Group divisible designs with two associate classes and $(\lambda_1, \lambda_2) = (1, 2)$

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Abstract

A group divisible design GDD$(v = v_1 + v_2 + \cdots + v_g, g, k; \lambda_1, \lambda_2)$ is an ordered pair $(V, \mathcal{B})$ where V is a v-set of symbols and $\mathcal{B}$ is a collection of k-subsets (called blocks) of V satisfying the following properties: the v-set is divided into g groups of sizes $v_1, v_2, \ldots, v_g$; each pair of symbols from the same group occurs in exactly λ_1 blocks in $\mathcal{B}$; and each pair of symbols from different groups occurs in exactly λ_2 blocks in $\mathcal{B}$. In this paper we give necessary conditions on m and n for the existence of a GDD$(v = m+n, 2, 3; 1, 2)$, along with sufficient conditions for each $m \leq \frac{n}{2}$. Furthermore, we introduce some construction techniques to construct some GDD$(v = m + n, 2, 3, 1, 2)$s when $m > \frac{n}{2}$, namely, a GDD$(v = 9 + 15, 2, 3; 1, 2)$ and a GDD$(v = 25 + 33, 2, 3; 1, 2)$.

1 Introduction

A *group divisible design* GDD$(v = v_1 + v_2 + \cdots + v_g, g, k; \lambda_1, \lambda_2)$ is an ordered pair $(V, \mathcal{B})$ where V is a v-set of symbols and $\mathcal{B}$ is a collection of

†This research was carried out under financial support from the Center of Excellence in Mathematics, the Commission on Higher Education, Thailand.

k-subsets (called *blocks*) of V satisfying the following properties: the v-set is divided into g groups of sizes $v_1, v_2, \ldots, v_g$; each pair of symbols from the same group occurs in exactly λ_1 blocks in $\mathcal{B}$; and each pair of symbols from different groups occurs in exactly λ_2 blocks in $\mathcal{B}$. Symbols occurring in the same group are known to statisticians as *first associates*, and symbols occurring in different groups are called *second associates*. The existence of such GDDs has been of interest over the years, going back to at least the work of Bose and Shimamoto in 1952 who began classifying such designs [12]. More recently, much work has been done on the existence of such designs when $\lambda_1 = 0$ (see [2] for a summary).

Most interestingly, if the number of groups is less than the block size then the construction of such GDDs is notoriously difficult. For $k = 3$ this existence problem was completely solved by Sarvate, Fu and Rodger [4, 5] in the case where all groups have the same size. In this paper we focus on an existence of a GDD$(v = m + n, 2, 3; 1, 2)$ for any m and n. Since we are dealing with GDDs with two groups and block size 3, we will use GDD$(m, n; \lambda_1, \lambda_2)$ for GDD$(v = m + n, 2, 3; \lambda_1, \lambda_2)$ from now on, and we refer to the blocks as *triples*. Punnim and Sarvate have written the first draft in this direction and later became part of [1]. In particular they have completely determined all pairs of integers (n, λ) for which a GDD$(1, n; 1, \lambda)$ exists. Other work on the existence problem of a GDD$(m, n; \lambda_1, \lambda_2)$ for possible m, n, λ_1 and λ_2 includes work on a GDD$(m, n; \lambda, 1)$ [11] and a GDD$(m, n; \lambda, 2)$ when $\lambda \neq 1$ [14]. In this paper we investigate the existence of a GDD$(m, n; \lambda, 2)$ for the remaining case $\lambda = 1$. The sufficient conditions for its existence seem to be complicated while the necessary conditions can be easily obtained by describing it graphically as follows.

Let λK_v denote the graph on v vertices in which each pair of vertices is joined by λ edges. Let G_1 and G_2 be graphs. The graph $G_1 \vee_\lambda G_2$ is formed from the union of G_1 and G_2 by joining each vertex in G_1 to each vertex in G_2 with λ edges. A G-decomposition of a graph H is a partition of the edges of H such that each element of the partition induces a copy of G. Thus the existence of a GDD$(m, n; \lambda_1, \lambda_2)$ is easily seen to be equivalent to the existence of a K_3-decomposition of $\lambda_1 K_m \vee_{\lambda_2} \lambda_1 K_n$. In particular the existence of a GDD$(m, n; 1, 2)$ is equivalent to a K_3-decomposition of $K_m \vee_2 K_n$.

2 Necessary Conditions

Let $(V = V_1 \cup V_2, \mathcal{B})$ be a GDD$(m, n; \lambda_1, \lambda_2)$ where V_1 is an m-set and V_2 is an n-set. Then there exists a K_3-decomposition of $\lambda_1 K_m \vee_{\lambda_2} \lambda_1 K_n$ where

$V(K_m) = V_1$ and $V(K_n) = V_2$. It is easy to see that the graph $\lambda_1 K_m \vee_{\lambda_2} \lambda_1 K_n$ is of order $m+n$ and size $\lambda_1\binom{m}{2} + \lambda_1\binom{n}{2} + \lambda_2 mn$. Furthermore, the graph contains m vertices of degree $\lambda_1(m-1)+\lambda_2 n$ and n vertices of degree $\lambda_1(n-1) + \lambda_2 m$. Triples in $\mathcal{B}$ can be partitioned into sets $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_{12}$ and $\mathcal{B}_{21}$ in a natural way, namely

$\mathcal{B}_1$ is the set of triples consisting of three elements from V_1.

$\mathcal{B}_2$ is the set of triples consisting of three elements from V_2.

$\mathcal{B}_{12}$ is the set of triples consisting of one element from V_1 and two elements from V_2.

$\mathcal{B}_{21}$ is the set of triples consisting of one element from V_2 and two elements from V_1.

Put $t_1 = |\mathcal{B}_1|$, $t_2 = |\mathcal{B}_2|$, $t_{12} = |\mathcal{B}_{12}|$, and $t_{21} = |\mathcal{B}_{21}|$. We obtain the following relations.

1. $b = |\mathcal{B}| = \frac{1}{3}[\lambda_1\binom{m}{2} + \lambda_1\binom{n}{2} + \lambda_2 mn]$, that is, $6 \mid \lambda_1[m(m-1) + n(n-1)] + 2\lambda_2 mn$.
2. $r_1 = \frac{1}{2}[\lambda_1(m-1) + \lambda_2 n]$ and $r_2 = \frac{1}{2}[\lambda_1(n-1) + \lambda_2 m]$, that is, $2 \mid \lambda_1(m-1) + \lambda_2 n$ and $2 \mid \lambda_1(n-1) + \lambda_2 m$, where r_1 represents the number of blocks in $\mathcal{B}$ containing any fixed element of V_1 and r_2 the number of blocks in $\mathcal{B}$ containing any fixed element of V_2.
3. $3t_1 + t_{21} = \lambda_1\binom{m}{2}$, $3t_2 + t_{12} = \lambda_1\binom{n}{2}$ and $2t_{12} + 2t_{21} = \lambda_2 mn$.
4. $t_1 + t_2 + t_{12} + t_{21} = b = \frac{1}{6}[\lambda_1[m(m-1) + n(n-1)] + 2\lambda_2 mn]$.

In particular if $\lambda_1 = 1$ and $\lambda_2 = 2$, then we obtain the following necessary conditions.

Theorem 2.1. *Let m and n be positive integers. If there exists a* GDD$(m, n; 1, 2)$ *then there exist non-negative integers h and k such that* $(m, n) \in \{(6k+1, 6h+3), (6k+3, 6h+1), (6k+3, 6h+3)\}$.

Proof. Since $2 \mid m-1+2n$, $2 \mid n-1+2m$ and $6 \mid [m(m-1)+n(n-1)]+4mn$, we have that both m and n must be odd satisfying $m(m-1)+n(n-1) + 4mn \equiv 0 (\mathrm{mod}\ 6)$. Three possible cases are verified. If $m \equiv 1 (\mathrm{mod}\ 6)$, then, $n(n-1) + 4n \equiv 0 (\mathrm{mod}\ 6)$; thus $n \equiv 3 (\mathrm{mod}\ 6)$. If $m \equiv 3 (\mathrm{mod}\ 6)$, then $n(n-1) \equiv 0 (\mathrm{mod}\ 6)$; thus $n \equiv 1, 3 (\mathrm{mod}\ 6)$. Lastly, if $m \equiv 5 (\mathrm{mod}\ 6)$, then $20 + n(n-1) + 20n \equiv n^2 + n + 2 \equiv 0 (\mathrm{mod}\ 6)$; so, there is no n satisfying the congruence. Therefore, the only possible values of m and n are $(m, n) \in \{(6k+1, 6h+3), (6k+3, 6h+1), (6k+3, 6h+3)\}$. □

3 Sufficient Conditions

It can be noted from the necessary conditions that GDD$(n, n; 1, 2)$ does not exist. Suppose for the remainder of this paper that $m < n$. In Section 3.1 we prove the existence of a GDD$(m, n; 1, 2)$ when $m \leq \frac{n}{2}$. In Sections 3.2 and 3.3 we construct two GDD$(m, n; 1, 2)$s for which $\frac{n}{2} < m < n$. This can appears to be a more complicated problem.

A GDD$(m, n; 1, 2)$ is said to be *gregarious* if each triple intersects each group. Thus a gregarious GDD$(m, n; 1, 2)$ is a GDD$(m, n; 1, 2)$ in which $t_1 = t_2 = 0$. A necessary condition for an existence of such designs is that $(n - m)^2 = n + m$. This simple necessary condition was proved to be sufficient in [3]. If $t_1 + t_2 = 1$, necessary conditions lead to $m = (4t + 1)(2t + 3)$ and $n = (2t + 1)(4t + 7)$ for a non-negative integer t, in which the first two pairs are $(3, 7)$ and $(25, 33)$. Similarly, if $t_1 + t_2 = 2$, we have $m = (2t + 1)(4t + 9)$ and $n = (4t + 3)(2t + 5)$ for a non-negative integer t with the first pair $(9, 15)$. Since $(m, n) = (3, 7)$ satisfies $m \leq \frac{n}{2}$, the first nontrivial cases for $t_1 + t_2 = 1$ and $t_1 + t_2 = 2$ are GDD$(25, 33; 1, 2)$ and GDD$(9, 15; 1, 2)$, respectively. We construct these two GDDs using a graph labeling and latin squares.

3.1 GDD$(m, n, 1, 2)$ when $m \leq \frac{n}{2}$

When $\lambda_1 = \lambda_2 = 1$, a GDD$(m, n; 1, 1)$ is a *Steiner triple system* and is denoted by STS(v) where $v = m + n$. Let $(V, \mathcal{B})$ be an STS(v). Then the number of triples $b = |\mathcal{B}| = v(v - 1)/6$. A *parallel class* in an STS(v) is a set of disjoint triples whose union is the set V. A parallel class contains $v/3$ triples, and hence an STS(v) having a parallel class can exist only when $v \equiv 3 \pmod 6$. A *Kirkman triple system*, denoted by KTS(v) is an $STS(v)$, namely $(V, \mathcal{B})$, with the set $\mathcal{B}$ can be partitioned into parallel classes. Note that there are exactly $(v - 1)/2$ parallel classes for a KTS(v). Here is a well-known result, also see [9].

Theorem 3.1. *Let v be a positive integer.*

1. *An STS(v) exists if and only if $v \equiv 1$ or $3 \pmod 6$.*
2. *A KTS(v) exists if and only if $v \equiv 3 \pmod 6$.*

For any integer v, a *difference triple* is a subset of three distinct elements $\{x, y, z\}$ of $\{1, 2, \ldots, v - 1\}$ such that $x + y \equiv \pm z \pmod v$, and its corresponding *base block* is the triple $\{0, x, x + y\}$. In 1896, Heffter [7] posted a problem called *Heffter's Difference Problem* and it was solved by Peltesohn in 1939 [10], namely:

- The sets $\{1, 2, \ldots, \frac{v-1}{2} = 3k\}$ and $\{1, 2, \ldots, \frac{v-1}{2} = 3k+1\} \setminus \{\frac{v}{3} = 2k+1\}$ can be partitioned into difference triples if $v = 6k+1$ and $v = 6k+3$ respectively, except for $v = 9$.

Let $(S, \mathcal{T})$ be an STS. An *automorphism* of $(S, \mathcal{T})$ is a bijection $\alpha : S \to S$ such that $t = \{x, y, z\} \in \mathcal{T}$ if and only if $t\alpha = \{x\alpha, y\alpha, z\alpha\} \in \mathcal{T}$. An STS($v$) is *cyclic* if it has an automorphism that is a permutation consisting of a single cycle of length v. Let $V = \{0, 1, 2, \ldots, v-1\}$ and $D(v)$ be a set of difference triples that are solution to Heffter's Difference Problem. Consider the collection of base blocks obtained from the difference triples in $D(v)$. For $v = 6k+1$, there are exactly k base blocks $B_1, B_2, \ldots, B_k$. Let $\mathcal{B}_i$ be the set of $6k+1$ blocks obtained from the base block B_i. Thus $(V, \mathcal{B})$, where $\mathcal{B} = \bigcup_{i=1}^{k} \mathcal{B}_i$, forms an STS($6k+1$). Note that for each $i = 1, 2, \ldots, k$, $\mathcal{B}_i$ contains $6k+1$ blocks and for $j = 0, 1, \ldots, v-1$, there are exactly three blocks in $\mathcal{B}_i$ containing j. The result is summarized in the following theorem, see details in [9].

Theorem 3.2. *For all $v \equiv 1 \pmod 6$, there exists a cyclic STS(v).*

Example 3.3. This example is included to illustrate the use of difference triples to construct cyclic STSs and will also be used in the construction of a GDD in Example 3.7. For $v = 13$, the set $\{1, 2, \ldots, 6\}$ can be partitioned into difference triples $\{1, 3, 4\}$ and $\{2, 5, 6\}$, and its corresponding base blocks are $B_1 = \{0, 1, 4\}$ and $B_2 = \{0, 2, 7\}$. This yields

$$\begin{aligned}
\mathcal{B}_1 = \; & \{\{0,1,4\}, \{1,2,5\}, \{2,3,6\}, \{3,4,7\}, \{4,5,8\}, \{5,6,9\}, \{6,7,10\}, \\
& \{7,8,11\}, \{8,9,12\}, \{9,10,0\}, \{10,11,1\}, \{12,11,2\}, \{12,0,3\}\} \\
& \text{and} \\
\mathcal{B}_2 = \; & \{\{0,2,7\}, \{1,3,8\}, \{2,4,9\}, \{3,5,10\}, \{4,6,11\}, \{5,7,12\}, \\
& \{6,8,0\}, \{7,9,1\}, \{8,10,2\}, \{9,11,3\}, \{10,12,4\}, \{12,0,5\}, \\
& \{12,1,6\}\}.
\end{aligned}$$

Hence, $(\{0, 1, \ldots, 12\}, \mathcal{B}_1 \cup \mathcal{B}_2)$ forms an STS(13). □

The following notations will be used for our constructions.

1. Let $\{x, y, z\}$ be a triple and $a \notin \{x, y, z\}$ a symbol. Then $a * \{x, y, z\}$ will produce three triples $\{a, x, y\}, \{a, x, z\}, \{a, y, z\}$. Similarly if $\mathcal{T}$ is a set of triples from X and $a \notin X$, then $a * \mathcal{T}$ is defined as

$$a * \mathcal{T} = \{a * T : T \in \mathcal{T}\}.$$

2. Let (x, y, z) be an ordered triple and let a, b and c be three distinct symbols none of which is in $\{x, y, z\}$. Then $< a, b, c > \star(x, y, z)$ will produce three triples $\{a, x, y\}, \{b, x, z\}, \{c, y, z\}$. Similarly if $\mathcal{T}$ is a

set of ordered triples from X and $a, b, c \notin X$ are distinct symbols, then $< a, b, c > \star\mathcal{T}$ is defined as

$$< a, b, c > \star\mathcal{T} = \{< a, b, c > \star T : T \in \mathcal{T}\}.$$

Now we are ready to show that necessary conditions for the existence of a GDD$(m, n; 1, 2)$ with $m \leq \frac{n}{2}$ are sufficient.

Lemma 3.4. *If* $n \equiv 3 \pmod 6$, $m \equiv 1, 3 \pmod 6$ *and* $m \leq \frac{n}{2}$, *then there exists a GDD*$(m, n; 1, 2)$.

Proof Let V_1 and V_2 be an m-set and n-set, respectively. Since $n \equiv 3 \pmod 6$, there exists a KTS(n). Say $n = 6h+3$ and $V_1 = \{a_1, a_2, \ldots, a_m\}$. Let $(V_2, \mathcal{B}')$ be a KTS(n) with parallel classes $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{3h+1}$. Put

$$\mathcal{B}'' = \bigcup_{i=1}^{m}(a_i * \mathcal{P}_i) \cup P_{m+1} \cup \ldots \cup \mathcal{P}_{3h+1}.$$

Furthermore, since $m \equiv 1, 3 \pmod 6$, there exists an STS(m). Let $(V_1, \mathcal{B}''')$ be an STS(m). Hence, setting $\mathcal{B} = \mathcal{B}'' \cup \mathcal{B}'''$ yields a GDD$(m, n; 1, 2)$ as desired. □

Example 3.5. To construct a GDD$(3, 9; 1, 2)$, let $V_1 = \{x, y, z\}$ and $V_2 = \{0, 1, \ldots, 8\}$. Then $\mathcal{B}' = \bigcup_{i=1}^{4} \mathcal{P}_i$ is a KTS(9) where parallel classes $\mathcal{P}_i$ are as follows:

$$\begin{aligned}
\mathcal{P}_1 = &\ \{\{0,1,2\},\{3,4,5\},\{6,7,8\}\}\\
\mathcal{P}_2 = &\ \{\{0,3,6\},\{1,4,7\},\{2,5,8\}\}\\
\mathcal{P}_3 = &\ \{\{0,4,8\},\{2,3,7\},\{1,5,6\}\}\\
\mathcal{P}_4 = &\ \{\{0,5,7\},\{1,3,8\},\{2,4,6\}\}
\end{aligned}$$

Now, put $\mathcal{B}'' = (x * \mathcal{P}_1) \cup (y * \mathcal{P}_2) \cup (z * \mathcal{P}_3) \cup \mathcal{P}_4$. Explicitly, $\mathcal{B}'' =$
$\{\{x,0,1\},\{x,0,2\},\{x,1,2\},\{x,3,4\},\{x,3,5\},\{x,4,5\},\{x,6,7\},\{x,6,8\},\{x,7,8\},$
$\{y,0,3\},\{y,0,6\},\{y,3,6\},\{y,1,4\},\{y,1,7\},\{y,4,7\},\{y,2,5\},\{y,2,8\},\{y,5,8\},$
$\{z,0,4\},\{z,0,8\},\{z,4,8\},\{z,2,3\},\{z,2,7\},\{z,3,7\},\{z,1,5\},\{z,1,6\},\{z,5,6\},$
$\{0,5,7\},\{1,3,8\},\{2,4,6\}\}$.
Now $\mathcal{B}''' = \{x, y, z\}$ is an STS(3). Hence, $\mathcal{B} = \mathcal{B}'' \cup \mathcal{B}'''$ forms a GDD$(3, 9; 1, 2)$ as in Lemma 3.4. □

Lemma 3.6. *If* $n \equiv 1 \pmod 6$, $m \equiv 3 \pmod 6$ *and* $m \leq \frac{n}{2}$, *then there exists a GDD*$(m, n; 1, 2)$.

Proof Let V_1 and V_2 be an m-set and n-set, respectively. Say $n = 6h+1$ and $V_1 = \{a_{i1}, a_{i2}, a_{i3} \mid i = 1, 2, \ldots, \frac{m}{3}\}$. Since $n \equiv 1 \pmod 6$, we have a cyclic STS(n). Let $B_1, B_2, \ldots, B_h$ be base blocks of V_2. Note that for our construction, we fix the order of elements in each base block. Let $\mathcal{B}_i$ be the set of $6h+1$ ordered triples obtained from the ordered based block B_i, namely, if $B_i = (b_1, b_2, b_3)$ then $\mathcal{B}_i = \{(b_1+j, b_2+j, b_3+j) | j = 1, \ldots, 6h+1\}$ where the adding modulo n. Since $m \leq \frac{n}{2}$, we have $\frac{m}{3} \leq h$. Put

$$\mathcal{B}^* = \bigcup_{i=1}^{\frac{m}{3}} (< a_{i1}, a_{i2}, a_{i3} > \star \mathcal{B}_i) \cup \mathcal{B}_{\frac{m}{3}+1} \cup \ldots \cup \mathcal{B}_h.$$

Furthermore, since $m \equiv 3 \pmod 6$, there exists an STS(m). Let $(V_1, \mathcal{B}')$ be an STS(m). Hence, setting $\mathcal{B} = \mathcal{B}' \cup \mathcal{B}^*$ yields a GDD($m, n; 1, 2$) as desired. □

Example 3.7. To construct a GDD($3, 13; 1, 2$), we use $V_1 = \{x, y, z\}$ and $V_2 = \{0, 1, \ldots, 12\}$. Base blocks B_1 and B_2, together with the set of blocks $\mathcal{B}_1$ and $\mathcal{B}_2$ are obtained in Example 3.3. Put $\mathcal{B}^* = (< x, y, z > \star \mathcal{B}_1) \cup \mathcal{B}_2$. To spell this out, $< x, y, z > \star \mathcal{B}_1 = \{\{x, 0, 1\}, \{y, 0, 4\}, \{z, 1, 4\}, \{x, 1, 2\}, \{y, 1, 5\}, \{z, 2, 5\}, \{x, 2, 3\}, \{y, 2, 6\}, \{z, 3, 6\}, \{x, 3, 4\}, \{y, 3, 7\}, \{z, 4, 7\}, \{x, 4, 5\}, \{y, 4, 8\}, \{z, 5, 8\}, \ldots, \{x, 12, 0\}, \{y, 12, 3\}, \{z, 0, 3\}\}$.
Furthermore, $\mathcal{B}' = \{\{x, y, z\}\}$ simply gives us an STS(3). Hence, $\mathcal{B} = \mathcal{B}' \cup \mathcal{B}^*$ yields a GDD($3, 13; 1, 2$). □

Theorem 3.8. *Let m and n be positive integers such that $m \leq \frac{n}{2}$. There exists a* GDD($m, n; 1, 2$) *if and only if there exist non-negative integers h and k such that* $(m, n) \in \{(6k+1, 6h+3), (6k+3, 6h+1), (6k+3, 6h+3)\}$.

Proof It follows from Theorem 2.1, Lemma 3.4 and Lemma 3.6. □

3.2 GDD($9, 15; 1, 2$)

Given $X = \{v_1, v_2, \ldots, v_n\}$ a set of n vertices, both notations $K_n(X)$ and $K_n(v_1, \ldots, v_n)$ stand for the complete graph on the vertex set X. Now let $X = \{x_0, x_1, \ldots, x_8\}$ and $Y = \{0, \ldots, 8\} \cup \{a, b, c, d, e, f\}$. Consider $K_9(X) \vee_2 K_{15}(Y)$. Since $t_1 + t_2 = 2$, there are only two triples from the same group. We give a design such that $\mathcal{B}_1 = \emptyset$ and $\mathcal{B}_2 = \{\{a, b, c\}, \{d, e, f\}\}$. Other triples must be in $\mathcal{B}_{12}$ or $\mathcal{B}_{21}$. We first construct triples in $\mathcal{B}_{12}$ which correspond to an edge in $K_{15}(Y) - K_3(a, b, c) - K_3(d, e, f)$ with a vertex in X. To do that, we decompose $K_{15}(Y) - K_3(a, b, c) - K_3(d, e, f)$ into nine subgraphs H_i each having degree sequence $1^8 2^7$.

For $i = 0, \ldots, 8$, consider F_i a subgraph of $K_{15}(Y) - K_3(a, b, c) - K_3(d, e, f)$ as in Figure 1. Note that vertices labeled by integers modulo 8.

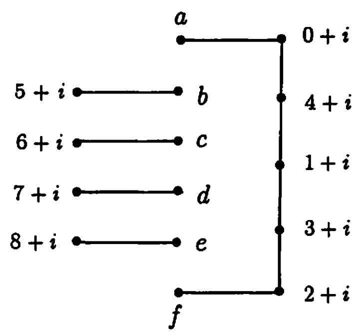

Figure 1: F_i, a subgraph of $K_{15}(Y) - K_3(a,b,c) - K_3(d,e,f)$.

For $i = 0, 1, \ldots 8$, we create each subgraph H_i by adding a single edge to F_i; that is $H_i = F_i + e_i$ where

$$\begin{array}{lll} e_0 = cf & e_1 = be & e_2 = cd \\ e_3 = ad & e_4 = bf & e_5 = ae \\ e_6 = af & e_7 = bd & e_8 = ce \end{array}$$

Then H_i has 15 vertices with the degree sequence $1^8 2^7$, however, note that not all the H_is are isomorphic. It can be directly verified that $K_{15}(Y) - K_3(a,b,c) - K_3(d,e,f)$ is decomposed into $\{H_0, H_1, \ldots, H_8\}$. Each edge uv in H_i gives rise to the triple $\{x_i, u, v\}$. Hence,

$$\mathcal{B}_{12} = \{\{x_i, u, v\} | uv \in E(H_i), i = 0, \ldots, 8\}.$$

The vertex x_i must meet vertices in $K_{15}(Y)$ twice, however, there are 8 vertices of degree one in each H_i. Those vertices must meet x_i again in $\mathcal{B}_{21}$. The following symmetric partial Latin square of order 9 with the i^{th}

row contains all elements of degree one in H_i.

$\oplus$	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
x_0		7	b	8	e	d	5	a	6
x_1	7		8	c	0	f	d	6	a
x_2	b	8		0	a	1	e	f	7
x_3	8	c	0		1	b	2	e	f
x_4	e	0	a	1		2	c	3	d
x_5	d	f	1	b	2		3	c	4
x_6	5	d	e	2	c	3		4	b
x_7	a	6	f	e	3	c	4		5
x_8	6	a	7	f	d	4	b	5	

This symmetric partial Latin square yields the triangles

$$\mathcal{B}_{21} = \{\{x_i, x_j, x_i \oplus x_j\} | 0 \leq i, j \leq 8\}.$$

In conclusion, we have that $(X \cup Y, \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_{12} \cup \mathcal{B}_{21})$ is a GDD(9,15;1,2). Hence, the next theorem records our desired result.

Theorem 3.9. *There exists a GDD(9,15;1,2).*

3.3 GDD(25,33;1,2)

Here we let $X = \{x_0, x_1, \ldots, x_{24}\}$ and $Y = \{0, \ldots, 24\} \cup \{a, b, c\} \cup \{z_1, \ldots, z_5\}$. Note that $|X| = 25$ and $|Y| = 33$. Consider $K_{25}(X) \vee_2 K_{33}(Y)$. Since $t_1 + t_2 = 1$, we give a design for $t_1 = 0$ and $t_2 = 1$. Let $\{a, b, c\}$ be the only triple from the same group. That is, $\mathcal{B}_1 = \emptyset$ and $B_2 = \{\{a, b, c\}\}$. Other triples must be constructed from an edge in $K_{25}(X)$ with a vertex in $K_{33}(Y)$ or an edge in $K_{33}(Y) - K_3(a, b, c)$ with a vertex in $K_{25}(X)$.

A ρ-labeling of a graph is an injection from the vertices of the graph with q edges to the set $\{0, 1, \ldots, 2q\}$, where if the edge labels induced by the absolute value of the difference of the vertex labels are $\{a_1, a_2, \ldots, a_a\}$, then $a_i = i$ or $a_i = 2q + 1 - i$. Rosa introduced this kind of labeling in 1967 [13] and proved the following result.

Theorem 3.10. *For a graph R with q edges, the complete graph K_{2q+1} can be decomposed into copies of R if and only if R has a ρ-labeling.*

Lemma 3.11. *There exists a graph decomposition of $K_{33} - K_3$ into 25 isomorphic spanning subgraphs.*

Proof. We will partition $K_{33}(Y) - K_3(a, b, c)$ where $Y = \{0, \ldots, 24\} \cup \{a, b, c\} \cup \{z_1, \ldots, z_5\}$ into 25 isomorphic subgraphs. First consider $K_{25}(0, 1,$

$\ldots, 24)$ a complete subgraph of G. By Theorem 3.10, $K_{25}(0, 1, \ldots, 24)$ can be decomposed into copies of $R = C_7 \cup 5K_2$ if R has a ρ-labeling. Figure 2 illustrates R with its ρ-labeling.

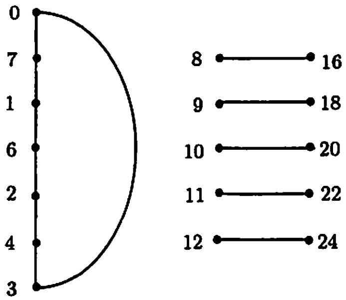

Figure 2: $R_0 = C_7 \cup 5K_2$ with a ρ-labeling.

For $i = 0, 1, \ldots, 24$, we obtain R_i the i^{th} copy of R in $K_{25}(0, 1, \ldots, 24)$ by adding i to each vertex in R modulo 25. Now we extend each copy of R_i in $K_{25}(0, 1, \ldots, 24)$ to $F_i \cong C_7 \cup 13K_2$ in K_{33} as in Figure 3.

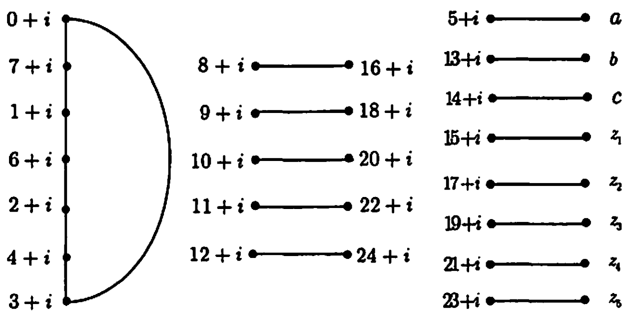

Figure 3: The subgraph $F_i = C_7 \cup 13K_2$ in $K_{33}(V(G))$.

Besides, for $i = 0, 1, \ldots, 24$, we create each subgraph H_i by adding a single edge to F_i; that is $H_i = F_i + e_i$ where

$e_0 = z_1z_5$	$e_1 = z_2z_4$	
$e_2 = cz_1$	$e_3 = bz_1$	$e_4 = az_1$
$e_5 = z_1z_2$	$e_6 = z_3z_5$	
$e_7 = az_2$	$e_8 = bz_2$	$e_9 = cz_2$
$e_{10} = z_2z_3$	$e_{11} = z_1z_4$	
$e_{12} = bz_3$	$e_{13} = cz_3$	$e_{14} = az_3$
$e_{15} = z_3z_4$	$e_{16} = z_2z_5$	
$e_{17} = cz_4$	$e_{18} = az_4$	$e_{19} = bz_4$
$e_{20} = z_4z_5$	$e_{21} = z_1z_3$	
$e_{22} = az_5$	$e_{23} = bz_5$	$e_{24} = cz_5$

Then $H_i \cong C_7 \cup P_4 \cup 11K_2$ has 33 vertices with the degree sequence $1^{24}2^9$. It can be directly verified that $K_{33}(Y) - K_3(a,b,c)$ decomposes into $\{H_0, H_1, \ldots, H_{24}\}$. □

Each of the 25 isomorphic subgraphs in the previous lemma induces a set of triples containing a vertex in X. Each edge uv in H_i gives rise to the triple $\{x_i, u, v\}$. Hence,

$$\mathcal{B}_{12} = \{\{x_i, u, v\} \mid uv \in E(H_i), i = 0, \ldots, 24\}.$$

It now remains to determine $\mathcal{B}_{21}$. For $i = 0, 1, \ldots, 24$, the vertex $x_i \in X$ must occur in the same triple with each vertex in Y twice, however, there remain 24 vertices of degree one in each H_i. These vertices must meet x_i again in $\mathcal{B}_{21}$. The following symmetric partial Latin square of order 25 with the i^{th} row contains all elements of degree one in H_i.

$\oplus$	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	x_{16}	x_{17}	x_{18}	x_{19}	x_{20}	x_{21}	x_{22}	x_{23}	x_{24}
x_0		16	z_2	17	c	18	5	19	$z3$	20	8	21	9	22	10	23	11	b	12	24	13	z_4	14	a	15
x_1	16		17	a	18	z_5	19	6	20	b	21	9	22	10	23	11	24	12	z_1	13	0	14	c	15	z_3
x_2	z_2	17		18	z_3	19	a	20	7	21	b	22	10	23	11	24	12	0	13	z_5	14	1	15	z_4	16
x_3	17	a	18		19	c	20	z_5	21	8	22	z_3	23	11	24	12	0	13	1	14	z_2	15	2	16	z_4
x_4	c	18	z_3	19		20	z_4	21	z_5	22	9	23	z_2	24	12	0	13	1	14	2	15	b	16	3	17
x_5	18	z_5	19	c	20		21	z_3	22	a	23	10	24	z_4	0	13	1	14	2	15	3	16	b	17	4
x_6	5	19	a	20	z_4	21		22	c	23	z_1	24	11	0	b	1	14	2	15	3	16	4	17	z_2	18
x_7	19	6	20	z_5	21	z_3	22		23	z_4	24	c	0	12	1	b	2	15	3	16	4	17	5	18	z_1
x_8	z_3	20	7	21	z_5	22	c	23		24	a	0	z_1	1	13	2	z_4	3	16	4	17	5	18	6	19
x_9	20	b	21	8	22	a	23	z_4	24		0	z_5	1	z_1	2	14	3	z_3	4	17	5	18	6	19	7
x_{10}	8	21	b	22	9	23	z_1	24	a	0		1	z_4	2	c	3	15	4	z_5	5	18	6	19	7	20
x_{11}	21	9	22	z_3	23	10	24	c	0	z_5	1		2	b	3	z_2	4	16	5	a	6	19	7	20	8
x_{12}	9	22	10	23	z_2	24	11	0	z_1	1	z_4	2		3	z_5	4	c	5	17	6	a	7	20	8	21
x_{13}	22	10	23	11	24	z_4	0	12	1	z_1	2	b	3		4	a	5	z_2	6	18	7	z_5	8	21	9
x_{14}	10	23	11	24	12	0	b	1	13	2	c	3	z_5	4		5	z_1	6	z_2	7	19	8	z_4	9	22
x_{15}	23	11	24	12	0	13	1	b	2	14	3	z_2	4	a	5		6	z_5	7	c	8	20	9	z_1	10
x_{16}	11	24	12	0	13	1	14	2	z_4	3	15	4	c	5	z_1	6		7	b	8	z_3	9	21	10	a
x_{17}	b	12	0	13	1	14	2	15	3	z_3	4	16	5	z_2	6	z_5	7		8	z_1	9	a	10	22	11
x_{18}	12	z_1	13	1	14	2	15	3	16	4	z_5	5	17	6	z_2	7	b	8		9	c	10	z_3	11	23
x_{19}	24	13	z_5	14	2	15	3	16	4	17	5	a	6	18	7	c	8	z_1	9		10	z_2	11	z_3	12
x_{20}	13	0	14	z_2	15	3	16	4	17	5	18	6	a	7	19	8	z_3	9	c	10		11	z_1	12	b
x_{21}	z_4	14	1	15	b	16	4	17	5	18	6	19	7	z_5	8	20	9	a	10	z_2	11		12	c	13
x_{22}	14	c	15	2	16	b	17	5	18	6	19	7	20	8	z_4	9	21	10	z_3	11	z_1	12		13	z_2
x_{23}	a	15	z_4	16	3	17	z_2	18	6	19	7	20	8	21	9	z_1	10	22	11	z_3	12	c	13		14
x_{24}	15	z_3	16	z_4	17	4	18	z_1	19	7	20	8	21	9	22	10	a	11	23	12	b	13	z_2	14	

The above symmetric partial Latin square yields the triangles

$$\mathcal{B}_{21} = \{\{x_i, x_j, x_i \oplus x_j\} | \ 0 \le i \ne j \le 24\}.$$

Therefore we successfully conclude our next desired result.

Theorem 3.12. *There exists a GDD(25,33;1,2).*

4 Acknowledgement

The authors would like to thank the anonymous reviewers for their valuable comments and suggestions to improve the quality of the paper.

References

[1] A. Chaiyasena, S.P. Hurd, N. Punnim and D. G. Sarvate, Group Divisible Designs with two association classes, *J. Combin. Math. Combin. Comput.* to appear.

[2] C. J. Colbourn and D. H. Dinitz (eds), *Handbook of Combinatorial Designs(2^{nd} Edition)* (Chapman and Hall, CRC Press, Boca Raton, 2007).

[3] S. I. El-Zanati, N. Punnim and C. A. Rodger, Gregarious GDDs with Two Associate Classes, *Graphs Combin.* 26 (2010) No.6,775–780.

[4] H. L. Fu, C. A. Rodger and D.G. Sarvate, The Existence of Group Divisible Designs with First and Second Associates, Having Block Size 3, *Ars Combin.* 54 (2000), 33–50.

[5] H. L. Fu and C. A. Rodger, Group Divisible Designs with Two Associate Classes: $n = 2$ or $m = 2$, *J. Combin. Theory (A)* 83 (1998), 94–117.

[6] J.A. Gallian, A Dynamic Survey of Graph Labeling, *Electron. J. Combin.* 16 (2009), #DS6.

[7] L. Heffter, Über Nachbarconfigurationen, Tripesysteme and metacyklische Grupen, *Deutsche Mathem. Vereining. Jahresber.*, 5 (1896), 67–69.

[8] A. Kotzig and A. Rosa, Nearly Kirkman triple system, *Proceedings of the Fifth SE Conference on Combinatorics, Graph Theory and Computing, Boca Raton, FL*, (1974), 607–614.

[9] C. C. Lindner and C. A. Rodger, *Design Theory* (CRC Press, Boca Raton, 1997).

[10] R. Peltesohn, Eine Lösung der beiden Heffterschen Differenzenprobleme, *Composito Math.* 6 (1939), 251–257.

[11] N. Pabhapote and N. Punnim, Group Divisible Designs with Two Associate Classes and $\lambda_2 = 1$, *Int. J. Math. Math. Sci.* (2011), Article ID 148580, 10 pages, doi:10.1155/2011/148580.

[12] D. Raghavarao, *Construction and Combinatorial Problems in Design of Experiments* (Dover, New York, 1988).

[13] A. Rosa, On Certain Valuations of the Vertices of a Graph, *Theory of Graphs (Internat. Symposium, Rome, July 1966), Gordon and Breach, N.Y. and Dunod Paris* (1967), 349–355.

[14] C. Uiyyasathian and W. Lapchinda, Group Divisible Designs with Two Associate Classes and $\lambda_2 = 2$, *Int. J. Pure Appl. Math.* 55(2009) No.4, 561–568.