

Computing the Ramsey Number $R(K_5 - P_3, K_5)$

Jesse A. Calvert
Department of Mathematics,
Washington University
St. Louis, MO 63105
jcalvert@wustl.edu

Michael J. Schuster
Department of Mathematics, North
Carolina State University
Raleigh, NC 27695
mjschust@ncsu.edu

Stanisław P. Radziszowski
Department of Computer Science, Rochester Institute of Technology
Rochester, NY 14623
spr@cs.rit.edu

Abstract. We give a computer-assisted proof of the fact that $R(K_5 - P_3, K_5) = 25$. This solves one of the three remaining open cases in Hendry's table, which listed the Ramsey numbers for pairs of graphs on 5 vertices. We find that there exist no $(K_5 - P_3, K_5)$ -good graphs containing a K_4 on 23 or 24 vertices, where a graph F is (G, H) -good if F does not contain G and the complement of F does not contain H . The unique $(K_5 - P_3, K_5)$ -good graph containing a K_4 on 22 vertices is presented.

1 Introduction

For simple graphs G and H , a (G, H) -good graph is a graph F that contains no subgraph G and whose complement contains no subgraph H . A $(G, H; n)$ -good graph is a (G, H) -good graph on n vertices. We will denote the set of all (G, H) -good graphs by $\mathcal{R}(G, H)$ and, similarly, the set of all $(G, H; n)$ -good graphs by $\mathcal{R}(G, H; n)$. The minimum number of vertices n such that no $(G, H; n)$ -good graph exists is the Ramsey number $R(G, H)$.

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The best known bounds for various types of Ramsey numbers are listed in the dynamic survey *Small Ramsey Numbers* by the third author [8]. For a comprehensive overview of Ramsey numbers and general graph theory terminology not defined in this paper we recommend a widely used textbook by West [9]. P_k denotes a path on k vertices, and $K_5 - P_3$ can be seen either as a K_5 with two adjacent edges removed or a K_4 with an additional vertex connected to two of its vertices.

In 1989, Hendry [5] compiled a table of known values and bounds on Ramsey numbers for connected graphs G and H on five vertices. For the Ramsey number $R(K_5 - P_3, K_5)$ the Hendry's table gives the bound $R(K_5 - P_3, K_5) \leq 28$; a lower bound of 25 can be obtained from the result $R(K_4, K_5) = 25$ [7]. In the 2009 REU (NSF Research Experiences for Undergraduates Program) Black, Leven and Radziszowski [1] showed that the upper bound can be reduced to $R(K_5 - P_3, K_5) \leq 26$. The main goal of the 2010 REU was to show that $R(K_5 - P_3, K_5) = 25$ or $R(K_5 - P_3, K_5) = 26$, which was accomplished using a combination of combinatorial reasoning and computation. The computations required to show that $R(K_5 - P_3, K_5) = 25$ were easily completed on a standard desktop computer. However, the computation of the number of $(K_5 - P_3, K_5)$ -good graphs containing K_4 on less than 25 vertices was much longer. We found that there were no $(K_5 - P_3, K_5)$ -good graphs on 24 or 23 vertices, exactly one on 22 vertices, and millions on 21.

The general question of characterizing graphs G , H and extensions G' of G , for which the equality $R(G, H) = R(G', H)$ holds, is very difficult. Only a few such cases are known, and some of them are presented in Section 5. Our detailed study of $(K_5 - P_3, K_5)$ -good graphs seems to provide evidence that, at least sometimes, avoiding larger graph G' may be not much stronger than avoiding G . We expect that many other interesting cases exist for which $R(G, H) = R(G', H)$.

Section 2 presents two enumerations of smaller graphs needed later in the paper, the algorithm foundations and computations showing the main results are described in Sections 3 and 4, respectively, and finally Section 5 points out how our result relates to a general 1989 theorem by Burr, Erdős, Faudree and Schelp [3].

2 Enumerations for $\mathcal{R}(K_5 - P_3, K_5)$

In order to study $\mathcal{R}(K_5 - P_3, K_5)$, it is useful to have enumerations of the sets $\mathcal{R}(K_4 - P_3, K_5)$ and $\mathcal{R}(K_5 - P_3, K_4)$. It is known that $R(K_4 - P_3, K_5) = 14$ and $R(K_5 - P_3, K_4) = 18$ [4]. We have generated the corresponding sets of graphs using a simple vertex by vertex extension algorithm, and McKay's *nauty* package [6] to eliminate isomorphs. The

1092 nonisomorphic graphs in $\mathcal{R}(K_4 - P_3, K_5)$ and the 3454499 nonisomorphic graphs in $\mathcal{R}(K_5 - P_3, K_4)$ were enumerated. The results agreed with the computations reported in [1], and the data is summarized in Tables I and II (two typographical errors in [1] were corrected). We include the tables here in full since they are needed to see the context of computations performed to obtain our results.

n	$ \mathcal{R}(K_5 - P_3, K_4; n) $	#edges	#graphs with K_4	#edges
2	2	0-1	0	
3	4	0-3	0	
4	10	1-6	1	6
5	26	2-8	2	6-7
6	92	3-12	8	6-12
7	391	5-16	29	7-12
8	2228	7-21	149	8-16
9	15452	9-27	751	10-19
10	107652	12-31	3946	12-24
11	557005	15-36	10649	15-28
12	1455946	18-40	6780	18-32
13	1184231	33-45	0	
14	130816	41-50	0	
15	640	50-55	0	
16	2	60	0	
17	1	68	0	

Table I. Statistics of $\mathcal{R}(K_5 - P_3, K_4)$.

The last two columns of Table I give the counts and the corresponding edge ranges of all $(K_5 - P_3, K_4)$ -good graphs which contain K_4 as a subgraph, i.e. of all graphs which are $(K_5 - P_3, K_4)$ -good but not (K_4, K_4) -good. We will show in Section 4 that a similar type of distribution occurs in $\mathcal{R}(K_5 - P_3, K_5)$.

In Table II, the last two columns present counts and the corresponding edge ranges of all $(K_4 - P_3, K_5)$ -good graphs which contain K_3 as a subgraph, or equivalently, those graphs which are $(K_4 - P_3, K_5)$ -good but not (K_3, K_5) -good.

n	$ \mathcal{R}(K_4 - P_3, K_5; n) $	#edges	#graphs with K_3	#edges
2	2	0-1	0	
3	4	0-3	1	3
4	8	0-4	1	3
5	15	1-6	2	3-4
6	36	2-9	4	3-6
7	78	3-12	7	4-7
8	190	4-16	11	5-9
9	308	6-17	18	6-12
10	326	8-20	13	8-13
11	110	10-22	5	10-15
12	13	12-24	1	12
13	1	26	0	

Table II. Statistics of $\mathcal{R}(K_4 - P_3, K_5)$.

3 Properties of $(K_5 - P_3, K_5)$ -good Graphs

Since $R(K_4, K_5) = 25$ [7], any $(K_5 - P_3, K_5; 25)$ -good graph F contains at least one K_4 . Let x be the vertex of this K_4 with the smallest degree. We denote by F_x^+ the graph induced by the neighborhood of vertex x and by F_x^- the graph induced by the anti-neighborhood of vertex x (non-neighbors of x , not including x). Note that F_x^+ must be a $(K_4 - P_3, K_5)$ -good graph, while F_x^- must be a $(K_5 - P_3, K_4)$ -good graph.

Lemma 1 *For $n \geq 4$, if F is a $(K_5 - P_3, K_5; n)$ -good graph containing a K_4 , then the sum of the degrees of the vertices in any K_4 contained in F is at most $n + 8$.*

Proof: Let d_1, d_2, d_3 , and d_4 be the degrees of vertices of a K_4 in F . The neighborhoods of each vertex in this K_4 must be disjoint, otherwise we have a $K_5 - P_3$ subgraph in F . Hence

$$\sum_{i=1}^4 (d_i - 3) + 4 \leq n, \text{ or } \sum_{i=1}^4 d_i \leq n + 8$$

□

For a given n , we can determine the maximum of the minimum degree vertex x in the K_4 under consideration, and thus the possible values of

$|V(F_x^+)|$ and $|V(F_x^-)|$. Note that $|V(F_x^+)| + |V(F_x^-)| = n - 1$ in each case. All possibilities for $n \geq 22$ are summarized in Table III, where column 2 shows the upper bound of Lemma 1, and column 3 is the upper bound $\lfloor (n + 8)/4 \rfloor$ on the minimum degree vertex x in the K_4 under consideration.

n	$n + 8$	$\lfloor n/4 \rfloor + 2$	$ V(F_x^+) $	$ V(F_x^-) $
25	33	8	7	17
			8	16
24	32	8	6	17
			7	16
			8	15
23	31	7	5	17
			6	16
			7	15
22	30	7	4	17
			5	16
			6	15
			7	14

Table III. Possible parameters of $(K_5 - P_3, K_5; n)$ -good graphs containing K_4 , for $n \geq 22$.

Let F be a $(K_5 - P_3, K_5; 25)$ -good graph containing K_4 . By Table I and Table II, there are only 3 possible graphs for F_x^- and 18 possible graphs (with a K_3) for F_x^+ . The computation we ran determined the possible ways these graphs can be connected. Given a vertex v in F_x^+ , we define the *cone* of v as the set of all the vertices adjacent to v in F_x^- . There are many restrictions we can place on these cones with the given parameters:

- (C1) *The cones of any two vertices in any K_3 in F_x^+ must be disjoint. Otherwise, the vertices of this K_3 , x and any vertex in the intersection of two of these cones will create a $K_5 - P_3$.*
- (C2) *The complement in $V(F_x^-)$ of the union of the cones of any two non-adjacent vertices, a and b , must not contain an independent set of order 3. Otherwise this independent set together with a and b will be an independent set of order 5.*
- (C3) *The complement in $V(F_x^-)$ of the cones of any three non-adjacent vertices, a , b and c , must not contain an independent set of order 2*

(that is, it must be complete). Otherwise we again have an independent set of order 5 with the vertices a, b, c , and any two non-adjacent vertices in the complement.

- (C4) *The intersection of the cones of any two adjacent vertices, a and b , must not contain an edge.* Otherwise the vertices a, b, x and the vertices connected by this edge will create a $K_5 - P_3$.

These constraints are not exhaustive enough to fully characterize $(K_5 - P_3, K_5)$ -good graphs, but they are sufficiently restrictive that we will be able to prove that $R(K_5 - P_3, K_5) = 25$ and find the sole $(K_5 - P_3, K_5; 22)$ -good graph.

4 Computation of $R(K_5 - P_3, K_5)$

As shown in Figure 1, all possible neighborhoods of x in $V(F)$ on 25 vertices can be constructed from a triangle and two independent vertices. The highly similar substructure of F_x^+ will allow us to eliminate constructions without having to attempt to arrange all 7 or 8 cones. For the three $(K_5 - P_3, K_4)$ -good graphs on 16 and 17 vertices we find arrangements of cones on five vertices, either a triangle and two non-adjacent vertices or a triangle and two adjacent vertices. The first three vertices form a triangle and its cones are subject to condition (C1). If the last two vertices are adjacent their cones must satisfy condition (C4), otherwise (C2). The last two vertices are also independent to each vertex in the triangle, and must satisfy condition (C2) with these vertices as well. Finally, if the last two vertices are independent they, along with any of the vertices in the triangle, must satisfy condition (C3).

Theorem 1

- (1) $R(K_5 - P_3, K_5) = 25$.
- (2) *There are no $(K_5 - P_3, K_5; 22)$ -good graph containing a K_4 on 23 or 24 vertices, and there is a unique $(K_5 - P_3, K_5; 22)$ -good graph which contains a K_4 .*

Proof: The proof is computational.

- (1) For the three $(K_5 - P_3, K_4)$ -good graphs on $|V(F_x^-)| = 16$ and 17 vertices there were no valid arrangements of five cones. In fact, there is no valid arrangement of 4 cones for the graph on 17 vertices. Therefore the result follows.

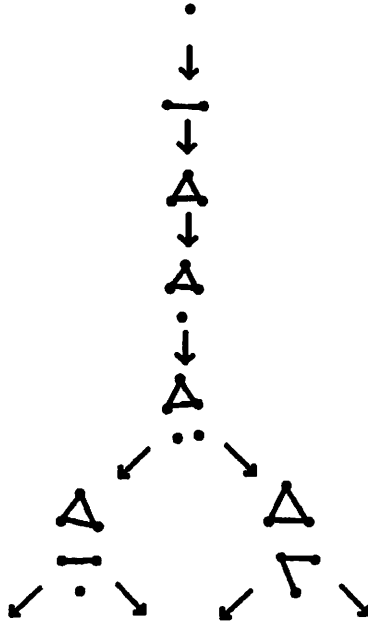


Figure 1: Two constructions of F_x^+ on up to 6 vertices form a subgraph of all graphs induced by the apexes of cones needed in computations. All F_x^+ must contain K_3 with two isolated vertices.

- (2) When we run the algorithm building cone arrangements for $|V(F_x^-)| = 15$ we find that there is no valid arrangement of 6 cones. This eliminates the possibility of creating a $(K_5 - P_3, K_5; n)$ -good graph with a K_4 for $n = 23, 24, 25$. For $|V(F_x^-)| = 14$ we find that there is one valid arrangement of 7 cones. This is for $F_x^+ = C_3 \cup C_4$, and it gives us exactly one $(K_5 - P_3, K_5; 22)$ -good graph with a K_4 , whose adjacency matrix is given in Figure 2.

□

The main computations were performed at least twice with independent implementations by the first two authors. They agreed on the number of possible cone arrangements in all cases and on the final results.

1	0	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	0	1	1	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0
3	1	1	0	1	0	0	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0
4	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1
5	1	0	0	0	0	1	1	0	1	0	1	0	1	1	0	0	0	0	1	0	1	0
6	1	0	0	0	1	0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	0
7	1	0	0	0	1	0	0	1	0	1	0	1	0	1	0	0	0	1	0	0	1	0
8	1	0	0	0	0	1	1	0	1	0	0	1	0	0	1	0	1	1	0	0	0	1
9	0	1	0	0	1	0	0	1	0	1	0	1	1	0	1	0	0	0	0	1	1	0
10	0	1	0	0	0	1	1	0	1	0	1	0	0	0	1	1	0	1	0	1	0	0
11	0	1	0	0	1	0	1	0	0	1	0	1	0	0	0	1	1	0	1	0	0	1
12	0	1	0	0	0	1	0	1	1	0	1	0	0	1	0	0	1	0	0	0	1	1
13	0	0	1	0	1	0	1	0	1	0	0	0	0	1	1	0	0	1	1	0	0	1
14	0	0	1	0	1	1	0	0	0	0	0	1	1	0	0	1	0	1	0	1	0	1
15	0	0	1	0	0	0	0	1	1	1	0	0	1	0	0	1	0	0	1	0	1	1
16	0	0	1	0	0	1	0	0	0	1	1	0	0	1	1	0	0	0	0	1	1	1
17	0	0	1	0	0	0	1	1	0	0	1	1	0	0	0	0	0	1	1	1	1	0
18	0	0	0	1	0	1	0	1	0	1	0	0	1	1	0	0	1	0	1	1	0	0
19	0	0	0	1	1	1	0	0	0	0	1	0	1	0	1	0	1	1	0	0	1	0
20	0	0	0	1	0	0	1	0	1	1	0	0	0	1	0	1	1	1	0	0	1	0
21	0	0	0	1	1	0	0	0	1	0	0	1	0	0	1	1	1	0	1	1	0	0
22	0	0	0	1	0	0	1	1	0	0	1	1	1	1	1	0	0	0	0	0	0	0

Figure 2: The unique $(K_5 - P_3, K_5; 22)$ -good graph with a K_4 . Vertices 1 through 4 form K_4 , x is the first vertex, vertices 5 through 8 induce C_4 , and vertices 9 through 22 are those in $V(F_x^-)$

5 Some Related Ramsey Numbers

Burr, Erdős, Faudree and Schelp [3] proved a theorem showing that certain small extensions of complete graphs don't increase the Ramsey number. Let $\widehat{K}_{n,p}$ be the unique graph obtained by connecting a new vertex v to p vertices of a K_n .

Theorem 2 [3] For $m, n \geq 3$ and $m + n \geq 8$,

$$R(\widehat{K}_{m,p}, \widehat{K}_{n,q}) = R(K_m, K_n)$$

$$\text{with } p = \left\lceil \frac{m}{n-1} \right\rceil \text{ and } q = \left\lceil \frac{n}{m-1} \right\rceil.$$

Note that this theorem also implies $R(\widehat{K}_{m,p}, K_n) = R(K_m, K_n)$. For the case $m = 4$ and $n = 5$, this theorem shows that $R(\widehat{K}_{4,1}, \widehat{K}_{5,2}) = 25$, which does not prove $R(\widehat{K}_{4,2}, K_5) = R(K_5 - P_3, K_5) = 25$. However, using Theorem 1 and slightly modifying the proof of Theorem 2, we can further show that $R(\widehat{K}_{4,2}, \widehat{K}_{5,2}) = 25$.

Theorem 3 *All of the following Ramsey numbers are equal to 25.*

- (1) $R(K_4, \widehat{K}_{5,2})$,
- (2) $R(\widehat{K}_{4,1}, K_5)$,
- (3) $R(\widehat{K}_{4,1}, \widehat{K}_{5,2})$,
- (4) $R(\widehat{K}_{4,2}, K_5)$,
- (5) $R(\widehat{K}_{4,2}, \widehat{K}_{5,2})$.

Proof:

(1-3) Directly from Theorem 2.

(4) Proved in Section 4.

(5) Take a $(\widehat{K}_{4,2}, \widehat{K}_{5,2})$ -good coloring of a K_{25} . Then there must be a blue K_5 . By Theorem 2 and the fact that $R(K_3, K_5) = 14$, we have that $R(\widehat{K}_{3,1}, \widehat{K}_{5,2}) = 14$. Therefore, in the 20 vertices not contained in the blue K_5 there must be a red $\widehat{K}_{3,1}$. Each vertex of the blue K_5 is adjacent in blue to at least one vertex of the red $\widehat{K}_{3,1}$. So some vertex of the red $\widehat{K}_{3,1}$ is adjacent in blue to at least 2 vertices of the blue K_5 , creating a blue $\widehat{K}_{5,2}$.

□

To conclude, we note that the difficulty of Theorem 3.4, namely the title case of this paper, is apparently far greater than that of all other cases covered by Theorem 3. Better understanding of this difference could lead to an improvement of Theorem 2 covering all manageable small extensions of complete graphs.

Note. Recently, Boza [2] obtained the equality $R(K_5 - P_3, K_5) = 25$ using an approach very different from ours.

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