On k-Edge-Magic Cubic Graphs

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Abstract

Let G be a (p, q)-graph in which the edges are labeled $k, k+1, \ldots, k+q-1$, where $k \geq 0$. The vertex sum for a vertex v is the sum of the labels of the incident edges at v. If the vertex sums are constant, modulo p, then G is said to be k-edge-magic. In this paper we investigate some classes of cubic graphs which are k-edge-magic. We also provide a counterexample to a conjecture that any cubic graph of order $p \equiv 2 \pmod 4$ is k-edge-magic for all k.

1 Introduction

All graphs in this paper are simple graphs with no loops or multiple edges.

Stewart in [23, 24] defined that a graph is supermagic if the edges are labeled $1, 2, 3, \ldots, q$ so that the vertex sums are constant. He showed that K_3, K_4, K_5 are not supermagic and when $n \equiv 0 \pmod 4$, K_n is not supermagic. For $n > 5$, K_n is supermagic if and only if $n \equiv 0 \pmod 4$. For a generalization of this result, see [11].

Hartsfield and Ringel in [6] provided some classes of supermagic graphs. Ho and Lee in [7] extended the result of Stewart to regular complete k-partite graphs. Recently, Shiu, Lam and Cheng in [18] considered a class of

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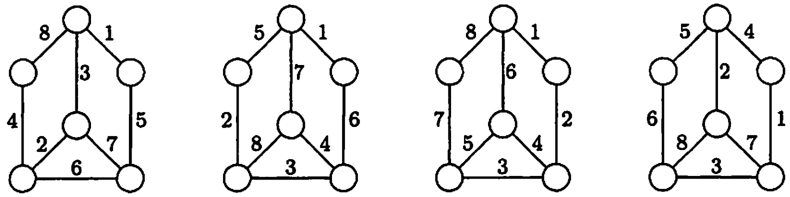
supermagic graphs which are disjoint union of $K_{3,3}$. A general construction of supermagic graphs is considered in [20].

Definition 1. Let G be a (p, q)-graph in which the edges are labeled $k, k+1, \ldots, k+q-1$, where $k \geq 0$. The vertex sum S for a vertex v is the sum of the labels of the incident edges at v. If the vertex sums are constant, modulo p, then G is said to be ***k-edge-magic*** (in short ***k-EM***).

From the definition, it is obvious that if G is supermagic, then G is 1-edge-magic.

Example 1. Figure 1 shows a graph G with 6 vertices and 8 edges which is 1-edge-magic with different constant sums.



$S \equiv 0 \pmod 6$ $S \equiv 1 \pmod 6$ $S \equiv 3 \pmod 6$ $S \equiv 5 \pmod 6$

Figure 1: The $(6, 8)$-graph which is 1-EM

Example 2. The following maximal outerplanar graphs with 6 vertices are 1-edge-magic.

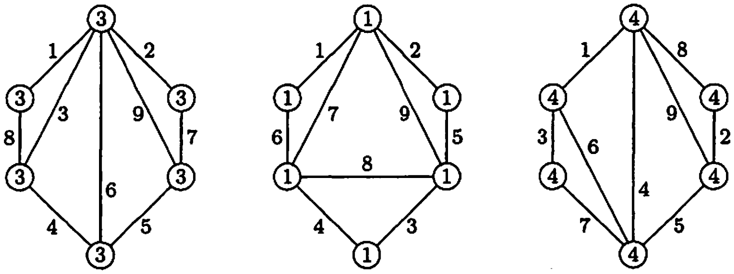


Figure 2: The 1-EM maximal outerplanar graphs with 6 vertices

If $k = 1$, then G is said to be ***edge-magic***. The concept of edge-magic graphs was introduced by Lee, Seah and Tan in [11]. A necessary condition for a (p, q)-graph to be edge-magic is $q(q+1) \equiv 0 \pmod p$. However, this condition is not sufficient. There are infinitely many connected graphs such as trees and cycles satisfying this condition that are not edge-magic.

The Cartesian product of two paths is frequently called the ***grid*** graph. The Cartesian product of two cycles is called the torus graph. It was shown in [16] that the torus graph $C_m \times C_n$ is edge-magic for all $m, n > 2$.

Lee, Pigg and Cox in [10] showed that $C_n \times K_2$ is edge-magic if and only if n is odd and greater or equal to 3.

Schaffer and Lee in [16] have shown that if G and H are both odd-order, regular, edge-graceful graphs, where G is d-regular with m vertices, H is k-regular with n vertices, and $\gcd(d, n) = \gcd(k, m) = 1$, then $G \times H$ is edge-graceful. In particular, they showed that the torus graph $C_{2i+1} \times C_{2j+1}$ is edge-graceful.

In [9], Kwong and Lee investigated fans, ladder graphs and pagoda graphs which are all k-EM. They showed that ladder graphs $P_n \times P_2$ are not k-EM for any k when n is 3, 4 and 5. They conjecture that it is true for all $n \geq 6$. In [14], Lee, Sun and Wen investigated some k-EM complete bipartite graphs.

Lee, Pigg and Cox in [10] conjectured that every connected simple cubic graph G with $p \equiv 2 \pmod 4$ is edge-magic. They also showed that the conjecture is true for prisms and other cubic graphs.

It is natural to extend the conjecture to k-edge-magic as follow:

Conjecture 1.1. *Any cubic graph of order* $p \equiv 2 \pmod 4$ *is* k*-edge-magic for all* k.

In this paper, we investigate several classes of cubic graphs which are k-edge-magic and show their peculiar behavior. We also provide a counterexample for the conjecture 1.1. For more conjectures and open problems on edge-magic graphs the reader is referred to [10, 11, 12, 16]. The reader should see the survey article of Gallian [4] for various labeling problems.

2 General Theory of k-edge-magic Graphs

We list a couple theorems for k-edge-magic graphs here. Even the proof can easily be found, we write it down for completeness.

Proposition 2.1. *A necessary condition for a* (p, q)*-graph to be* k*-edge-magic is* $q(q + 2k - 1) \equiv 0 \pmod p$.

Proof. The sum of the labels of all edges is

$$q\frac{k + (k + q - 1)}{2}.$$

Since every edge is counted twice in the vertex sums, the result follows. □

Proposition 2.2. *If a (p, q)-graph G is k-edge-magic then it is $(pt + k)$-edge-magic for all $t \geq 0$.*

Proof. The vertex sum is calculated in modulo p. □

Example 3. Figure 3 demonstrates, for $k = 0, 1, 2, 3, 4, 5$, the k-edge-magic labelings in the complete bipartite graph $K_{3,3}$. By Lemma 2.2, it is k-edge-magic for all k.

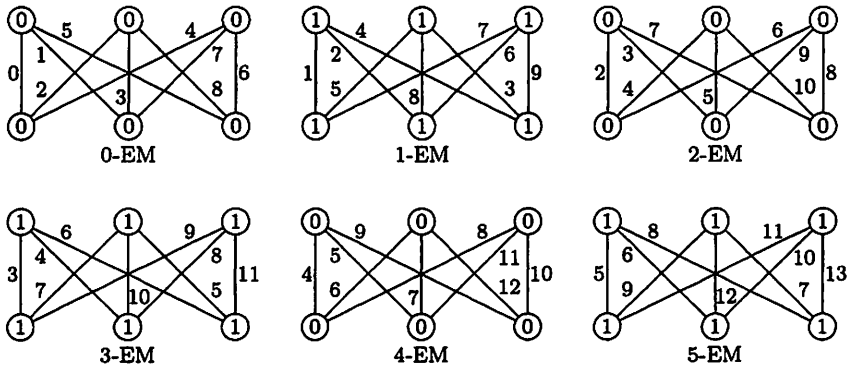


Figure 3: Complete bipartite graph $K(3, 3)$ is k-EM for $k = 0, 1, \ldots, 5$

An r-regular graph with p vertices has $q = \frac{rp}{2}$ edges. When r is odd, since q is an integer, we know that p must be even. By the above two propositions, we can quickly rule out some r-regular graphs of a certain order to be k-edge-magic.

Theorem 2.3. *When r is odd, an r-regular graph with order $p \equiv 0 \pmod 4$ is not k-edge-magic for all k.*

Proof. Since $p \equiv 0 \pmod 4$, let $p = 4s$ for some positive integer s. Thus, the number of edges is $q = \frac{rp}{2} = 2rs$. Assume that this graph is k-edge-magic for some k. The necessary condition from Proposition 2.1 requires

$$(2rs)(2rs + 2k - 1) \equiv 0 \pmod{4s}.$$

This implies $2s = 0 \pmod{4s}$ when r is odd, which is impossible. This completes the proof. □

Therefore, in this article, we only consider the cubic graphs with order $p \equiv 2 \pmod 4$.

The following result was first proved in [8].

Lemma 2.4. *For any r-regular graph, if it is k-edge-magic for some integer k, then it is k-edge-magic for all integer k.*

Proof. If an r-regular graph admits a k-edge-magic labeling, then, by adding one to every edge label, we create a $(k+1)$-edge-magic labeling since each vertex is of the same order r. Similarly, subtracting one from every edge label creates a $(k-1)$-edge-magic labeling. □

Since a cubic graph has the one-implies-all property as Lemma 2.4, due to the nature of modulo p operation, we can focus on finding a 1-edge-magic labeling.

Theorem 2.5. *If a cubic graph of order p, G, is k-edge-magic, and p and 3 are coprime, then for any integer S, G is h-edge-magic with sum S, where h is an integer.*

Proof. Since G is cubic and k-edge-magic with sum s modulo p, if we add 1 or subtract 1 to all edge labels, then, obviously, G becomes $(k+1)$-edge-magic with sum $(S+3)$ modulo p or $(k-1)$-edge-magic with sum $(S-3)$ modulo p. Since $\gcd(p,3) = 1$, if we continue this process, we can reach any integer S as the vertex sum modulo p. □

3 A Couple Special k-edge-magic Labelings

In this section, we construct a special k-edge-magic labeling of cubic graphs.

Let G be a cubic graph of order p. If G is k-edge-magic, then according to Theorem 2.3 holds that $p \equiv 2 \pmod 4$. Assume that $p = 2n$, where $n \equiv 1 \pmod 2$. Then $q = 3n$.

To find a 1-edge-magic labeling for G, we need to label all $3n$ edges by the number $\{1, 2, 3, \ldots, 3n\}$. Here we divide these numbers into three groups: $\{1, 2, 3, \ldots, n\}$, $\{n+1, n+2, n+3, \ldots, 2n\}$ and $\{2n+1, 2n+2, 2n+3, \ldots, 3n\}$, namely, group I, II and III, respectively. Since we do all the operation under modulo $2n$, the first and third groups are identical.

3.1 Möbius Ladders

The concept of Möbius ladder was introduced by Guy and Harary in [5]. It is a cubic circulant graph with an even number n of vertices, formed from an n-cycle by adding edges (called "rungs") connecting opposite pairs of vertices in the cycle. For the Möbius ladder, namely ML$(2n)$, let the vertices be denoted, in order, by $\{a_1, a_2, \ldots, a_{2n}\}$ and the edges are then $(a_1, a_2), (a_2, a_3), \ldots, (a_{2n}, a_1)$ and $(a_1, a_{n+1}), (a_1, a_{n+1}), \ldots, (a_n, a_{2n})$.

In [17], Sedláček proved that the Möbius ladder ML($2n$) is supermagic if and only if n is odd. This implies that

Theorem 3.1. *The Möbius ladder ML($2n$) is k-edge-magic for all k if and only if n is odd.*

But, we give a new proof here by using the labeling introduced in the beginning of the section to demonstrate how to use this special labeling.

Proof. For convenience, we modulo $2n$ for the subscript of each vertex a_i so that $a_{2n} = a_0$, $a_{2n+1} = a_1$, and so on.

Now, we label the edge (a_{2i-1}, a_{2i}) by i for all $i = 1, 2, \ldots, n$. This occupies all the numbers from group I. We also label the edge (a_{n+2i-1}, a_{n+2i}) by $2n + i$ for all $i = 1, 2, \ldots, n$. Again, it uses all the number from group III. Finally, we label the edge (a_i, a_{n+i}) by $2n-i+1$ for all $i = 1, 2, 3, \ldots, n$. It is easy to see that the numbers of group II are used here.

Because the subscripts are modulo $2n$, before we calculate the vertex sum, we need a couple observations here. First, when $i = n+1, n+2, n+3, \ldots, 2n$, the edge (a_i, a_{n+i}) is the same edge as (a_{i-n}, a_i). Thus, the label of the edge (a_i, a_{n+i}) for $i = n+1, n+2, n+3, \ldots, 2n$ is $2n - (i - n) + 1$. Second, since n must be odd, the two vertex subscripts of the edge (a_{n+2i-1}, a_{n+2i}) for $i = 1, 2, \ldots, n$ are even and odd, respectively. Moreover, for $\frac{n+1}{2} \leq i \leq n$, the edge (a_{n+2i-1}, a_{n+2i}) is the same as (a_{2i-n-1}, a_{2i-n}) with the label $2n + i$.

We can see the sum of the vertex a_{2t} is the sum of three adjacent edges (a_{2t-1}, a_{2t}), (a_{2t}, a_{2t+1}), and (a_{2t}, a_{n+2t}). Note the second edge has the subscripts in (*even*, *odd*) format. Thus, as discussed in the above paragraph, the label is

$$2n + \frac{2t+n+1}{2} \text{ if } 1 \leq t \leq \frac{n}{2}$$

or

$$2n + \frac{2t-n+1}{2} \text{ if } \frac{n}{2} \leq t \leq n.$$

So, the vertex sum is

$$(t) + \left(2n + \frac{2t+n+1}{2}\right) + (2n - 2t + 1) \text{ if } 1 \leq t \leq \frac{n}{2}$$

or

$$(t) + \left(2n + \frac{2t-n+1}{2}\right) + (2n - (2t-n) + 1) \text{ if } \frac{n}{2} \leq t \leq n.$$

Both sums can be simplified into $4n + \frac{n+1}{2} + 2$, a constant.

Similarly, by a similar argument, the sum of the vertex a_{2t+1} is also $4n + \frac{n+1}{2} + 2$, the same constant. This provides ML($2n$) a 1-edge-magic labeling. By Lemma 2.4, ML($2n$) is k-edge-magic for all k if n is odd. □

Example 4. Figure 4 gives edge-magic labelings for ML(6) and ML(10).

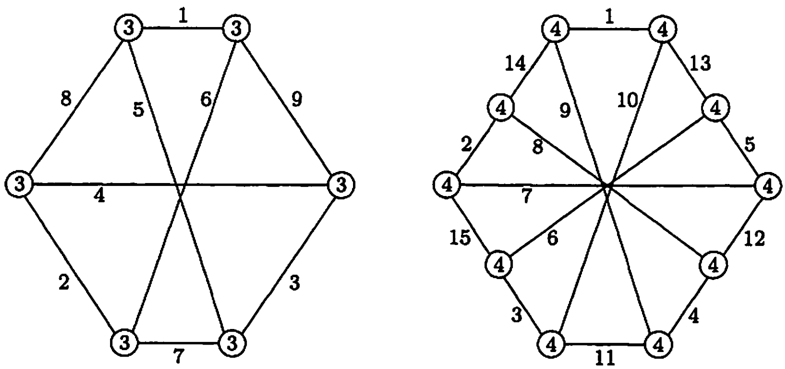


Figure 4: Edge-magic labelings for ML(6) and ML(10)

3.2 Cylinder Graphs

In [10], Lee, Pigg and Cox proved that the cylinder graph $C_n \times P_2$ is edge-magic. Here we use the idea of the previous labeling to re-prove it again.

Theorem 3.2. *The cylinder graph $C_n \times P_2$ is a k-edge-magic graph for all k if and only if n is odd.*

Proof. First we name the vertices of the inner cycle $a_1, a_2, \ldots, a_n$ and the vertices of the outer cycle $b_1, b_2, \ldots, b_n$. The edges are (a_i, a_{i+1}), (b_i, b_{i+1}) and (a_i, b_i) for all $i = 1, 2, \ldots, n$. Note here that, for convenience, we modulo n for the subscript of each vertex a_i and b_i so that $a_n = a_0$, $a_{n+1} = a_1$, $b_n = b_0$, $b_{n+1} = b_1$, and so on.

Now, we label the edges (a_{2i-1}, a_{2i}) and (b_{2i-1}, b_{2i}) by i modulo $2n$ for all $i = 1, 2, \ldots, n$. This occupies all numbers in both groups I and III. We also label the edge (a_i, b_i) by $2n - i + 1$ for all $i = 1, 2, 3, \ldots, n$, which uses all numbers in group II.

Note that, when $\frac{n}{2} \le i \le n$, the edge (a_{2i-1}, a_{2i}) is the same as (a_{2i-n-1}, a_{2i-n}). Also, if $1 \le i \le \frac{n}{2}$, the vertex subscripts of (a_{2i-1}, a_{2i}) is in the form of (*odd*, *even*) and $\frac{n}{2} \le i \le n$, the vertex subscripts are of the form (*even*, *odd*).

We can see the sum of the vertex a_{2t} is the sum of three adjacent edges (a_{2t-1}, a_{2t}), (a_{2t}, a_{2t+1}), and (a_{2t}, b_{2t}), which is

$$(t) + \left(\frac{2t+n+1}{2}\right) + (2n - 2t + 1) = 2n + \frac{n+1}{2} + 1,$$

a constant.

Similarly, by a similar argument, the sum of the vertex a_{2t+1} is also $2n + \frac{n+1}{2} + 1$, the same constant.

Since the edge (b_i, b_{i+1}) is labeled the same as the edge (a_i, a_{i+1}) modulo $2n$, the sum of the vertex b_i is the same as the vertex a_i. This provides $C_n \times P_2$ a 1-edge-magic labeling. By Lemma 2.4, $C_n \times P_2$ is k-edge-magic for all k if n is odd. □

Example 5. Figure 5 gives edge-magic labelings for $C_3 \times P_2$ and $C_5 \times P_2$.

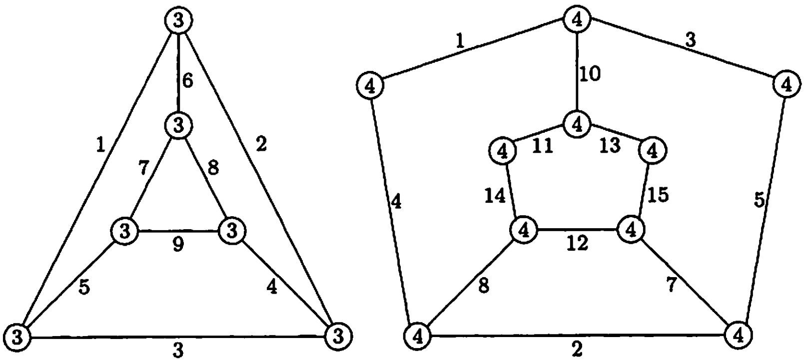


Figure 5: Edge-magic labelings for $C_3 \times P_2$ and $C_5 \times P_2$

The special labeling we used for $C_{2t+1} \times P_2$ consists of two sub-cycles of length $2t+1$. Each vertex in one of the sub-cycles has two edges labeled by numbers in group I and one edge labeled by a number in group II. But, a vertex in another sub-cycle is adjacent by two edges labeled by numbers in group III and one edge labeled by a number in group II. We can obtain a new edge-magic labeling for $C_{2t+1} \times P_2$ by switching the labels in group II and III as follow:

Theorem 3.3. *For the graph $C_{2t+1} \times P_2$, consider the edge-magic labeling constructed in the proof of Theorem 3.2. Assume that the labeling has the vertex sum S modulo $4t+2$. If we add $2t+1$ to all edge labels in group II and subtract $2t+1$ to all edge labels in group III, then we obtain another 1-EM labeling with sum $S + (2t+1)$ modulo $4t+2$.*

Proof. Since every vertex with edges labeled by numbers in group III has two such kind of edges, when subtract $2t+1$ twice into the vertex sum, under modulo $4t+2$, it is not changed. But, when adding $2t+1$ to the third edge, it adds $2t+1$ into the vertex sum S modulo $4t+2$. At the same time, since the vertex in another sub-cycle contains only one edge labeled by a number in group II, the vertex sum adds $2t+1$. Therefore, we obtain another 1-EM labeling with sum $S+(2t+1)$ modulo $4t+2$. □

Example 6. The figure 6 shows the exchange between numbers in group II and group III for the graph $C_7 \times P_2$.

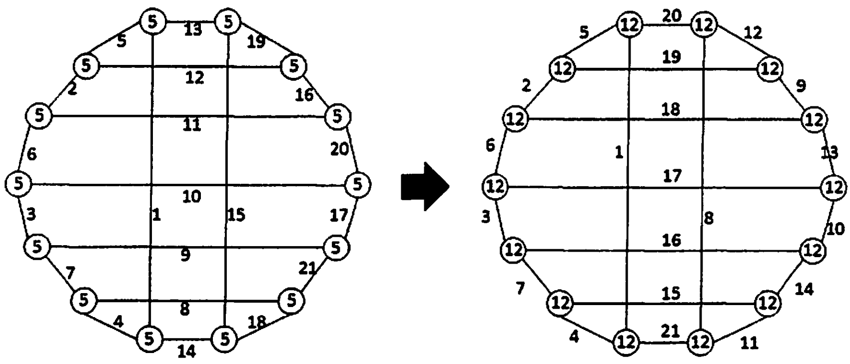


Figure 6: Exchange between group II and group III

3.3 Construct a New k-EM Graph from a Known One

For a k-edge-magic graph, assume that the edge (a,b) and the edge (c,d) share the same label modulo p, say l_1, l_2. It is easy to see that if we remove these two edges and add two edges (a,d) and (c,b) and label them arbitrarily with the label l_1 and l_2, the vertex sum of the new graph remains the same. Similarly, if we remove edges (a,b) and (c,d) and add two edges (a,c) and (b,d) and label them with l_1 and l_2, we also construct a new k-edge-magic graph. We summarize it as:

Theorem 3.4. *For a k-edge-magic graph, if edge (a,b) and the edge (c,d) share the same label modulo p, then if we remove these two edges and add two edges (a,d) and (c,b), then the new graph is still k-edge-magic.*

We provide a couple examples to demonstrate this construction.

Example 7. Our first example is to turn ML(14) to $C_7 \times P_2$. We swapped the edges labeled with 0 and 14 that are the same modulo 14. See Figure 7.

Example 8. The second example is also to start from ML(14). Here we swap two edges labeled by 1 and 15 that are the same modulo 14. See Figure 8.

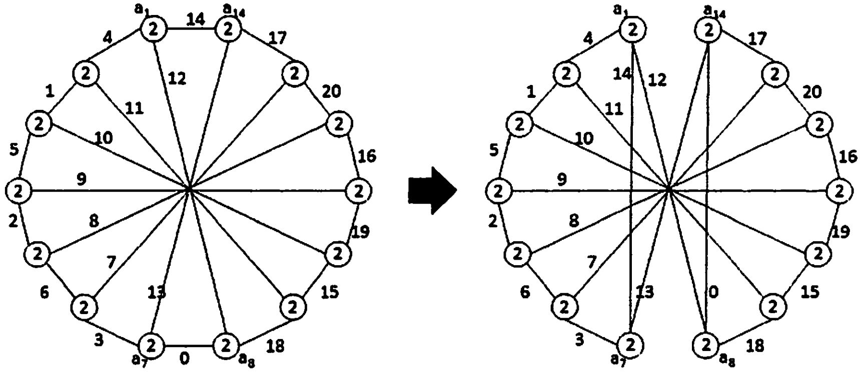


Figure 7: From ML(14) to $C_7 \times P_2$

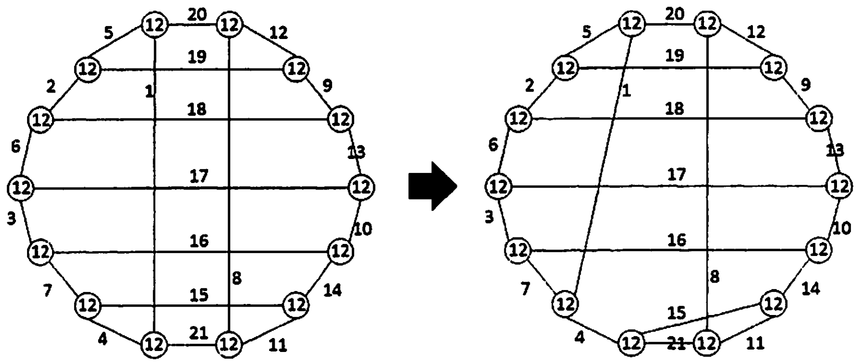


Figure 8: From ML(14) to a new edge-magic graph

4 Some k-edge-magic Cubic Graphs

While trying to prove the Conjecture 1.1, we find more low-number-of-vertices examples of k-edge-magic cubic graphs.

Example 9. The Figure 9 shows that the Petersen graph is k-edge-magic for all k.

Example 10. Figure 10 gives edge-magic labelings for the turtle shell graphs TS(6) and TS(10).

It is well-known that ML(6) and TS(6) are the only two cubic graphs with order 6. We summarize

Corollary 4.1. *All the cubic graphs with order* 6 *are* k*-edge-magic for all* k.

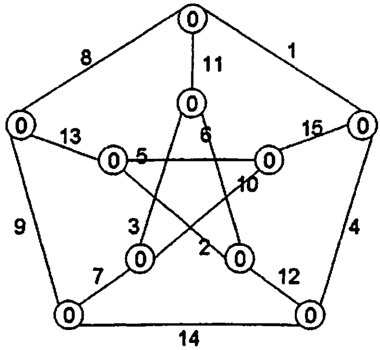


Figure 9: Edge-magic labeling of Petersen graph

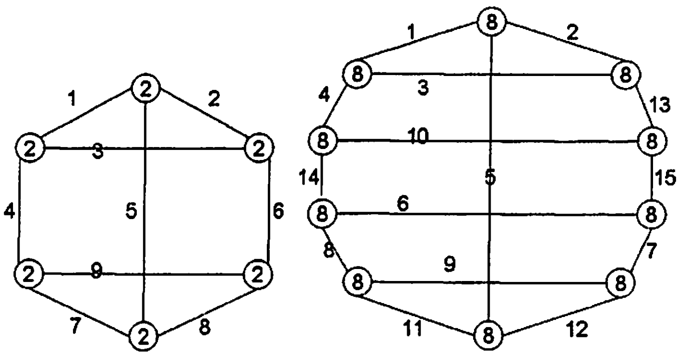


Figure 10: Edge-magic labelings for TS(6) and TS(10)

Example 11. There are 19 connected and 2 non-connected cubic graphs of order 10. When we examine them, 18 out of 19 connected cubic graphs are k-edge-magic. We have shown ML(10), TS(10), $C_5 \times P_2$, and the Petersen graph. In Figures 11, we demonstrate all other 14 k-edge-magic cubic graphs of order 10 here by giving k-edge-labelings.

5 Counterexamples of the Conjecture

Unfortunately, one of a connected order 10 cubic graph destroys the conjecture 1.1.

To demonstrate this counterexample, we need the definition of the Mod(k)-edge-magic graphs. In [2], Chopra, Dios and Lee define

Definition 2. Let $k \geq 2$ and G be a (p, q)-graph in which the edges are

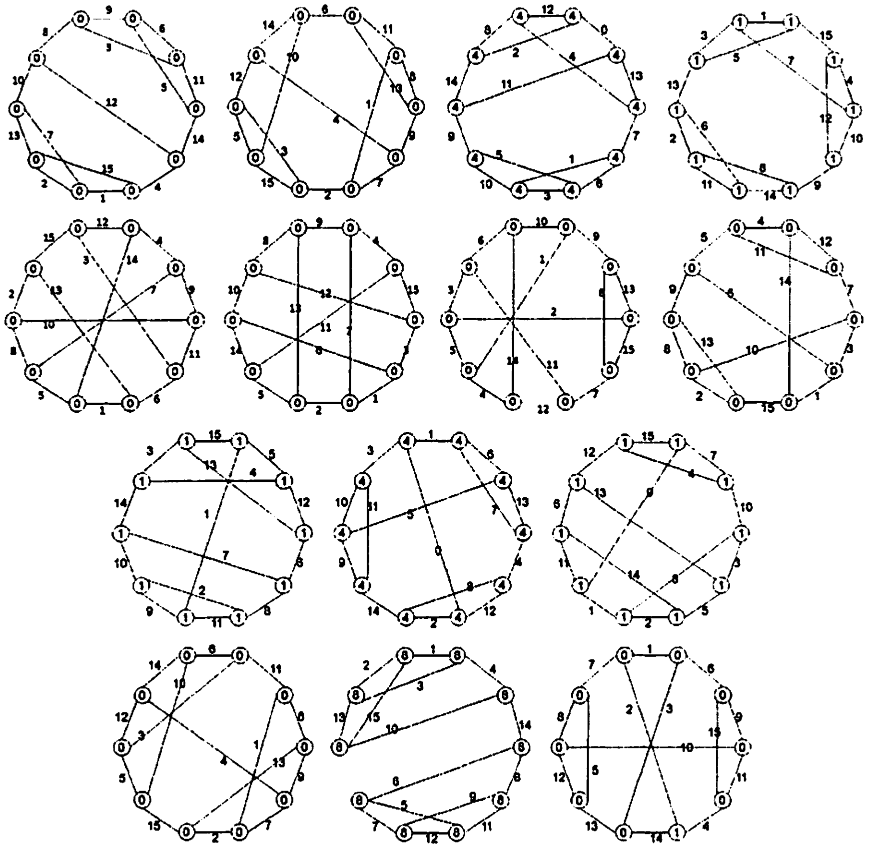


Figure 11: 14 order 10 edge-magic cubic graphs

labeled $1, 2, \ldots, q$. The vertex sum for a vertex v is the sum of the labels of the incident edges at v. If the vertex sums are constant, mod k, then G is said to be Mod(k)-edge-magic (in short Mod(k)-EM).

Lemma 5.1. *For a graph with order p, if it is* 1*-edge-magic, then it is Mod(k)-edge-magic if k divides p.*

Proof. An 1-edge-magic labeling is under modulo p. If k divides p, then it is obvious a Mod(k)-edge-magic labeling. □

Immediately from Lemma 2.4 and Lemma 5.1 we obtain:

Lemma 5.2. *For a cubic graph with order p, if it is k-edge-magic for some integer k, then it is Mod($\frac{p}{2}$)-edge-magic.*

Theorem 5.3. *The graph in Figure 12 is not Mod(5)-edge-magic.*

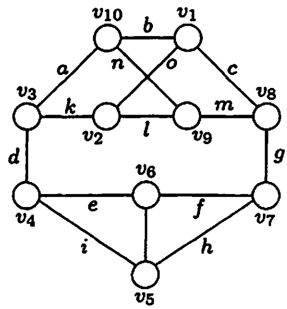


Figure 12: The counterexample

Proof. We assume that the graph in the Figure 12 is Mod(5)-edge-magic.

Since the graph is cubic, by adding 1 or subtracting 1 to each edge, the labeling remains Mod(5)-edge-magic with the vertex sum increased or decreased by 3. Because $\gcd(3,5) = 1$, we can obtain a new Mod(5)-edge-magic labeling with the vertex sum to be any integer. Thus, we can assume the vertex sum of the Mod(5)-edge-magic to be 0 without loss of generality.

By v_9 and v_{10}, we have

$$a + b \equiv l + m \pmod 5. \tag{1}$$

By v_1 and v_2, we have

$$b + c \equiv k + l \pmod 5. \tag{2}$$

By v_5 and v_6, we have

$$h + i \equiv f + e \pmod 5. \tag{3}$$

From equations (1) and (2), we have $a - c \equiv m - k \pmod 5$, that is,

$$a + k \equiv c + m \pmod 5. \tag{4}$$

Therefore, by v_3 and v_8 with equation (4), we have $g \equiv d \pmod 5$. By looking at v_4 and v_7, we have

$$h + f \equiv e + i \pmod 5. \tag{5}$$

From equations (3) and (5), we have $i - f \equiv f - i \pmod 5$, that is, $i \equiv f \pmod 5$. Then we have $h \equiv e \pmod 5$. Moreover, by v_4 and v_6, we have $g \equiv j \equiv d \pmod 5$. Let us summarize what we found here:

Note here the labels of the edges $\{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o\}$ equals $\{1, 2, 3, 4, 0, 1, 2, 3, 4, 0, 1, 2, 3, 4, 0\}$ under modulo 5.

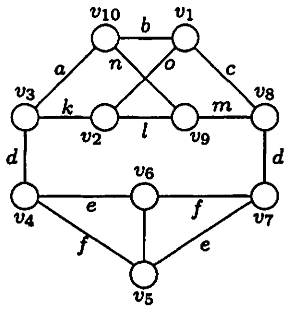


Figure 13: The counterexample

First, we notice that d has already been used three times. Thus, no matter which number in the set $\{0,1,2,3,4\}$ is assigned to d, no other symbol can be assigned to the same number.

Second, we see that there are two copies of pair $\{e,f\}$. After assigning numbers to them, those numbers can be assigned to others only once.

Since the vertex sum is 0 modulo 5, by the vertex v_4, we have $d+e+f \equiv 0 \pmod 5$. If d is labeled i, where $i \in \{1,2,3,4\}$, modulo 5, then $e+f \equiv 5-i \pmod 5$. The only two solutions for $\{e,f\}$ in $\mathbb{Z}_5$ for this equation are $\{5-i,0\}$ and $\{\frac{5-i}{2},\frac{5-i}{2}\}$. Because there are totally four e and f edges, but we can label the same number at most three times, we cannot label $\{e,f\}$ by $\{\frac{5-i}{2},\frac{5-i}{2}\}$. Therefore, the pair $\{e,f\}$ must be labeled $\{5-i,0\}$.

From the vertex v_3, v_4 and v_8, we have

$$a+k \equiv e+f \equiv c+m \pmod 5. \tag{6}$$

Due to the symmetry of the graph, without loss of generality, we can assume that $\{a,k\}$ are labeled the same as $\{e,f\}$. It forces $\{c,m\}$ to be labeled by the same number $\frac{5-i}{2}$. Now, there are only one $\frac{5-i}{2}$ and three copies of $5-\frac{5-i}{2}$ left for labeling the edges $\{b,n,o,l\}$.

From v_1, we have $b+o+c \equiv 0 \pmod 5$. This implies that

$$b+o \equiv -c \pmod 5.$$

But, we can only use just one $\frac{5-i}{2}$ or three $-\frac{5-i}{2}$ to label b and o. Thus, the sum of edges b and o must be either 0, or $-(5-i)$ modulo 5. Both cannot be equal to $-c$, where c is either $\frac{5-i}{2}$ or $-\frac{5-i}{2}$. This is a contradiction.

Now, we assume that d is labeled 0. By equation (6), we have

$$e+f \equiv a+k \equiv c+m \equiv 1+4 \text{ or } 2+3 \pmod 5. \tag{7}$$

We might have three different sets, $\{2,2,3,3\}$, $\{1,2,3,4\}$, or $\{1,1,4,4\}$, of numbers modulo 5 left to label edges $\{b,n,o,l\}$. Similarly, by the previous argument, $b+o$ cannot be possibly equal to $-c$ when $b,n,o,l \in \{2,2,3,3\}$ or $\{1,1,4,4\}$.

When $b, n, o, l \in \{1, 2, 3, 4\}$, if $a \equiv c \pmod 5$, then by v_{10} and v_1, we have $n \equiv o \pmod 5$. Also, if $a \not\equiv c \pmod 5$, then by v_{10} and v_9, we have $b \equiv l \pmod 5$. Both situations are impossible.

Therefore, the graph in the Figure 12 is not Mod(5)-edge-magic. □

Corollary 5.4. *The cubic graph in Figure 12 is not k-edge-magic for all k.*

Proof. Assume that the cubic graph in Figure 12 is k-edge-magic for some k. Since $\gcd(3, 10) = 1$, by Theorem 2.5, it must be h-edge-magic with sum 0 modulo 10. Thus, by Lemma 5.2, it is Mod(5)-edge-magic. This is a contradiction. □

By a similar argument or after an exhaustive search using a computer, two non-connected cubic graphs of order 10 are not k-edge-magic for all k.

Remark. In 2003 [15], Lee, Wang and Wen found that the graph in Figure 12 is not edge-magic by using computer search. Also, without publishing his result, Raridan recently used the computer to reconfirm the same result.

We have seen a lot of edge-magic cubic graphs. But, we only find few counterexamples. Thus, our question now is what the sufficient condition for a connected cubic graph to be k-edge-magic is or can you classify all connected edge-magic cubic graphs? Also, is there any non-connected cubic graph which is k-edge-magic for some k?

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