Existence of Balanced Arrays and the Positive Semi-definiteness of the Moment Matrix

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Dedicated to Professor Roger B. Eggleton, for his contributions to combinatorics.

Abstract

In this paper, we obtain a new set of conditions which are necessary for the existence of balanced arrays of strength eight with two levels by making use of the positive semi-definiteness of the matrix of moments. We also demonstrate, using illustrative examples, that the maximum number of constraints derived using these results are better than those obtained earlier.

1 Introduction and Preliminaries

For the sake of completeness, we first present some basic concepts and definitions concerning balanced arrays (B-arrays). If $\underline{\alpha}$ is any column vector of an m-rowed matrix T, then the symbols $\lambda(\underline{\alpha})$, $P(\underline{\alpha})$, and $w(\underline{\alpha})$ denote, respectively, the number of times $\underline{\alpha}$ occurs in T, the column vector obtained by permuting the elements of $\underline{\alpha}$, and the weight of the column vector $\underline{\alpha}$ (the weight of $\underline{\alpha}$ is the number of non-zero elements in it). It is quite obvious that $w[P(\underline{\alpha})] = w(\underline{\alpha})$. In this paper, we confine ourselves to bi-level arrays.

Definition 1. A *balanced array* (B-array) T with m rows (factors, constraints), N columns (runs, treatment-combinations), two symbols (say, 0 and 1), and of strength t is merely a matrix T of size $(m \times N)$ with two elements 0 and 1 such that in every $(t \times N, t \leq m)$ submatrix T^* (clearly, there are $\binom{m}{t}$ such sub-matrices) of T, the following condition holds: $w(\underline{\alpha}) = w[P(\underline{\alpha})] = \mu_i$ (say), where $\underline{\alpha}$ is any $(t \times 1)$ vector of T^* with i 1s $(0 \leq i \leq t)$ in it.

Remarks: Clearly, the above definition can be extended to B-arrays with s symbols. The vector $\underline{\mu}' = (\mu_0, \mu_1, \ldots, \mu_t)$ is called the *index set* of T, and clearly $N = \sum_{i=0}^{t} \binom{t}{i} \mu_i$. In this paper, we will restrict ourselves to arrays with $t = 8$.

Definition 2. If $\mu_i = \mu$ for each i, then the B-array is reduced to an *orthogonal array* (O-array) with index set μ. In this case, $N = \mu \cdot 2^t$.

In this paper, we will restrict ourselves to $t = 8$.

B-arrays tend to unify numerous combinatorial structures such as balanced incomplete block (BIB) designs, group divisible designs, nested BIB designs, rectangular designs, etc. B-arrays have been extensively used in the construction of optimal balanced fractional factorial designs. Different values of t give rise to factorial designs of different resolutions. For example, B-arrays with $t = 8$ give us, under certain conditions, balanced fractional factorial designs of resolution IX which allow us to estimate all the effects up to and including four-factor interactions under the assumption that higher-order interactions are negligible. It is well-known that O-arrays, a special case of B-arrays, have been extensively used in information and coding theory, in medicine, in quality control in industry, etc. To gain further insight into the importance of these arrays in statistical design of experiments and in combinatorics, the interested reader may consult the list of references (by no means an exhaustive one) at the end of this paper, and also further references cited therein.

Thus, the existence and construction of these arrays are quite important from the point of view of applications in real-life situations as well as to the study of combinatorial structures. To construct a B-array for an arbitrary given set of parameters m and $\underline{\mu}'$ is a very complex and difficult problem. In particular, we confine ourselves here to the problem of obtaining the maximum number of constraints m for a given $\underline{\mu}'$ with $t = 8$, which is a non-trivial problem. Such problems for O-arrays and B-arrays have been addressed, among others, by Bose and Bush [1], Chopra et. al [4, 5, 6, 7, 8], Hedayat et. al [9], Rafter and Seiden [13], Rao [14], Saha et. al [15], Seiden and Zemach [16], Yamamoto et. al [18], etc.

First of all, we obtain some inequalities involving m and $\underline{\mu}'$ for $t = 8$. Given $\underline{\mu}'$ (ie. given N), these inequalities only involve m (the unknown parameter). Clearly, each of these inequalities must be satisfied for $m = 8$, which serves as a check on the correctness of these inequalities. The challenging situation arises when $m \geq 9$. If any of these inequalities is contradicted for a certain value of m (say, $m = k + 1$ where $k \geq 8$), then the maximum value of m is k. These inequalities are necessary conditions, but not sufficient, for the existence of B-arrays. We make use of these inequalities to obtain the maximum value of m for a given $\underline{\mu}'$.

2 New Necessary Conditions for B-arrays with $t = 8$

The following results can be easily established.

Lemma 1. *A B-array T with $t = 8$, $m = 8$, and arbitrary $\underline{\mu}'$, always exists.*

Lemma 2. *A B-array T with $t = 8$ is also of lower strength k, where* $0 \leq k \leq 8$.

Note. It is not difficult to see that, when considered as an array of strength k, the elements of its index set are linear combinations of the elements $\mu_0, \mu_1, \mu_2, \ldots, \mu_8$. Let $A(j,k)$ be the jth element $(0 \leq j \leq k)$ of the parameter vector of T when it is considered as an array of strength k, where $A(j,k)$ in terms of μ_i is given by

$$A(j,k) = \sum_{i=0}^{8-k} \binom{8-k}{i} \mu_{i+j}, \quad \text{where } j = 0, 1, 2, \ldots, k, \ (k \leq 8). \qquad (2.1)$$

From (2.1), it is obvious that $A(j,8) = \mu_j$ $(0 \leq j \leq 8)$, $A(j,0) = N = A(0,0)$, and $A(8,8) = \mu_8$.

The next lemma expresses the moments of the weights of the columns of T as a polynomial function in terms of its parameters m, and in terms of the elements of the index set $\underline{\mu}'$.

Lemma 3. *Let T be a B-array of strength eight with parameters m and $\underline{\mu}'$. Let x_j $(0 \leq j \leq m)$ be the number of columns of weight j in T. Then, the following must hold:*

$$L_0 = \sum_{j=0}^{m} x_j = N,$$

$$L_k = \sum_{j=0}^{m} j^k x_j = \sum_{r=1}^{k} a_r m_r A(r,r), \quad (1 \leq k \leq 8), \tag{2.2}$$

where $m_r = m(m-1)(m-2)\cdots(m-r+1)$ and a_r are known which appear while deriving $\sum j^k x_j$ with $1 \leq k \leq 8$.

Clearly, L_k is the moment of order k around zero for the columns of weight k.

Remark. To facilitate the computations, we provide the values of a_r $(1 \leq r \leq 8)$; the elements of the vector $(a_1, a_2, \ldots, a_8)$ are respectively: (1), $(1,1)$, $(1,3,1)$, $(1,7,6,1)$, $(1,15,25,10,1)$, $(1,31,90,65,15,1)$, $(1,63,301,350,140,21,1)$, and $(1,287,966,1701,1050,266,28,1)$.

Next, we consider the following matrix for $t = 2\mu$ (even, μ being a positive integer):

$$M_{2\mu} = \begin{pmatrix} L_0 = N & L_1 & L_2 & \cdots & L_\mu \\ L_1 & L_2 & L_3 & \cdots & L_{\mu+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ L_\mu & L_{\mu+1} & L_{\mu+2} & \cdots & L_{2\mu} \end{pmatrix}, \tag{2.3}$$

It is a symmetric matrix and is non-negative definite (n.n.d), which can be seen by observing the non-negative definiteness of the quadratic form $\sum(\alpha_0 + \alpha_1 j + \alpha_2 j^2 + \cdots + \alpha_\mu j^\mu)^2 x_j$ in variables $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_\mu$. We now state the main results.

Theorem 1. *Let x_j $(0 \leq j \leq m)$ be the number of columns of weight j in T, where T is a B-array of strength eight with m constraints and having index set $\underline{\mu}' = (\mu_0, \mu_1, \ldots, \mu_8)$. For T to exist, the following inequalities* *must be satisfied:*

$$L_0L_2 \geq L_1^2, \tag{2.4}$$

$$L_0L_2L_4 + 2L_1L_2L_3 \geq L_0L_3^2 + L_1^2L_4 + L_2^3, \tag{2.5}$$

$$\begin{aligned} L_0L_2L_4L_6 &+ 2L_0L_3L_4L_5 + L_1^2L_5^2 + 2L_1L_2L_3L_6 + 2L_1L_3L_4^2 \\ &+2L_2^2L_3L_5 + L_2^2L_4^2 + L_3^4 \geq \\ &L_0L_2L_5^2 + L_0L_3^2L_6 + L_0L_4^3 + L_1^2L_4L_6 + 2L_1L_3^2L_5 \\ &+2L_1L_2L_4L_5 + L_2^3L_6 + 3L_2L_3^2L_4. \end{aligned} \tag{2.6}$$

Proof. (Outline). For $t = 8$, we select $\mu = 4$ in (2.3) and we obtain the following (5×5) matrix:

$$M_8 = \begin{pmatrix} L_0 & L_1 & L_2 & L_3 & L_4 \\ L_1 & L_2 & L_3 & L_4 & L_5 \\ L_2 & L_3 & L_4 & L_5 & L_6 \\ L_3 & L_4 & L_5 & L_6 & L_7 \\ L_4 & L_5 & L_6 & L_7 & L_8 \end{pmatrix}.$$

Since M_8 is n.n.d., all of its leading principal minors have determinant greater than or equal to 0. Now, consider the following leading principal minors:

$$\begin{pmatrix} L_0 & L_1 \\ L_1 & L_2 \end{pmatrix}, \quad \begin{pmatrix} L_0 & L_1 & L_2 \\ L_1 & L_2 & L_3 \\ L_2 & L_3 & L_4 \end{pmatrix}, \quad \begin{pmatrix} L_0 & L_1 & L_2 & L_3 \\ L_1 & L_2 & L_3 & L_4 \\ L_2 & L_3 & L_4 & L_5 \\ L_3 & L_4 & L_5 & L_6 \end{pmatrix}.$$

We obtain (2.4), (2.5), and (2.6), respectively, by expanding the determinant of each of the leading principal minors mentioned above. □

Theorem 2. *Let x_j $(0 \leq j \leq m)$ be the number of columns of weight j in T, where T is a B-array of strength eight with m constraints and having index set $\underline{\mu}' = (\mu_0, \mu_1, \ldots, \mu_8)$. For T to exist, the following inequality must* *be satisfied:*

$$\sum_{i=0}^{4} L_i A_i \geq \sum_{i=0}^{4} L_i B_i, \quad \text{where} \tag{2.7}$$

$$\begin{aligned}
A_0 &= L_2L_6(L_4L_8 + 2L_5L_7) + L_3(L_3L_7^2 + 2L_4L_5L_8 + 2L_5L_6^2) \\
&\quad + L_4^2(2L_5L_7 + L_6^2) + L_5^4, \\
B_0 &= L_2(L_4L_7^2 + L_5^2L_8 + L_6^3) + L_3(L_3L_6L_8 + 2L_4L_6L_7 + 2L_5^2L_7) \\
&\quad + L_4(L_4^2L_8 + 3L_5^2L_6), \\
A_1 &= L_1(L_4L_7^2 + L_5^2L_8 + L_6^3) + 2L_2(L_3L_6L_8 + L_4L_6L_7 + L_5^2L_7) \\
&\quad + 2L_3(L_3L_6L_7 + L_4^2L_8 + L_4L_5L_7 + L_5^2L_6) + 4L_4^2L_5L_6, \\
B_1 &= L_1L_6(L_4L_8 + 2L_5L_7) + 2L_2(L_3L_7^2 + L_4L_5L_8 + L_5L_6^2) \\
&\quad + 2L_3(L_3L_5L_8 + L_4L_5L_7 + 2L_4L_6^2) + 2L_4(L_4^2L_7 + L_5^3), \\
A_2 &= L_2(L_2L_7^2 + 2L_3L_5L_8 + L_4^2L_8) + L_2L_6(2L_4L_6 + L_5^2) \\
&\quad + L_3^2(2L_5L_7 + L_6^2) + L_4^2(4L_3L_7 + 3L_5^2) + 2L_3L_4L_5L_6, \\
B_2 &= L_2L_6(L_2L_8 + 2L_3L_7) + L_4(4L_2L_5L_7 + 3L_3^2L_8 + 4L_3L_5L_6) \\
&\quad + 2L_3L_5^3 + 3L_4^3L_6, \\
A_3 &= L_3(L_3^2L_8 + 3L_4^2L_6 + 3L_4L_5^2), \\
B_3 &= 2L_3^2(L_4L_7 + L_5L_6) + 4L_4^3L_5, \\
A_4 &= L_4^4, \quad \text{and} \quad B_4 = 0.
\end{aligned}$$

Proof. (Outline). As in the proof of Theorem 1, we now consider M_8, which is a principal minor of M_8. Since M_8 is n.n.d., its determinant is greater than or equal to 0 (in particular). A direct calculation of the determinant of M_8 yields (2.7). □

Remarks. It is not difficult to see that each of the L_i $(0 \leq i \leq 8)$ is a polynomial function in m and $\underline{\mu}'$. For a given $\underline{\mu}'$, it is merely a polynomial function of the number of constraints m. Thus, (2.4) to (2.7) are inequalities in m, for given $\underline{\mu}'$. Each of these inequalities must be satisfied (for a given $\underline{\mu}'$) for B-arrays with $t = m = 8$ (which acts as a sort of check on the correctness of these inequalities). For arrays with $t = 8$ and $m \geq 9$, if any one inequality is contradicted, then this would imply that the B-array for that value of m (say, m^*) does not exist, and does not exist for any $m > m^*$ also.

3 Further Discussions

In order to check the existence of any B-array with $t = 8$ for a given m and $\underline{\mu}'$, a computer program was prepared involving the results of (2.4) to (2.7). If any of these inequalities is contradicted for a given m and $\underline{\mu}'$, then the B-array does not exist for that m and $\underline{\mu}'$. In order to obtain the max(m) for a given $\underline{\mu}'$, we use (2.4) to (2.7), which are merely inequalities in m. Starting with $m = 9$, we test each value of m, and if a contradiction occurs for some $m = k + 1$ (say), then the maximum constraints for such an array is k. The problem of determining the existence of a B-array with a given number of runs N is a very difficult and complex problem, since a given N usually corresponds to a very large number of $\underline{\mu}'$ values. Next, we present some illustrative examples for arrays with $t = 8$ and compare the present results with earlier ones found within the mathematical literature.

Example 1. Here, we compare the current results with the ones found in Dios/Chopra [8], and in Chopra/Bsharat [4]. For the arrays with $\underline{\mu}' = (1,3,6,4,1,7,5,1,2)$, $(1,3,2,2,1,5,5,2,2)$ and $(1,4,3,3,2,8,4,1,1)$ found in Dios/Chopra [8], we have $m \leq 12$, $m \leq 10$, and $m \leq 9$, respectively. However, the above inequalities are improved upon to $m \leq 9$, $m \leq 9$, and $m \leq 8$ respectively, which are obtained by using (2.7). Here, $m \leq 8$ is the optimal inequality and cannot be further improved upon. The values of $\underline{\mu}'$ considered in Chopra/Bsharat [4] are $(1,1,1,1,1,6,1,1,1)$, $(1,1,3,1,1,6,6,2,2)$, $(1,1,1,1,1,6,3,1,1)$, and $(1,1,1,1,1,4,6,3,1)$ which yielded $m \leq 10$, $m \leq 12$, $m \leq 11$, and $m \leq 11$, respectively. However, these inequalities are improved upon (by the use of (2.7)) to $m \leq 8$ (optimal value for m is 8), $m \leq 9$ (a considerable improvement over $m \leq 12$), $m \leq 8$, and $m \leq 10$, respectively.

Example 2. Next, we compare our current results with the ones found in Chopra/Low/Dios [7]. For the arrays with $\underline{\mu}' = (8,8,8,8,6,8,8,8,8)$, $(4,4,4,4,4,4,4,4,3)$, $(7,8,8,8,8,8,8,8,8)$ and $(6,5,6,6,6,5,5,5,5)$, we get $m \leq 83$, $m \leq 50$, no max(m) available, and no max(m) available, respectively. However, by using (2.4) to (2.7), the above inequalities are improved upon to $m \leq 10$, $m \leq 12$, $m \leq 15$, and $m \leq 21$, respectively. Thus, we find considerable improvement in our current results over those previously found.

Note. Even though the present results produce considerable improvements (for the arrays in examples above) over the ones given earlier, we do not claim that these are uniformly better for every array. The problem of finding a set of conditions producing the "best" and optimal inequalities satisfied by the number of constraints m for each B-array with index set

μ' is a very difficult and complex problem. Such conditions are not even available for O-arrays (a subset of B-arrays).

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