On f-Edge Covered Critical Graphs *

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Abstract Let $G(V, E)$ be a simple graph, and let f be an integer function defined on V with $1 \leq f(v) \leq d(v)$ to each vertex $v \in V$. An f-edge covered colouring is an edge colouring C such that each colour appears at each vertex v at least $f(v)$ times. The maximum number of colours needed to f-edge covered colour G is called the f-edge covered chromatic index of G and denoted by $\chi'_{fc}(G)$. Any simple graph G has an f-edge covered chromatic index equal to δ_f or $\delta_f - 1$, where $\delta_f = \min\{\lfloor \frac{d(v)}{f(v)} \rfloor : v \in V(G)\}$. Let G be a connected and not complete graph with $\chi'_{fc} = \delta_f - 1$. If for each $u, v \in V$ and $e = uv \notin E$, we have $\chi'_{fc}(G + e) > \chi'_{fc}(G)$, then G is called an f-edge covered critical graph. In this paper, some properties on f-edge covered critical graph are discussed. It is proved that if G is an f-edge covered critical graph, then for each $u, v \in V$ and $e = uv \notin E$ there exists $w \in \{u, v\}$ with $d(w) \leq \delta_f(f(w) + 1) - 2$ such that w is adjacent to at least $\max\{d(w) - \delta_f f(w) + 1, (f(w) + 2)d(w) - \delta_f(f(w) + 1)^2 + f(w) + 3\}$ vertices which are all δ_f-vertices in G.

Keywords: f-edge cover colouring; f-edge covered critical graph; f-edge cover chromatic index.
AMS subject classification: 05C15, 05C25

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*This work is supported by GIIFSDU (yzc09073), NSFC (10871119, 10901097), RFDP (200804220001) of China.
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1 Introduction

Our terminologies and notations will be standard. The reader is referred to [1] for the undefined terms. Graphs in this paper are simple and connected, unless otherwise stated, i.e., they have no loops or multiple edges. We use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of a graph G, respectively. Let $N_G(v)$ denote the neighborhood of v and let $d(v) = |N_G(v)|$ be the degree of v in G. Let f be a positive integer function defined on $V(G)$ such that $1 \leq f(v) \leq d(v)$ for any $v \in V$. A *k-edge coloring* C of a graph G is an assignment of k colours, $\{1, 2, \cdots, k\}$, to the edges of G. The colouring C is *proper* if no two adjacent edges have the same colour. Unless otherwise stated, the edge colouring of graphs in this paper are not necessarily proper. An *f-edge covered colouring* is an edge colouring C such that each colour appears at each vertex v at least $f(v)$ times. The maximum number of colours needed to f-edge covered colour G is called *the f-edge covered chromatic index* of G and denoted by $\chi'_{fc}(G)$. In our daily life many problems on optimization and network design, e.g., coding design, building blocks, the file transfer problem on computer networks, schedule problems and so on [6], are related to the f-edge cover colouring which is firstly presented by Song and Liu [7].

Let
$$\delta_f = \min\{\lfloor \frac{d(v)}{f(v)} \rfloor : v \in V(G)\},$$
in which $\lfloor \frac{d(v)}{f(v)} \rfloor$ is the largest integer not greater than $\frac{d(v)}{f(v)}$. Since $d(v) \geq f(v)\chi'_{fc}(G)$, $\chi'_{fc}(G) \leq \delta_f$. If G is a multigraph and $u, v \in V(G)$, let $E(uv)$ be the set of edges joining u and v in G. The multiplicity $\mu(uv)$ of edge uv is the size of $E(uv)$. Set $\mu(v) = \max\{\mu(uv) : u, v \in V(G)\}$, $\mu(G) = \max\{\mu(v) : v \in V(G)\}$, which are respectively called the multiplicity of a vertex v and the multiplicity of graph G. Song and Liu studied the bound of f-edge cover chromatic index of multigraphs and obtained the following result [7]. If there is no confusion, we use V, E, $N(v)$ and δ_f instead of $V(G)$, $E(G)$, $N_G(v)$ and $\delta_f(G)$.

Theorem 1.1. [7] *Let G be a multigraph. Let $f(v)$, $d(v)$, and δ_f be defined as earlier. Then*
$$\min_{v \in V}\{\lfloor \frac{d(v) - \mu(v)}{f(v)} \rfloor \leq \chi'_{fc}(G) \leq \delta_f.$$

When G is a simple graph, we have $\mu(v) = 1$ for each $v \in V$. Therefore the following corollary holds.

Corollary 1.2. *Let G be a graph. Let $f(v)$, $d(v)$, and δ_f be defined as earlier. Then*

$$\delta_f - 1 \leq \min_{v \in V}\{\lfloor \frac{d(v)-1}{f(v)} \rfloor\} \leq \chi'_{fc}(G) \leq \delta_f.$$

Clearly, the f-edge cover colouring is trivial if $\delta_f = 1$. So, in the following we can suppose that $\delta_f \geq 2$. From the above corollary, we can see that the f-edge cover chromatic index of any graph G $\delta_f \geq 2$ must be equal to δ_f or $\delta_f - 1$. This immediately gives us a simple way of classifying graphs into two types according to $\chi'_{fc}(G)$. More precisely, we say that G is of f_c-class 1 if $\chi'_{fc}(G) = \delta_f$, and that G is of f_c-class 2 if $\chi'_{fc}(G) = \delta_f - 1$. In general, the problem of determining the f-edge cover chromatic index of graphs is NP-hard because deciding edge covering chromatic index of graphs is NP-complete [10], which is the special case of our general edge covering problem.

If $f(v) = 1$ for any $v \in V(G)$, then the f-edge cover colouring problem reduces to the classical edge covering colouring of G which is an edge colouring such that the edges assigned the same colour formed an edge cover of G. The *edge cover chromatic index* $\chi'_c(G)$ of G is the maximum size of a partition of $E(G)$ into edge covers of G. By Corollary 1.2,

$$\delta - 1 \leq \chi'_c(G) \leq \delta,$$

which is also an important result of Gupta [2].

Let G be a connected and not complete graph with $\chi'_{fc}(G) = \delta_f(G) - 1$. If for any $u, v \in V$ and $e = uv \notin E$, $\chi'_{fc}(G + e) = \delta_f(G)$, then G is called an *f-edge covered critical graph*. $e = uv \notin E$ is called a *critical edge* of G.

We call v a *δ_f-vertex* if $d(v) = \delta_f f(v)$. In [11], Wang, et. al. studied some properties on the f-edge covered critical graph and obtained the following result.

Theorem 1.3. *Let G be an f-edge covered critical graph. Then for each $u, v \in V$ and $e = uv \notin E$, there exists $w \in \{u, v\}$ with $d(w) \leq \delta_f(f(w) + 1) - 2$ such that w is adjacent to at least $d(w) - \delta_f f(w) + 1$ vertices which are all δ_f-vertex in G.*

If $f(v) = 1$ for all $v \in V$, then the δ_f-vertex is the vertex with minimum degree δ. Song and Liu [8] studied the properties of edge covered critical graph and gave a result which can also be obtained by Theorem 1.3.

Corollary 1.4. *Let G be an edge covered critical graph. Then for each $u, v \in V$ and $e = uv \notin E$, there exists $w \in \{u, v\}$ with $d(w) \leq 2\delta - 2$ such that w is adjacent to at least $d(w) - \delta + 1$ vertices of degree δ in G.*

In this paper, we also studied some properties on the f-edge covered critical graph. The following theorem is our main result.

Theorem 1.5. *Let G be an f-edge covered critical graph. Then for each $u, v \in V$ and $e = uv \notin E$, there exists $w \in \{u, v\}$ with $d(w) \leq \delta_f(f(w)+1)-2$ such that w is adjacent to at least $(f(w)+2)d(w)-\delta_f(f(w)+1)^2+f(w)+3$ vertices which are all δ_f-vertex in G.*

The following result obtained by Song and Liu [8] can be easily obtained by Theorem 1.5.

Corollary 1.6. *Let G be an edge covered critical graph. Then for each $u, v \in V$ and $e = uv \notin E$, there exists $w \in u, v$ with $d(w) \leq 2\delta - 2$ such that w is adjacent to at least $3d(w) - 4\delta + 4$ vertices of degree δ in G.*

By Theorem 1.3 and Theorem 1.5, we have the following corollary.

Corollary 1.7. *Let G be an edge covered critical graph. Then for each $u, v \in V$ and $e = uv \notin E$, there exists $w \in u, v$ with $d(w) \leq 2\delta - 2$ such that w is adjacent to at least $\max\{d(w) - \delta_f f(w) + 1, (f(w) + 2)d(w) - \delta_f(f(w) + 1)^2 + f(w) + 3\}$ vertices of degree δ in G.*

2 Definitions and preliminary results

Given a k-edge colouring C of a graph G, we shall denote by $c_i(v)$ the number of colour i represented at v. For any $v \in V$ and $1 \leq i \leq k$, let

$$\sigma_i(v) = \max\{0, f(v) - c_i(v)\}, \ \sigma(v) = \sum_{i=1}^{k} \sigma_i(v).$$

For any $v \in V$ and $1 \leq i \leq k$, if $c_i(v) > f(v)$, i is called an *u-proper colour of v*, and each edge coloured i incident v is called an *u-proper edge of v*. Correspondingly, *f-proper colour of v* and *f-proper edge of v* are defined when $c_i(v) = f(v)$, *l-proper colour of v* and *l-proper edge of v* are defined when $c_i(v) < f(v)$. Clearly, C is an f-edge cover colouring of G if and only if $\sigma(v) = 0$ for all vertex $v \in V$. Let G be an f-edge covered critical graph. Then for each $u, v \in V$ and $e = uv \notin E$ there exists an f-edge cover colouring of $G + e$ with δ_f colours. Restricting this colouring to G, we get a δ_f-edge colouring C of G which satisfies:

(1) $\sigma(x) = 0$, for all $x \in V - \{u, v\}$.

(2) $\sigma(u) = 1$ or $\sigma(v) = 1$.

Let C be a k-edge colouring of G and $\sigma = \Sigma_{v \in V} \sigma(v)$. If we can recolour the edges of G with k colours and get a new colouring C' of G such that $\sigma' < \sigma$, we say that C' is an *improved colouring* of G. If C can not be improved, we call C a *good colouring* of G. Clearly, for an f-edge covered critical graph G and a critical edge $e = uv \notin E$, there must be a good colouring of G with δ_f colours satisfies (1) and (2).

Let C be an edge colouring of a graph G, and let α and β be two of the used colours. Let $G(v; \beta, \alpha)$ be the component containing v of the subgraph of G induced by the edges coloured α and β. A (β, α) exchange chain L of G is a sequence $(v_0, e_1, v_1, e_2, \cdots, v_{r-1}, e_r, v_r)$ of vertices and edges of G in which

(a) For $1 \leq i \leq r$, the vertices v_{i-1} and v_i are distinct and both incident with the edge e_i,

(b) The edges are all distinct and are coloured alternately by α and β,

(c) e_1 is coloured by α and $\alpha(v_0) > \beta(v_0)$. Similarly, let γ denote the colour of e_r and $\overline{\gamma}$ denote the other colour of $\{\alpha, \beta\}$. Then $\gamma(v_r) > \overline{\gamma}(v_r)$.

A (β, α) exchange chain L is called *minimal* if there does not exist another (β, α) exchange chain L' which is starting at the same vertex as L and $L' \subset L$.

A connected graph G is called Eulerian if each vertex of G has even degree. Let $E(i) = \{e \in E : C(e) = i\}$. A subgraph H of G is called an *obstruction* (about C), if $H = G(v_0; \beta, \alpha)$ is Eulerian, $|E(H)|$ is odd, $\alpha(x) = \beta(x) = f(x)$ for all $x \in V(H) - \{v\}$ and $\alpha(v) = \beta(v) + 2 = f(v) + 1$. We shall use the following lemma which can be found in [11].

Lemma 2.1. *Let C be a given good colouring of an f-edge covered critical graph G with δ_f colours, and a critical edge $e = uv \notin E$. If $w \in \{u, v\}$, $\sigma(w) = 1$, $\alpha(w) > f(w)$ and $\beta(w) < f(w)$, then $H = G(w; \alpha, \beta)$ is an obstruction and $d(w) \leq \delta_f(f(w) + 1) - 2$.*

3 The Proof of Theorem 1.5

In this section, we will prove Theorem 1.5. Let $v \in V(G)$ and $N_d(w) = \{w\} \cup \{y \in V(G) : yw \in E(G),\ yw \textit{ is coloured by an u-proper colour of } w\}$. Firstly, we give two important lemmas.

Lemma 3.1. *Let C be a given good colouring of an f-edge covered critical graph G with δ_f colours, and a critical edge $e = uv \notin E$. If $w \in \{u, v\}$ such that $\sigma(w) = 1$, then for any two vertices $u_1, u_2 \in N_d(w)$, the u-proper colours of u_1 and the u-proper colours of u_2 are all different and all the u-proper colours of each vertex in $N_d(w) - \{w\}$ are the f-proper colours of w.*

Proof. Since G is an f-edge covered critical graph, there is a good colouring C of G with δ_f colours and for each $e = uv \notin E$ there exists $w \in \{u, v\}$, $\sigma(w) = 1$. Without loss of generality, suppose that $w = u$, so we suppose $\alpha(u) < f(u)$.

If $x \in N_d(w)$ and α is an u-proper colour of x, then $x \neq u$ and $\beta = c(ux)$ is an u-proper colour of u. Let $H = G(u; \alpha, \beta)$. Then H contains x, and $\alpha(x) > f(x)$, contradicting to Lemma 2.1. So all the u-proper colours of each vertex in $N_d(w)$ are the u-proper colours or f-proper colours of w.

Let $u_1, u_2 \in N_d(u)$, $u_1 \neq u_2$, γ is one of the common u-proper colours of u_1 and u_2. There are three cases.

Case 1. $u \in \{u_1, u_2\}$, Without loss of generality, we set $u = u_1$, $\beta = c(uu_2)$. If $\beta = \gamma$, it is clearly that β is a common u-proper colour of u and u_2. Recolour uu_2 with α, then $\alpha(u) \geq f(u)$, $\beta(u) \geq f(u)$ and for any $v \in V - \{u\}$, $\sigma(v)$ is not changed, a contradiction. If $\beta \neq \gamma$, then β is a l-proper colour of u_2. In $G - uu_2$, choose a minimal (β, γ) exchange chain L starting at u_2. L must be in one of the following three cases.

(1) L does not stop at u_2 or u.

(2) L stops at u.

(3) L stops at u_2 with an edge coloured by γ.

We first exchange the two colours on L. If one of the cases (1) and (2) happens, recolour uu_2 by α, so we can get an improved colouring, a contradiction. When (3) happens, recolour uu_2 be γ, then $\gamma(u) \geq f(u) + 2$ and $\alpha(u) \leq f(u) - 1$. Thus the new $G(u; \gamma, \alpha)$ is not an obstruction, a contradiction.

Case 2. $u \notin \{u_1, u_2\}$, and $c(uu_1) = c(uu_2) = \beta$. By the analysis of Case 1, γ is not one of the u-proper colours of u. Choose two minimal (α, γ) exchange chain starting at u_1 and u_2, respectively. If they both stop with an edge coloured by γ at u_1 and u_2, respectively, then go on the choices by the edges coloured by α incident with u_1 and u_2. Since $\alpha(u) < f(u)$ and $\gamma(u) = f(u)$, there exists a (α, γ) exchange chain doesn't stop at u. Suppose that (α, γ) exchange chain L_1 starting at u_1 doesn't stop at u.

It is clearly that L_1 doesn't stop at u_1 and doesn't contain u. We first exchange the two colours on L_1, so α is an u-proper colour of u_1 and uu_1 is an u-proper edge of u in the new colouring. So the new colouring we obtained is a good colouring. Let $H = G(u;\alpha,\beta)$. Then H contains u_1 and $\alpha(u_1) \geq f(u_1)+1$. By Lemma 2.1, $G(u;\alpha,\beta)$ is not an obstruction, a contradiction.

Case 3. $u \notin \{u_1,u_2\}$, and $\beta_1 = c(uu_1) \neq c(uu_2) = \beta_2$. With the analysis above, $\gamma \neq \beta_1, \beta_2$ and the minimal (β_1,γ) exchange chain L_1 must stop at u_1 with an edge coloured by γ. Exchange the two colours of L_1 and recolour uu_1 by γ, then γ is one of the common u-proper colours of u and u_2, and uu_2 is an u-proper edge of u, a contradiction.

From the analysis above, we prove that for any two vertices $u_1, u_2 \in N_d(w)$, the u-proper colours of u_1 and the u-proper colours of u_2 are all different and all the u-proper colours of each vertex in $N_G(w) - \{w\}$ are the f-proper colours of w. The proof is completed. ■

Lemma 3.2. *Let C be a given good colouring of an f-edge covered critical graph G with δ_f colours, and a critical edge $e = uv \notin E$. If $w \in \{u,v\}$, $\sigma(w) = 1$, then w is adjacent to at least $(f(w)+2)d(w) - \delta_f(f(w)+1)^2 + f(w)+3$ vertices which are all δ_f-vertex in G.*

Proof. Set k be the number of the δ_f-vertex adjacent to w in G. Then there is at most k δ_f-vertex in $N_d(w) - \{w\}$. It is easy to prove that each colour is represented at v at most $f(v)+1$ times for any $v \in N_d(w)$. By Lemma 3.1, for any two vertices $u_1, u_2 \in N_d(w)$, the u-proper colours of u_1 and the u-proper colours of u_2 are all different and all the u-proper colours of each vertex in $N_d(w) - w$ are the f-proper colours of w. Then we have

$$d(w) - (\delta_f f(w) - 1) + \sum_{y \in N_d(w)-\{w\}} (d(y) - \delta_f f(y)) \leq \delta_f - 1$$

and

$$|N_d(w)| = (f(w)+1)(d(w) - \delta_f f(w) + 1) + 1.$$

So we have

$$d(w) - (\delta_f f(w) - 1) + (f(w)+1)(d(w) - \delta_f f(w) + 1) - k \leq \delta_f - 1.$$

Thus

$$k \geq (f(w)+2)d(w) - \delta_f(f(w)+1)^2 + f(w) + 3.$$

It is to say that w is adjacent to at least $(f(w)+2)d(w) - \delta_f(f(w)+1)^2 + f(w)+3$ vertices which are all δ_f-vertex in G. The proof is completed. ■

Proof of Theorem 1.5. Since G is an f-edge covered critical graph, for each $u, v \in V$ and $e = uv \notin E$, $G + e$ can be f-edge cover coloured by δ_f colours. So there exists a good colouring of G with δ_f colours, and satisfies $w \in \{u, v\}$, $\sigma(w) = 1$. By Lemma 3.1 and Lemma 3.2, $d(w) \leq \delta_f(f(w) + 1) - 2$ and w is adjacent to at least $(f(w)+2)d(w) - \delta_f(f(w)+1)^2 + f(w) + 3$ vertices which are all δ_f-vertex in G. ■

Other interesting results on the f-edge cover colouring and f-edge colouring can be found in [3], [4], [5], [9] and [12]. Finally, we present the following problems.

Problem 3.3. *Can we find other properties of f-edge covered critical graphs?*

Problem 3.4. *What kind of graphs have the property that $\chi'_{fc}(G) = \delta_f$?*

Acknowledgments

The authors are indebted to anonymous referees for their valuable comments.

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