On the Dirac-type Conjecture for Anti-directed Hamiltonian Digraphs

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Abstract

Let D be a directed graph. An *anti-directed cycle* in D is a set of arcs which form a cycle in the underlying graph, but for which no two consecutive arcs form a directed path in D; this cycle is called an *anti-directed Hamilton cycle* if it includes all vertices of D. Grant [6] first showed that if D has even order n, and each vertex indegree and outdegree in D is a bit more than $2n/3$, then D must contain an anti-directed Hamilton cycle. More recently, Busch et al. [1] lowered the lead coefficient, by showing that there must be an anti-directed Hamilton cycle if all indegrees and outdegrees are greater than $9n/16$, and conjectured that such a cycle must exist if all indegrees and outdegrees are greater than $n/2$. We prove that conjecture holds for all directed graphs of even order less than 20, and are thus able to extend the above result to show that any digraph D of even order n will have an anti-directed Hamilton cycle if all indegrees and outdegrees are greater than $11n/20$.

1 Introduction

In what follows, for vertices u, v , we do not allow parallel edges uv in a graph, or parallel directed arcs $u \to v$ in a digraph. For any vertex v in graph G, we let $\deg(v)$ denote the degree of v, and let $\delta(G)$ denote the minimum degree over all vertices of G. Similarly for any vertex v in digraph D, we let $\deg^-(v)$ denote the indegree of v, let $\deg^+(v)$ denote the

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outdegree of v, and let $\delta(D)$ denote the minimum of all vertex indegrees and outdegrees.

Classic theorems by Dirac [4] and Ghouila-Houri [5] give sufficient conditions for a Hamilton cycle in a graph, and a directed Hamilton cycle in a directed graph.

Theorem 1 *[4] If G is a graph of order $n \geq 3$ and $\delta(G) \geq n/2$, then G contains a Hamilton cycle.*

Theorem 2 *[5] If D is a digraph of order n and $\delta(D) \geq n/2$, then G contains a directed Hamilton cycle.*

Let D be a directed graph. An *anti-directed cycle* in D is a set of arcs which form a cycle in the underlying graph, but for which no two consecutive arcs form a directed path in D (clearly such a cycle must have even length); this cycle is called an *anti-directed Hamilton cycle* if it includes all vertices of D. Our focus is on degree bounds similar to the ones in Theorems 1 & 2 that guarantee the existence of an anti-directed Hamilton cycle in a digraph.

The following theorem by Grant [6] gives a sufficient condition for the existence of an anti-directed Hamilton cycle in a directed graph D.

Theorem 3 *[6] If D is a digraph of even order n and minimum degree $\delta(D) \geq 2n/3 + \sqrt{n \times \ln(n)}$, then G contains an anti-directed Hamilton cycle.*

In his paper, Grant conjectured that the above theorem can be strengthened to assert that if D is directed of even order and $\delta(D) \geq n/2$ then D contains an anti-directed Hamilton cycle, but Cai [2] showed an infinite family of directed D with $\delta(D) = n/2$ that do not contain an anti-directed Hamilton cycle. Nevertheless, the Grant result was improved in Plantholt and Tipnis [7], and more recently even further in Busch et al. [1].

Theorem 4 *[1] If D is a digraph of even order n and $\delta(D) > 9n/16$, then G contains an anti-directed Hamilton cycle.*

Based on some additional evidence, they conjectured that the original Dirac-style conjecture by Grant is almost true; we refer to this as The 0.5 Conjecture.

Conjecture 1 *[1]* **The 0.5 Conjecture.** *If D is a digraph of even order n and $\delta(D) > n/2$, then G contains an anti-directed Hamilton cycle.*

Note that Theorem 4 verifies this Conjecture for $n \leq 14$.

We verify the Conjecture above for the cases when $n = 16$ and $n = 18$. Then, using another bound from [1] , we obtain the following improvement of Theorem 4; this lowers the coefficient from $\frac{9}{16}$ (= .5625) to $\frac{11}{20}$ (= .55) in the direction of the conjectured coefficient $\frac{1}{2}$.

Theorem 5 *If D is a digraph of even order n and $\delta(D) > 11n/20$, then G contains an anti-directed Hamilton cycle.*

2 Method Outline

Note that in any anti-directed Hamilton cycle of a digraph D, half the vertices have outdegree 2, and half have indegree 2. Now for any digraph of even order n, consider any partition of the vertices of D into $X \cup Y$, and with $|X| = |Y| = n/2$; we call this an *equipartition* of the vertices. Corresponding to such a partition, let $D(X, Y)$ be the spanning subdigraph of D whose arc set is the subset of the arcs of D that are directed from a vertex of X into a vertex of Y. Thus any anti-directed Hamilton cycle in D corresponds naturally to a unique equipartition of the vertices (by letting X be the set of vertices with outdegree 2 in the cycle), and a Hamilton cycle in the corresponding digraph $D(X, Y)$. Now let $B(X, Y)$ be the graph obtained from digraph $D(X, Y)$ by changing each arc to an undirected edge; we call $B(X, Y)$ the *bipartite equipartition graph* corresponding to partition (X, Y). Because by the structure of $D(X, Y)$ each cycle in $D(X, Y)$ is necessarily anti-directed, we make the following observation.

Observation 1 *Let D be a digraph of even order. Then D has an anti-directed Hamilton cycle if and only if it has a corresponding bipartite equipartition graph $B(X, Y)$ that has a Hamilton cycle.*

Our approach to the problem is as follows. Given a digraph D, we consider all corresponding bipartite equipartition graphs, and show that at least one of them has a Hamilton cycle. To do so, we rely strongly on the following result by Chvátal. For a set S of vertices in a graph, we let $N(S)$ denote the set of neighbors of S.

Theorem 6 *(Chvátal's Theorem) [3] Let B be a bipartite graph with order n, and partite sets X, Y, where $|X| = |Y| = n/2$. If B does not have a Hamilton cycle, then:*

If $n \equiv 2 (mod 4)$, for some $k < n/4$, there is a set S of k vertices, with $S \subseteq X$ or $S \subseteq Y$, such that

deg(v) $\leq k$ *for each vertex v in S, and $|N(S)| \leq k$.*

If $n \equiv 0 (mod 4)$, then either:

1. *for some $k < n/4$, there is a set S of k vertices, with $S \subseteq X$ or $S \subseteq Y$, such that* deg(v) $\leq k$ *for each vertex $v \in S$, and $|N(S)| \leq k$, or*
2. *there is a set $S = S_X \cup S_Y$ of $n/2$ vertices, where $|S_X| = n/4$ and $S_X \subseteq X$, and $|S_Y| = n/4$ and $S_Y \subseteq Y$, such that $|N(S_X)| \leq n/4$ and $|N(S_Y)| \leq n/4$ (and so* deg(v) $\leq$ n/4 *for each vertex in S).*

We call a set S of vertices that satisfies the conditions in Chvátal's Theorem a *blocking set* of deficient vertices.

Thus, Chvátal's Theorem shows that if a digraph D has no anti-directed Hamilton cycle, each of its corresponding equipartite bipartite graphs must have a certain number of low degree vertices (degree $\leq n/4$). Focusing on those low degree vertices, we say that a vertex in $B(X, Y)$ is a *deficient* vertex if its degree is at most $n/4$. Our approach to showing that a digraph D has an anti-directed Hamilton cycle is to count the number of deficient vertices that appear in $B(X, Y)$ as we range over all equipartitions (X, Y) of the vertices. We show that the number of these is insufficient by the Chvátal Theorem to make each $B(X, Y)$ non-hamiltonian, so by our earlier Observation, D has the desired anti-directed Hamilton cycle. We note in advance that with this argument, if $\delta(D) \geq t > n/2$, then we may assume that each vertex has indegree and outdegree exactly t, as this can only increase the number of times that a vertex has degree $k < n/4$ for each such k , as we range over all (X, Y) (this is equivalent to ignoring some adjacencies in our count, and showing the Hamilton cycle exists without using those adjacencies).

3 Cases $n = 16$ and $n = 18$

Note that Chvátal's Theorem gives different types of information for deficient vertices for the cases $n = 16$ and $n = 18$ because they differ mod 4, so we will use two different approaches. We begin with the case $n = 16$. By *the deficient degree multiset of B(X,Y)* we mean a listing, with repeats, of all vertex degrees that are at most $n/4$.

Lemma 1 *Let D be a digraph of order $n = 16$, with $\delta(D) \geq 9$. If there is an equipartition (X, Y) of the vertices of D whose corresponding bipartite graph $B = B(X,Y)$ has deficient degree multiset $\{2,2\}$ or $\{3,3,3\}$, then D has an anti-directed Hamilton cycle.*

Proof. By the previous comments, in the following argument we may assume that all indegrees and outdegrees are exactly 9. First suppose that the deficient degree multiset from B is $\{2,2\}$. By Chvátal's Theorem, we may assume without loss of generality that the vertices of degree 2, call them x_1 , x_2, are in X, and their neighbors in B are $\{y_1, y_2\}$. All vertices other than x_1 , x_2 have degree at least 5 in B. Since x_1 has outdegree 9 in D, $x_1 \to x_2$ is an arc in D. Let s be any vertex in Y - $\{ y_1, y_2\}$. Now let $X^* = X \cup \{s\}$ - $\{ x_2\}$, and let $Y^* = Y \cup \{ x_2\}$ - $\{s\}$, and consider the corresponding bipartite graph $B^* = B(X^*, Y^*)$. In B^*, $\deg(x_1) = 3$, $\deg(s) \geq 2$, and all other vertices of X^* have degree at least 4; among vertices in Y^*, $\deg(y_1) \geq 4$, $\deg(y_2) \geq 4$, $\deg(x_2) \geq 2$, and all other vertex degrees are at least 5. Thus by Chvátal's Theorem, B^* has a Hamilton cycle, and so D has an anti-directed Hamilton cycle as desired.

Now suppose that the deficient degree multiset from B is $\{3, 3, 3\}$. By Chvátal's Theorem, we may assume without loss of generality that the vertices of degree 3, call them x_1, x_2 and x_3, are all in X, and their neighbors in B are $\{ y_1, y_2, y_3\}$. All vertices other than x_1, x_2 and x_3 have degree at least 5 in B. Since x_1 has indegree 9 in D, either $x_1 \to x_2$ or $x_1 \to x_3$ is an arc in D; without loss of generality, assume $x_1 \to x_3$ is an arc. Choose a vertex s in $Y - \{y_1, y_2, y_3\}$ according to the following rule: if each of y_1, y_2, y_3 have degree at least 7 in B, let s be any vertex of $Y - \{ y_1, y_2, y_3\}$; otherwise, assume wlog that in B, $\deg(y_3) < 7$, and let s be a vertex from $Y - \{ y_1, y_2, y_3\}$ such that $s \to y_3$ is an arc in D (such a vertex must exist because y_3 has indegree 9 in D).

Now let $X^* = X \cup \{s\}$ - $\{ x_3\}$, and let $Y^* = Y \cup \{ x_3\}$ - $\{s\}$, and consider the corresponding bipartite graph $B^* = B(X^*, Y^*)$. In B^*, $\deg(x_1) = 4$, $\deg(x_2) \geq 3$, $d(s) \geq 2$, and all other vertices of X^* have degree at least 4; among vertices in Y^*, $\deg(y_1) \geq 4$, $\deg(y_2) \geq 4$, $\deg(y_3) \geq 5$, $\deg(x_3) \geq 2$, and all other vertex degrees are at least 5. Thus by Chvátal's Theorem, B^* has a Hamilton cycle, and so D has an anti-directed Hamilton cycle as desired. The result follows. ■

Theorem 7 *Let D be a directed graph of order $n = 16$. If each vertex indegree and outdegree is at least 9, then D has an anti-directed Hamilton* *cycle.*

Proof. As noted earlier, we may assume that all indegrees and outdegrees are exactly 9. We consider all $\binom{16}{8} = 12,870$ equipartitions of the vertices into (X, Y) , and the corresponding bipartite graph $B(X, Y)$ of each. Let n_2 denote the total number of degree 2 vertices as we range through all partitions, n_3 the number of degree 3, and n_4 the number of degree 4. Suppose that D does not have an anti-directed Hamilton cycle. Then each bipartite $B(X, Y)$ is not Hamiltonian, and so each contains at least one blocking set of vertices. By Chvátal's Theorem and Lemma 1, this means each contains one of the following:

1. a set of three deficient vertices , at least two of which are degree 2;
2. a set of four deficient vertices, at least three of which have degree at most three; or
3. at least eight vertices with degree at most four.

If a graph has more than one set satisfying the conditions above, we choose a representative set satisfying (1), (2) or (3) arbitrarily. Let a denote the number of $B(X, Y)$ whose representative set of vertices satisfy condition (1), b denote the number satisfying condition (2), and c the number satisfying (3). Then we must have:

$2a \leq n_2$,

$2a + 3b \leq n_2 + n_3$, and

$3a + 4b + 8c \leq n_2 + n_3 + n_4$.

Maximizing $a+b+c$ under these restrictions is a simple linear programming problem, with solution given by $a = 576, b = 5376$, and $c = 6811$, so $a + b + c = 12,763$. (Note that this tells us that in order to maximize the number of blocking sets, we should proceed in a way that is intuitively obvious: pair the degree 2 vertices together in 576 pairs, make 5376 triples with the degree 3 vertices, and 6,811 groups of 8 degree 4 vertices.) But then at most $12,763$ equipartion graphs $B(X, Y)$ can have a blocking set. However, because there are $12,870$ possible $B(X, Y)$, some of these must contain a Hamilton cycle. The result follows. ■

We turn our attention now to digraphs of order 18, and minimum outdegree and indegree at least 10. Again we can assume each outdegree and indegree is exactly 10, for any increase only lowers the number of deficient vertices in the corresponding $B(X, Y)$ graphs. The method used for $n = 16$ can be extended to this case. However, that argument now gets much more complicated with many subcases, in part because of the increase in the order and degree, but more so because of the different impact of Chvátal's Theorem when n is not a multiple of 4. We instead use the following idea. In our counting arguments, especially when n is not a multiple of 4, the maximum possible number of deficient bipartite graphs is obtained by assuming that the deficient vertices group together in the same set, be it X or Y. We will show that a sizeable number of times, deficient vertices occur simultaneously in X and Y; thus some deficient vertices are "wasted" when we apply the Chvátal Theorem, and we will be able to show that some of the bipartite graphs corresponding to equipartitions must be Hamiltonian.

So, let D be a directed graph with no parallel arcs, order 18, and each outdegree and indegree equal to 10. For any two vertices x, y, we wish to bound the number of equipartitions (X, Y) of the vertices for which $x \in X$, $y \in Y$, and both x and y are deficient (have degree 4 or less) in $B(X, Y)$. That number will vary, depending on common neighbors of x and y, but we first examine the possibilities. We partition the 16 vertices other than x, y into four sets:

$P = \{ v \mid x \to v \text{ and } v \to y \text{ are both arcs in } D \}$

$Q = \{ v \mid x \to v \text{ is an arc, but } v \to y \text{ is not an arc in D} \}$

$R = \{ v \mid x \to v \text{ is not an arc, but } v \to y \text{ is an arc in } D \}$

$S = \{ v \mid \text{neither } x \to v \text{ nor } v \to y \text{ are arcs in } D \}$

Lemma 2 *Let D be a directed graph with order 18, with each outdegree and indegree equal to 10, let x, y be vertices in D, and suppose $x \to y$ is <u>not</u> an arc in D. Let the 4-tuple (p, q, r, s) denote the cardinalities of the sets P, Q, R, S respectively described above. Then the number of equipartitions (X, Y) for which $x \in X$, $y \in Y$, and x, y are both deficient is given by the value k in the chart. The 4-tuples 10, 0, 0, 6 and 9, 1, 1, 5 are omitted from the chart; for these, $k = 0$.*

p,q,r,s	*8,2,2,4*	*7,3,3,3*	*6,4,4,2*	*5,5,5,1*	*4,6,6,0*
k	*420*	*840*	*1290*	*1270*	*2370*

Proof. The numbers above are obtained by straightforward but tedious calculation (checked by computer). For example, consider the situation when the 4-tuple is $(7, 3, 3, 3)$. Let us count the number of ways to choose the remaining 8 vertices for Y, and get both x, y deficient. From the sets P, Q, R, S, we can choose:

4 from P, 0 from Q, 2 from R, 2 from S (x has degree 4, so does y) in $\binom{7}{4}\binom{0}{0}\binom{3}{2}\binom{3}{2} = 315$ ways,

4 from P, 0 from Q, 3 from R, 1 from S (x has degree 4, y degree 3) in $35 * 1 * 1 * 3 = 105$ ways,

3 from P, 1 from Q, 3 from R, 1 from S (x has degree 4, so does y) in 315 ways,

3 from P, 0 from Q, 3 from R, 2 from S (x has degree 3, y degree 4) in 105 ways. Thus the total number of ways this can happen for the 4-tuple is 840. Other cases are similar. The result follows. ■

Lemma 3 *Let D be a directed graph with no parallel arcs, order* 18, *with each outdegree and indegree equal to* 10, *and let x, y be vertices in D, and suppose $x \to y$ __is__ an arc in D. Let the 4-tuple (p, q, r, s) denote the cardinalities of the sets P, Q, R, S respectively described above. Then the number of equipartitions (X, Y) for which $x \in X, y \in Y$, and x, y are both deficient is given by the value k in the chart. The 4-tuples* $(9,0,0,7), (8,1,1,6)$, *and* $(7,2,2,5)$ *have $k = 0$ and are omitted from the chart.*

p,q,r,s	*6,3,3,4*	*5,4,4,3*	*4,5,5,2*	*3,6,6,1*	*2,7,7,0*
k	*120*	*300*	*540*	*840*	*1290*

Proof. Now each of x, y adds one to its degree in $B(X, Y)$ from the arc $x \to y$; in order to choose a set Y so that both x and y will be deficient in $B(X, Y)$, we will need to select 8 remaining vertices for Y, and add at most 3 to each of the degrees of x *and* y. For example, if the 4-tuple is $(4, 5, 5, 2)$, from the sets P, Q, R, S we can choose:

3 from P, 0 from Q, 5 from R, 0 from S (x has degree 4, y degree 2) in $\binom{4}{3}\binom{5}{0}\binom{5}{5}\binom{2}{0} = 4$ ways,

3 from P, 0 from Q, 4 from R, 1 from S (x has degree 4, y degree 3) in $4 * 1 * 5 * 2 = 40$ ways,

3 from P, 0 from Q, 3 from R, 2 from S (x has degree 4, y degree 4) in $4 * 1 * 10 * 1 = 40$ ways,

2 from P, 0 from Q, 2 from R, 2 from S (x has degree 3, y degree 4) in $6 * 1 * 5 * 1 = 30$ ways,

2 from P, 0 from Q, 5 from R, 1 from S (x has degree 4, y degree 4) in $6 * 1 * 1 * 2 = 12$ ways,

2 from P, 1 from Q, 4 from R, 1 from S (x has degree 4, y degree 4) in $6 * 5 * 5 * 2 = 300$ ways,

1 from P, 0 from Q, 5 from R, 2 from S (x has degree 2, y degree 4) in $4 * 1 * 1 * 1 = 4$ ways,

1 from P, 1 from Q, 5 from R, 1 from S (x has degree 3, y degree 4) in $4 * 5 * 1 * 2 = 40$ ways,

2 from P, 1 from Q, 5 from R, 0 from S (x has degree 4, y degree 3) in $6 * 5 * 1 * 1 = 30$ ways,

1 from P, 2 from Q, 5 from R, 0 from S (x has degree 4, y degree 4) in $4 * 10 * 1 * 1 = 40$ ways,

for a total of 540 ways. Other cases follow similarly. ■

Theorem 8 *Let D be a directed graph of order $n = 18$ with no parallel edges. If each vertex indegree and outdegree is at least 10, then D has an anti-directed Hamilton cycle.*

Proof. Again, we may assume that all indegrees and outdegrees are exactly 10. We use the previous results to get a lower bound on the number of deficient vertex pairs that must appear in the bipartite graphs corresponding to all the equipartitions (X, Y). In particular, fix a vertex x, and consider the number of times that as we range over all partitions, we have $x \in X$, with x deficient in $B(X, Y)$, and a deficient vertex $y \in Y$. Because all vertex outdegrees and indegrees are 10, there are 100 directed paths of length 2 starting at x. Because x has outdegree = indegree = 10 in D, there are at least three vertices v such that $x \to v$ and $v \to x$ are both arcs. Thus, there are at most 97 directed paths of two arcs starting at x and ending at a vertex other than x. In Lemmas 2 and 3, for each choice of the second vertex y , the value p in the 4-tuple (p, q, r, s) gives the number of directed paths of length 2 from x to y. Thus, as we range over the 17 possible choices for y, the sum of the values of p in the 4-tuples must be at least 97.

For the seven vertices covered in Lemma 2, let $a_4, a_5, \ldots, a_9, a_{10}$ denote the number of times that the p-value is $4, 5, \ldots, 10$ respectively. Similarly, for the ten vertices covered by Lemma 3, let b_2, b_3, $\ldots$, b_8, b_9 denote the number of times the p-value is $2, 3, \ldots, 9$ respectively. Then as we range over all $B(X, Y)$ for which $x \in X$, the total of number of instances in which x is deficient and there is $y \in Y$ that is also deficient is at least

$$2370a_4+1770a_5+1290a_6+840a_7+420a_8+1290b_2+840b_3+540b_4+300b_5+120b_6$$

But we must have $\sum_{i=4}^{10} a_i = 7$, $\sum_{i=2}^{9} b_i = 10$, and for 2-paths,

$$\sum_{i=4}^{10} ia_i + \sum_{j=2}^{9} jb_j = 97.$$

Solving the corresponding linear program, we see that the minimum value for formula above is given by $a_9 = 7$, $b_3 = 6$, $b_4 = 4$, all other values 0. Thus, the total number of deficient $x - y$ pairs with $x \in X$ as we range over all $B(X,Y)$ is at least $840 * 6 + 540 * 4 = 7,200$; the sum then of all such pairs as we let x range over all 18 vertices is at least 129,600.

Now the total number of $B(X,Y)$ is $\binom{18}{9} = 48,620$. The number n_i of deficient vertices of degree i in these graphs for $i = 2,3,4$ is:

$n_2 = 218\binom{10}{2}\binom{7}{7} = 1620$, $n_3 = 218\binom{10}{3}\binom{7}{6} = 30,240$, and $n_4 = 218\binom{10}{4}\binom{7}{5} = 158,760$.

Now give a deficient vertex of degree d in $B(X,Y)$ "weight" $1/d$. Suppose that each $B(X,Y)$ does not have a Hamilton cycle. By Chvátal's Theorem, each $B(X,Y)$ must contain a blocking S with total weight at least 1. If more than one blocking set S satisfies the conditions of Chvátal's Theorem, we pick one with a minimum possible number of vertices arbitrarily, and call it "the" blocking set for $B(X,Y)$. But from earlier calculations, there are 129,600 pairs of deficient vertices appearing in the $B(X,Y)$ sets. Since there are 9 vertices in each of X,Y, it is possible that a single deficient vertex could appear for up to 9 of these in the calculations, but at least $129,600/9 = 14,400$ deficient vertices appear in the $B(X,Y)$ set that are not in the designated blocking sets.

The total weight of all deficient vertices is $1620 * (1/2) + 30,240 * (1/3) + 158,760 * (1/4) = 50,580$. But since 14,400 deficient vertices, each of weight at least 1/4 are not in the designated blocking sets, the total weight of the vertices in the blocking sets is at most $50,580 - 14,400/4 = 46,980$. But recall that the total number of partitions is 48,620, and since each of these must have a blocking set with total weight at least 1, we reach a contradiction. The result now follows. ■

Theorem 9 *The 0.5 Conjecture is true for all digraphs with order $n < 20$.*

Proof. This follows immediately from Theorems 4, 7 and 8. ■

4 Main Result

We require one more result, which appeared in [1] .

Theorem 10 *For any $\epsilon > 0$, every digraph D of even order*

$$n > \ln(4)/(\epsilon \ln((1+\epsilon)/(1-\epsilon)))$$

contains an anti-directed Hamilton cycle.

We are ready for our main result, stated earlier as Theorem 5.

Theorem 5. *Every digraph D with even order n and $\delta(D) > 11n/20$ contains an anti-directed Hamilton cycle.*

Proof. Applying Theorem 10 with $\epsilon = 0.05$ gives the result whenever $n \geq 278$, so assume $n \leq 276$. For a given D, as before we let n_k denote the number of occurrences of a vertex of degree k as we range over all equipartition bipartite graphs $B(X, Y)$ corresponding to D (so letting $d = \delta(D)$, we have $n_k = 2n\binom{d}{k}\binom{n-d-1}{n/2-d})$. Applying Chvátal's Theorem, the total number of blocking sets can be at most

$$n_2/2 + n_3/3 + \ldots + n_{(n-2)/4}/[(n-2)/4]$$

when n is congruent to $2(mod4)$, and a similar sum when $n \equiv 0(mod4)$. A computer run verifies that for all cases EXCEPT $n = 16, \delta(D) = 9$, and $n = 18, \delta(D) = 10$, the number of equipartitions $\binom{n}{n/2}$ exceeds the possible number of blocking sets, so that there is an anti-directed Hamilton cycle in D. The two cases with $n = 16$ and $n = 18$ are covered in Theorems 8 and 9, so the result follows. ■

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