are two such $\gamma(G)$-sets. We will show, by induction, that

$$DV^1_G(x_i) = \begin{cases} 2 & \text{if } i = 2 \\ 1 & \text{if } i \geq 4 \text{ and } i \text{ is even} \\ 0 & \text{if } i \text{ is odd} . \end{cases} \quad (3)$$

For $n = 4$ (the base case), the two γ-sets are $\{x_2, y_2, x_4\}$ and $\{x_2, y_2, y_4\}$, thus satisfying (3). Assume that (3) holds for G. Let D_1 and D_2 be $\gamma(G)$-sets, containing both x_2 and y_2, such that $x_n \in D_1$ and $y_n \in D_2$. Then D_1 extends to $D_1' = D_1 \cup \{y_{n+2}\}$ and D_2 extends to $D_2' = D_2 \cup \{x_{n+2}\}$, where D'_1 and D'_2 are $\gamma(G')$-sets. So, $DV^1_{G'}(x_i) = DV^1_G(x_i)$ for $1 \leq i \leq n$, $DV^1_{G'}(x_{n+1}) = 0$, and $DV^1_{G'}(x_{n+2}) = 1$. Thus

$$DV^1_{G'}(x_i) = \begin{cases} 2 & \text{if } i = 2 \\ 1 & \text{if } i \geq 4 \text{ and } i \text{ is even} \\ 0 & \text{if } i \text{ is odd}\ , \end{cases}$$

proving (3).

Case 2. $x_1 \in D$ *or* $y_1 \in D$*:* Denote by $DV^2(v)$ the number of such D's containing v. By Subcase 2.2 and Subcase 2.3 in the proof of Theorem 3.4, there are $2n + 2$ such $\gamma(G)$-sets; $n + 1$ such D's containing x_1, and $n + 1$ such D's containing y_1. We will show, by induction, that

$$DV^2_G(x_i) = \begin{cases} i & \text{if } i \equiv 0,2 \pmod 4 \text{ and } 2 \leq i \leq n \\ n+2-i & \text{if } i \equiv 1,3 \pmod 4 \text{ and } 1 \leq i \leq n-3 \\ 4 & \text{if } i = n-1\ . \end{cases} \tag{4}$$

Noting that no $\gamma(G)$-set contains both x_1 and y_1, we consider two subcases.

Subcase 2.1. $x_1 \in D$*:* Denote by $DV^{2,1}(v)$ the number of such D's containing v. For $n = 4$ (the base case), one can check that there are five such γ-sets: $\{x_1, x_2, y_4\}$, $\{x_1, y_2, x_4\}$, $\{x_1, y_3, x_4\}$, $\{x_1, y_3, y_4\}$, and $\{x_1, x_3, y_3\}$. Let $D_1, D_2, \cdots, D_{n+1}$ be $\gamma(G)$-sets containing x_1, where $\{x_{n-1}, y_{n-1}\} \subseteq D_{n+1}$. Then, for $1 \leq i \leq n$, each D_i extends to $D'_i = D_i \cup \{x_{n+2}\}$ if $y_n \in D_i$ and $D'_i = D_i \cup \{y_{n+2}\}$ if $x_n \in D_i$, where each D'_i $(1 \leq i \leq n)$ is a $\gamma(G')$-set; $D_{n+1} = \{x_i, y_j \mid i \equiv 1, j \equiv 3 \pmod 4 \text{ and } 1 \leq i,j \leq n-2\} \cup \{x_{n-1}, y_{n-1}\}$ does not extend to a $\gamma(G')$-set, but there exists a $\gamma(G')$-set $D'_{n+1} = \{x_i, y_j \mid i \equiv 1, j \equiv 3 \pmod 4 \text{ and } 1 \leq i,j \leq n\} \cup \{x_{n+1}, y_{n+1}\}$ which does not come from any $\gamma(G)$-set. Further, there exist two additional $\gamma(G')$-sets which do not come from any $\gamma(G)$-sets such as $D'_{n+2} = \{x_i, y_j \mid i \equiv 1, j \equiv 3 \pmod 4 \text{ and } 1 \leq i,j \leq n+1\} \cup \{x_{n+2}\}$ and $D'_{n+3} = \{x_i, y_j \mid i \equiv 1, j \equiv 3 \pmod 4 \text{ and } 1 \leq i,j \leq n+1\} \cup \{y_{n+2}\}$. So, noting that n is even, we have the following:

$$DV^{2,1}_{G'}(x_i) = \begin{cases} DV^{2,1}_G(x_i) & \text{if } i \equiv 0,2,3 \pmod 4 \text{ and } 1 \leq i \leq n-2 \\ DV^{2,1}_G(x_i) + 2 & \text{if } i \equiv 1 \pmod 4 \text{ and } 1 \leq i \leq n-2\ , \end{cases}$$

$$DV^{2,1}_{G'}(x_{n-1}) = \begin{cases} DV^{2,1}_G(x_{n-1}) - 1 & \text{if } n \equiv 0 \pmod 4 \\ DV^{2,1}_G(x_{n-1}) + 2 & \text{if } n \equiv 2 \pmod 4\ , \end{cases}$$

$$DV^{2,1}_{G'}(x_{n+1}) = \begin{cases} 3 & \text{if } n \equiv 0 \pmod 4 \\ 1 & \text{if } n \equiv 2 \pmod 4\ , \end{cases}$$

$DV^{2,1}_{G'}(x_n) = DV^{2,1}_G(x_n)$, and $DV^{2,1}_{G'}(x_{n+2}) = \frac{n}{2} + 1$.

Subcase 2.2. $y_1 \in D$: Denote by $DV^{2,2}(v)$ the number of such D's containing v. For $n = 4$ (the base case), one can check that there are five such γ-sets: $\{y_1, y_2, x_4\}$, $\{y_1, x_2, y_4\}$, $\{y_1, x_3, x_4\}$, $\{y_1, x_3, y_4\}$, and $\{y_1, x_3, y_3\}$. Let $\Gamma_1, \Gamma_2, \cdots, \Gamma_{n+1}$ be $\gamma(G)$-sets containing y_1, where $\{x_{n-1}, y_{n-1}\} \subseteq \Gamma_{n+1}$. Then, for $1 \le i \le n$, each Γ_i extends to $\Gamma'_i = \Gamma_i \cup \{x_{n+2}\}$ if $y_n \in \Gamma_i$ and $\Gamma'_i = \Gamma_i \cup \{y_{n+2}\}$ if $x_n \in \Gamma_i$, where each Γ'_i $(1 \le i \le n)$ is a $\gamma(G')$-set; $\Gamma_{n+1} = \{x_i, y_j \mid i \equiv 3, j \equiv 1 \pmod 4 \text{ and } 1 \le i,j \le n-2\} \cup \{x_{n-1}, y_{n-1}\}$ does not extend to a $\gamma(G')$-set, but there exists a $\gamma(G')$-set $\Gamma'_{n+1} = \{x_i, y_j \mid i \equiv 3, j \equiv 1 \pmod 4 \text{ and } 1 \le i,j \le n\} \cup \{x_{n+1}, y_{n+1}\}$ which does not come from any $\gamma(G)$-set. Further, there exist two additional $\gamma(G')$-sets which do not come from any $\gamma(G)$-sets such as $\Gamma'_{n+2} = \{x_i, y_j \mid i \equiv 3, j \equiv 1 \pmod 4 \text{ and } 1 \le i,j \le n+1\} \cup \{x_{n+2}\}$ and $\Gamma'_{n+3} = \{x_i, y_j \mid i \equiv 3, j \equiv 1 \pmod 4 \text{ and } 1 \le i,j \le n+1\} \cup \{y_{n+2}\}$. So, noting that n is even, we have the following:

$$DV^{2,2}_{G'}(x_i) = \begin{cases} DV^{2,2}_G(x_i) & \text{if } i \equiv 0,1,2 \pmod 4 \text{ and } 1 \le i \le n-2 \\ DV^{2,2}_G(x_i) + 2 & \text{if } i \equiv 3 \pmod 4 \text{ and } 1 \le i \le n-2\,, \end{cases}$$

$$DV^{2,2}_{G'}(x_{n-1}) = \begin{cases} DV^{2,2}_G(x_{n-1}) + 2 & \text{if } n \equiv 0 \pmod 4 \\ DV^{2,2}_G(x_{n-1}) - 1 & \text{if } n \equiv 2 \pmod 4\,, \end{cases}$$

$$DV^{2,2}_{G'}(x_{n+1}) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod 4 \\ 3 & \text{if } n \equiv 2 \pmod 4\,, \end{cases}$$

$DV^{2,2}_{G'}(x_n) = DV^{2,2}_G(x_n)$, and $DV^{2,2}_{G'}(x_{n+2}) = \frac{n}{2} + 1$.

Next, assume that (4) holds for G. Noting that $DV^2(v) = DV^{2,1}(v) + DV^{2,2}(v)$ and that n is even, by Subcase 2.1 and Subcase 2.2, we have

$$DV^2_{G'}(x_i) = \begin{cases} DV^2_G(x_i) & \text{if } i \equiv 0,2 \pmod 4 \text{ and } 1 \le i \le n-2 \\ DV^2_G(x_i) + 2 & \text{if } i \equiv 1,3 \pmod 4 \text{ and } 1 \le i \le n-2\,, \end{cases}$$

$DV^2_{G'}(x_{n-1}) = DV^2_G(x_{n-1}) + 1$, $DV^2_{G'}(x_n) = DV^2_G(x_n)$, $DV^2_{G'}(x_{n+1}) = 4$, and $DV^2_{G'}(x_{n+2}) = n + 2$, proving (4).

Now, noting that $DV(v) = DV^1(v) + DV^2(v)$ for $v \in V(P_2 \square P_n)$, where $n \ge 4$ is even and $n \ne 6$, combine (3) and (4) to obtain (1), proving (*ii*).

(iii) By Theorem 3.4, $P_2 \square P_6$ has an additional γ-set $\{x_2, y_2, x_5, y_5\}$. This, together with (1), for $x_i, y_i \in V(P_2 \square P_6)$, we obtain

$$DV(x_i) = DV(y_i) = \begin{cases} 8 - i & \text{if } i \text{ is odd and } 1 \le i \le 3 \\ 5 & \text{if } i = 2 \text{ or } i = 5 \\ i + 1 & \text{if } i \text{ is even and } 4 \le i \le 6\,, \end{cases}$$

which equals the domination value in (2). □

4 Total number of minimum dominating sets and domination value in $P_2\Box C_n$

For $n \geq 3$, consider $P_2\Box C_n$ as two copies of C_n with vertices labeled $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ with only the edges x_iy_i, for each i ($1 \leq i \leq n$), between two cycles (see Figure 4).

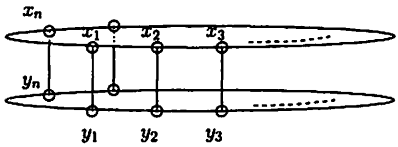

Figure 4: Labeling of vertices of $P_2\Box C_n$

We recall the following result.

Theorem 4.1. *[7] For $n \geq 3$,*

$$\gamma(P_2\Box C_n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 4 \\ \lceil \frac{n+1}{2} \rceil & \text{if } n \not\equiv 0 \pmod 4. \end{cases}$$

We introduce the following definition which will be used in the proof of Theorem 4.3.

Definition 4.2. Let G^1 and G^2 be disjoint copies of a graph G, and let D be a $\gamma(P_2\Box G)$-set. Let $\langle D \cap V(G^1)\rangle = \cup_{i=1}^{m_1}\mathcal{H}_i^1$, a disjoint union of connected components such that $|V(H_i^1)| \leq |V(H_{i+1}^1)|$ for $1 \leq i \leq m_1 - 1$; similarly, we write $\langle D \cap V(G^2)\rangle = \cup_{i=1}^{m_2}\mathcal{H}_i^2$. Let $\alpha = \max(|V(\mathcal{H}_{m_1}^1)|, |V(\mathcal{H}_{m_2}^2)|)$; we will denote by $\mathcal{H}_\alpha$ any $\mathcal{H}_i^j$ with $|V(\mathcal{H}_i^j)| = \alpha$, for $j = 1, 2$ ($1 \leq i \leq m_1$ or $1 \leq i \leq m_2$).

Example. The black vertices in Figure 5 form a $\gamma(P_2\Box C_{10})$-set D, where $\langle D\rangle$ contains $2\mathcal{H}_2$.

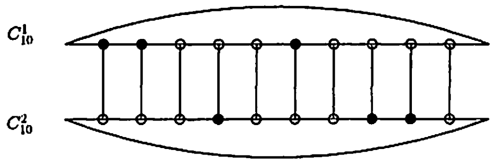

Figure 5: $2\mathcal{H}_2 \subseteq \langle D\rangle$, where D is a $\gamma(P_2\Box C_{10})$-set

Theorem 4.3. *Let $n \geq 3$. For each $v \in V(P_2 \Box C_n)$,*

$$DV(v) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod 4 \\ \frac{n+1}{2} & \text{if } n \equiv 1,3 \pmod 4 \text{ and } n \neq 3 \\ (\lceil \frac{n+1}{2} \rceil)^2 & \text{if } n \equiv 2 \pmod 4 \text{ and } n \neq 6 \\ 3 & \text{if } n = 3 \\ 17 & \text{if } n = 6 \ . \end{cases}$$

Proof. By Observation 2.2 and the vertex-transitivity, $DV(v) = DV(x_1)$ for each $v \in V(P_2 \Box C_n)$. Let D be a $\gamma(P_2 \Box C_n)$-set containing x_1, where $n \geq 3$; note that at least a vertex in $\{x_2, x_3, y_1, y_2, y_3\}$ belongs to D. Noting that each vertex dominates four vertices, we consider four cases.

Case 1. $n = 4k$, where $k \geq 1$: Since $\gamma(P_2 \Box C_{4k}) = 2k$ and $|V(P_2 \Box C_{4k})| = 8k$, each vertex is dominated by exactly one vertex (i.e., no vertex is doubly dominated). Thus there is a unique D containing x_1, i.e., $D = \{x_i, y_j \mid i \equiv 1, j \equiv 3 \pmod 4\}$, and hence $DV(x_1) = 1$.

Case 2. $n = 4k+1$, where $k \geq 1$: Here $\gamma(P_2 \Box C_{4k+1}) = 2k+1$. We will show that no D contains both x_1 and a vertex in $\{y_1, y_2, x_3\}$. First, we note that no D contains both x_1 and y_1: if $\{x_1, y_1\} \subseteq D$, then the part of $P_2 \Box C_{4k+1}$ not dominated by $\{x_1, y_1\}$ is a $P_2 \Box P_{4k-2}$, and $2k-1$ vertices of $D - \{x_1, y_1\}$ must dominate $P_2 \Box P_{4k-2}$. But $\gamma(P_2 \Box P_{4k-2}) = 2k$ by Theorem 3.1, and we reach a contradiction. Second, we note that no D contains both x_1 and y_2: if $\{x_1, y_2\} \subseteq D$, then the part of $P_2 \Box C_{4k+1}$ not dominated by $\{x_1, y_2\}$ is the graph H in Figure 6, and $2k-1$ vertices of $D - \{x_1, y_2\}$ must dominate H. If we let $S_0 = \{x_i, y_j \mid i \equiv 0, j \equiv$

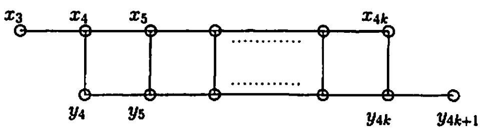

Figure 6: $H \subset P_2 \Box C_{4k+1}$

$2 \pmod 4$ and $4 \leq i, j \leq 4k-2\}$, then $|S_0| = 2(k-1)$, S_0 dominates $8(k-1)$ vertices, the part of H not dominated by S_0 is a P_4, and one vertex of $D - (S_0 \cup \{x_1, y_2\})$ must dominate P_4. But $\gamma(P_4) = 2$, and we reach a contradiction. (Similarly, no D contains both x_1 and y_{4k+1}.) Third, no D contains both x_1 and x_3: if $\{x_1, x_3\} \subseteq D$, then a vertex in $N[y_2] = \{x_2, y_1, y_2, y_3\}$ must belong to D. Since $\{x_1, y_1\} \not\subseteq D$ (and thus $\{x_3, y_3\} \not\subseteq D$ by the vertex-transitivity) and $\{x_1, y_2\} \not\subseteq D$, $x_2 \in D$.

If $R_0 := \{x_1, x_2, x_3\} \subseteq D$, then the part of $P_2 \Box C_{4k+1}$ not dominated by R_0, say H_1, must be dominated by $2k-2$ vertices in $D - R_0$. Since $|V(P_2 \Box C_{4k+1})| = 8k+2$ and $|N[R_0]| = 8$, $2k-2$ vertices in $D - R_0$ must dominate $8k-6$ vertices. But each vertex in $P_2 \Box C_{4k+1}$ dominates four vertices, and we reach a contradiction. (Similarly, no D contains both x_1 and x_{4k}.) So, we only need to consider D such that (i) $\{x_1, x_2\} \subseteq D$ (resp. $\{x_1, x_{4k+1}\} \subseteq D$) or (ii) no vertex in $N[x_1]$ is doubly dominated (i.e., $\{x_1, y_3\} \subseteq D$ and $\{x_1, y_{4k}\} \subseteq D$).

Subcase 2.1. $\{x_1, x_2\} \subseteq D$ (resp. $\{x_1, x_{4k+1}\} \subseteq D$): The part of $P_2 \Box C_{4k+1}$ not dominated by $\{x_1, x_2\}$, say H_2, must be dominated by $2k-1$ vertices in $D - \{x_1, x_2\}$. Since $|V(P_2 \Box C_{4k+1})| = 8k+2$ and $|N[\{x_1, x_2\}]| = 6$, $2k-1$ vertices in $D - \{x_1, x_2\}$ must dominate H_2 with $|V(H_2)| = 8k-4$, and thus there exists at most one γ-set containing both x_1 and x_2 (resp. x_1 and x_{4k+1}). Noting that $\{x_1\} \cup \{x_i, y_j \mid i \equiv 2, j \equiv 0 \pmod 4\}$ (resp. $\{x_i, y_j \mid i \equiv 1, j \equiv 3 \pmod 4\}$) is a γ-set, there is a unique D containing both x_1 and x_2 (resp. x_1 and x_{4k+1}).

Subcase 2.2. No vertex in $N[x_1]$ is doubly dominated: Since $x_1 \notin V(\mathcal{H}_2)$, by Subcase 2.1, there are $2k-1$ slots in which $\mathcal{H}_2$ can be placed.

By Subcase 2.1 and Subcase 2.2, we have $DV(x_1) = 2(1) + (2k-1) = 2k+1$.

Case 3. $n = 4k+2$, where $k \geq 1$: Here $\gamma(P_2 \Box C_{4k+2}) = 2k+2$. We will show that no D contains a $\mathcal{H}_\alpha$ for $\alpha \geq 4$. If $R_1 := \{x_1, x_2, x_3, x_4\} \subseteq D$, then the part of $P_2 \Box C_{4k+2}$ not dominated by R_1, say F_1, must be dominated by $2k-2$ vertices in $D - R_1$. Since $|V(P_2 \Box C_{4k+2})| = 8k+4$ and $|N[R_1]| = 10$, $2k-2$ vertices in $D - R_1$ must dominate F_1 with $|V(F_1)| = 8k-6$. But each vertex in $P_2 \Box C_{4k+2}$ dominates four vertices, and we reach a contradiction. We consider four subcases.

Subcase 3.1. $\mathcal{H}_3 \subseteq \langle D \rangle$: We denote by $DV^1(x_1)$ the number of such D's containing x_1. We note that the placement of $\mathcal{H}_3$ uniquely determines D: if $R_2 := \{x_1, x_2, x_3\} \subseteq D$, then the part of $P_2 \Box C_{4k+2}$ not dominated by R_2, say F_2, must be dominated by $2k-1$ vertices in $D - R_2$. Since $|V(P_2 \Box C_{4k+2})| = 8k+4$ and $|N[R_2]| = 8$, $2k-1$ vertices in $D - R_2$ must dominate F_2 with $|V(F_2)| = 8k-4$, and thus there exists at most one γ-set containing R_2. Noting that $\{x_1, x_2\} \cup \{x_i, y_j \mid i \equiv 3, j \equiv 1 \pmod 4 \text{ and } 3 \leq i, j \leq 4k+2\}$ is a γ-set, there is a unique D containing R_2. If $x_1 \in V(\mathcal{H}_3)$, there are three such D's, i.e., $\{x_1, x_2, x_3\} \subseteq D$, $\{x_{4k+2}, x_1, x_2\} \subseteq D$, and $\{x_{4k+1}, x_{4k+2}, x_1\} \subseteq D$. If $x_1 \notin V(\mathcal{H}_3)$, there are $2k-1$ slots in which $\mathcal{H}_3$ can be placed. So, $DV^1(x_1) = 3 + (2k-1) = 2k+2$.

Subcase 3.2. $2\mathcal{H}_2 \subseteq \langle D \rangle$: We denote by $DV^2(x_1)$ the number of such D's containing x_1. Since each vertex in $\mathcal{H}_2$ is doubly dominated, four vertices in $2\mathcal{H}_2$ are doubly dominated, and hence the placement of $2\mathcal{H}_2$ uniquely determines D. If $x_1 \in V(\mathcal{H}_2)$ (i.e., $\{x_1, x_2\} \subseteq D$ or $\{x_1, x_{4k+2}\} \subseteq D$), then there are $2k-1$ available slots to place the other $\mathcal{H}_2$. If $x_1 \notin V(\mathcal{H}_2)$, then there are $\binom{2k-1}{2}$ available slots to place $2\mathcal{H}_2$'s. Thus, $DV^2(x_1) = 2(2k-1) + \binom{2k-1}{2} = (2k-1)(k+1)$.

Subcase 3.3. $\mathcal{H}_2 \subseteq \langle D \rangle$ and $2\mathcal{H}_2 \not\subseteq \langle D \rangle$: We will show that no such D exists. Without loss of generality, suppose that $\{x_1, x_2\} \subseteq D$. In order for y_3 to be dominated, a vertex in $N[y_3] = \{x_3, y_2, y_3, y_4\}$ must be in D. By the hypothesis, $\{x_1, x_2, x_3\} \not\subseteq D$. First, suppose that $R_3 := \{x_1, x_2, y_2\} \subseteq D$. Then the part of $P_2 \Box C_{4k+2}$ not dominated by R_3, say F_3, must be dominated by $2k-1$ vertices in $D - R_3$. Since $|V(P_2 \Box C_{4k+2})| = 8k+4$ and $|N[R_3]| = 7$, $2k-1$ vertices in $D - R_3$ must dominate F_3 with $|V(F_3)| = 8k-3$. But each vertex in $P_2 \Box C_{4k+2}$ dominates four vertices, and we reach a contradiction. Second, suppose that $R_4 := \{x_1, x_2, y_3\} \subseteq D$. Then the part of $P_2 \Box C_{4k+2}$ not dominated by R_4, say F_4, is a graph isomorphic to H in Figure 6, and $2k-1$ vertices of $D - R_4$ must dominate $F_4 \cong H$, which is a contradiction by Case 2. Third, suppose that $R_5 := \{x_1, x_2, y_4\} \subseteq D$. Then the part of $P_2 \Box C_{4k+2}$ not dominated by R_5, say F_5, must be dominated by $2k-1$ vertices in $D - R_5$. Since $|V(P_2 \Box C_{4k+2})| = 8k+4$ and $|N[R_5]| = 10$, $2k-1$ vertices in $D - R_5$ must dominate F_5 with $|V(F_5)| = 8k-6$, and thus there exist two vertices in $N[F_5]$ that are doubly dominated. When $k = 1$, one can easily see that $y_5 \in D$ (i.e., $2\mathcal{H}_2 \subseteq \langle D \rangle$) or $x_6 \in D$ (i.e., $\mathcal{H}_3 \subseteq \langle D \rangle$); both cases contradict to the assumption. So we consider for $k \geq 2$. Without loss of generality, we may assume that at least one vertex in $N[y_4] \cap N[F_5] = \{x_4, y_5\}$ is doubly dominated. In order for x_4 to be doubly dominated, $x_5 \in D$. If $\{x_1, x_2, y_4, x_5\} \subseteq D$, then the part of $P_2 \Box C_{4k+2}$ not dominated by $\{x_1, x_2, y_4, x_5\}$ is the graph H' in Figure 7, and $2k-2$ vertices of $D - \{x_1, x_2, y_4, x_5\}$ must dominate H'. If we

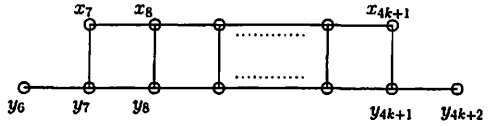

Figure 7: $H' \subset P_2 \Box C_{4k+2}$, where $k \geq 2$

let $S' = \{x_i, y_j \mid i \equiv 1, j \equiv 3 \pmod 4 \text{ and } 6 \leq i, j \leq 4k\}$, then $|S'| = 2k-3$, S' dominates $8k-12$ vertices, the part of H' not dominated by S' is a P_4, and one vertex of $D - (S' \cup \{x_1, x_2, y_4, x_5\})$ must dominate P_4. But $\gamma(P_4) = 2$ and we reach a contradiction. In order for y_5 to be doubly dominated, a vertex in $\{x_5, y_5, y_6\}$ must belong to D. Since $\{x_1, x_2, y_4, x_5\} \not\subseteq D$ and $\{x_1, x_2, y_4, y_5\} \not\subseteq D$, $y_6 \in D$. In this case, i.e.,

$\{x_1, x_2, y_4, y_6\} \subseteq D$, note that x_1, x_2, and y_5 are doubly dominated. In order for x_5 to be dominated, a vertex in $N[x_5] = \{x_4, x_5, x_6, y_5\}$ must be in D and each case results in at least two additional vertices to be doubly dominated, which is a contradiction. Thus, there is no $\gamma(P_2 \Box C_{4k+2})$-set containing exactly one $\mathcal{H}_2$.

Subcase 3.4. $\mathcal{H}_2 \not\subseteq \langle D \rangle$: We denote by $DV^3(x_1)$ the number of such D's containing x_1. First, suppose that $\{x_s, y_s\} \subseteq D$ for some s ($1 \leq s \leq 4k+2$). If $\{x_1, y_1\} \subseteq D$, then the part of $P_2 \Box C_{4k+2}$ not dominated by $\{x_1, y_1\}$ is $P_2 \Box P_{4k-1}$, and $2k$ vertices of $D - \{x_1, y_1\}$ must dominate $P_2 \Box P_{4k-1}$. By Theorem 3.4, there exist two such D's for $k \neq 1$ (i.e., $n \neq 6$) and there exist three such D's for $k = 1$ (i.e., $n = 6$). If $x_1 \in D$ and $\{y_1, y_2, y_{4k+2}\} \cap D = \emptyset$, then there are $2k$ available slots in which $\{x_s, y_s\} \subseteq D$ can be placed for some $s \neq 1$. Second, suppose that no two adjacent vertices belong to D. If we let $S_1 = \{x_i, y_j \mid i \equiv 1, j \equiv 3 \pmod 4 \text{ and } 1 \leq i, j \leq 4k\}$, then $|S_1| = 2k$ and the part of $P_2 \Box C_{4k+2}$ not dominated by S_1 is a P_4, so two vertices of $D - S_1$ must dominate P_4. Since no two adjacent vertices belong to D, if $S_1 \subseteq D$, then $\{x_{4k}, y_{4k+1}\} \subseteq D$ or $\{x_{4k}, y_{4k+2}\} \subseteq D$ or $\{x_{4k+1}, y_{4k+2}\} \subseteq D$, thus there are two pairs of vertices (not necessarily disjoint) in D that are at distance two apart. The number of ways of selecting 2 out of $2k+2$ available slots is $\binom{2k+2}{2} = (k+1)(2k+1)$. Thus, $DV^3(x_1) = 2 + 2k + (k+1)(2k+1) = (k+1)(2k+3)$ if $k \neq 1$, and $DV^3(x_1) = 11$ if $k = 1$.

Now, noting that $DV(x_1) = DV^1(x_1) + DV^2(x_1) + DV^3(x_1)$, we have $DV(x_1) = (2k+2)^2$ if $k \neq 1$, and $DV(x_1) = 17$ if $k = 1$.

Case 4. $n = 4k+3$, *where* $k \geq 0$: Here $\gamma(P_2 \Box C_{4k+3}) = 2k+2$. When $k = 0$, one can easily check that there are three γ-sets containing x_1, i.e., $\{x_1, y_1\}$, $\{x_1, y_2\}$, and $\{x_1, y_3\}$. So $DV(x_1) = 3$ for $x_1 \in V(P_2 \Box C_3)$. Next, we consider for $k \geq 1$. We will show that no D contains both x_1 and a vertex in $\{y_1, x_2, x_3\}$. First, note that no D contains both x_1 and y_1: If $\{x_1, y_1\} \subseteq D$, then the part of $P_2 \Box C_{4k+3}$ not dominated by $\{x_1, y_1\}$ is $P_2 \Box P_{4k}$, and $2k$ vertices of $D - \{x_1, y_1\}$ must dominate $P_2 \Box P_{4k}$. But $\gamma(P_2 \Box P_{4k}) = 2k+1$ by Theorem 3.1, and we reach a contradiction. Second, note that no D contains both x_1 and x_2: if $\{x_1, x_2\} \subseteq D$, then the part of $P_2 \Box C_{4k+3}$ not dominated by $\{x_1, x_2\}$, say H^*, must be dominated by $2k$ vertices. If we let $S^* = \{x_i, y_j \mid i \equiv 2, j \equiv 0 \pmod 4 \text{ and } 4 \leq i, j \leq 4k\}$, then $|S^*| = 2k-1$ and the part of $P_2 \Box C_{4k+3}$ not dominated by $S^* \cup \{x_1, x_2\}$ is a P_4, and one vertex of $D - (S^* \cup \{x_1, x_2\})$ must dominate P_4. But $\gamma(P_4) = 2$, and we reach a contradiction. (Similarly, no D contains both x_1 and x_{4k+3}.) Third, note that no D contains both x_1 and x_3: if $\{x_1, x_3\} \subseteq D$, then a vertex in $N[y_2] = \{x_2, y_1, y_2, y_3\}$ must belong to D. Since $\{x_1, y_1\} \not\subseteq D$, $\{x_3, y_3\} \not\subseteq D$, and $\{x_1, x_2\} \not\subseteq D$, we need to consider $\{x_1, x_3, y_2\} \subseteq D$: since $|V(P_2 \Box C_{4k+3})| = 8k+6$ and $|N[\{x_1, y_2, x_3\}]| = 8$, $2k-1$ vertices of $D - \{x_1, x_3, y_2\}$ must dominate $8k-2$ vertices, which is impossible since each vertex in $P_2 \Box C_{4k+3}$ dominates four vertices. (Similarly, $\{x_1, x_{4k+2}\} \not\subseteq D$.) So, we only need to consider D such that (i) $\{x_1, y_2\} \subseteq D$ (resp. $\{x_1, y_{4k+3}\} \subseteq D$) or (ii) no vertex in $N[x_1]$ is doubly dominated. So suppose that $\{x_1, y_2\} \subseteq D$. Then the part of $P_2 \Box C_{4k+3}$ that are not dominated by $\{x_1, y_2\}$, say H'', must be dominated by $2k$ vertices. Since $|V(P_2 \Box C_{4k+3})| = 8k+6$ and $|N[\{x_1, y_2\}]| = 6$, $2k$ vertices of $D - \{x_1, y_2\}$ must dominate H'' with $|V(H'')| = 8k$, and thus there exists at most one such D. Since $\{x_1, y_2\} \cup \{x_i, y_j \mid i \equiv 0, j \equiv 2 \pmod 4 \text{ and } 3 \leq i, j \leq 4k+3\}$ is a γ-set, if $\{x_1, y_2\} \subseteq D$, then there exists a unique such D. Similarly, there exists a unique D containing both x_1 and y_{4k+3}. If no vertex in $N[x_1]$ is doubly dominated (i.e, $\{x_1, y_3, y_{4k+2}\} \subseteq D$), then there are $2k$ slots in which a pair of vertices of D at distance two apart can be placed. Thus, $DV(x_1) = 2+2k$ if $k \geq 1$, and $DV(x_1) = 3$ if $k = 0$. □

As an immediate consequence of Theorem 4.3, Observation 2.1, Observation 2.2, and the vertex-transitivity of $P_2 \Box C_n$, we have the following.

Corollary 4.4. *For $n \geq 3$,*

$$\tau(P_2 \Box C_n) \begin{cases} 4 & \text{if } n \equiv 0 \pmod 4 \\ 2n & \text{if } n \equiv 1, 3 \pmod 4 \text{ and } n \neq 3 \\ n(n+2) & \text{if } n \equiv 2 \pmod 4 \text{ and } n \neq 6 \\ 9 & \text{if } n = 3 \\ 51 & \text{if } n = 6\,. \end{cases}$$

5 Open Problems

We end this paper with some open problems. One could ask the following questions.

1. In our terminology, Mynhardt [16] characterized vertices v in a tree T such that $DV(v) = \tau(T)$ or $DV(v) = 0$. Can we describe vertices satisfying $DV(v) = k$ for $k \neq 0, \tau(T)$?

2. For $e \in E(G)$, can we find the bounds of $\tau(G-e)$ in terms of $\tau(G)$? And, for $v \in V(G-e)$, how does $DV_{G-e}(v)$ change in terms of $DV_G(v)$?

3. For $w \in V(G)$, can we find the bounds of $\tau(G-w)$ in terms of $\tau(G)$? And, for $v \in V(G-w)$, how does $DV_{G-w}(v)$ change in terms of $DV_G(v)$?

4. For a given graph G, can we characterize subgraphs $H \subseteq G$ satisfying $DV_H(v) = DV_G(v)$ for each vertex $v \in V(H)$?

In parallel with the idea of $\tau(G)$, the anonymous referee suggested the following questions.

5. Can we compute the *number* of *ir*-sets (maximal irredundant sets of minimum cardinality), γ-sets (minimum dominating sets), γ_t-sets (minimum total dominating sets), i-sets (minimum independent dominating sets), β_0-sets (maximum independent sets), Γ-sets (minimal dominating sets of maximum cardinality), IR-sets (maximum irredundant sets) in a graph G?

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