Signed distance k-domatic numbers of graphs

[1]S.M. Sheikholeslami and [2]L. Volkmann

[1]Department of Mathematics
Azarbaijan University of Tarbiat Moallem
Tabriz, I.R. Iran
s.m.sheikholeslami@azaruniv.edu

[2]Lehrstuhl II für Mathematik
RWTH Aachen University
52056 Aachen, Germany
volkm@math2.rwth-aachen.de

Abstract

Let k be a positive integer and let G be a simple graph with vertex set $V(G)$. If v is a vertex of G, then the open k-neighborhood of v, denoted by $N_{k,G}(v)$, is the set $N_{k,G}(v) = \{u \mid u \neq v \text{ and } d(u,v) \leq k\}$. $N_{k,G}[v] = N_{k,G}(v) \cup \{v\}$ is the closed k-neighborhood of v. A function $f : V(G) \longrightarrow \{-1, 1\}$ is called a *signed distance k-dominating function* if $\sum_{u \in N_{k,G}[v]} f(u) \geq 1$ for each vertex $v \in V(G)$. A set $\{f_1, f_2, \ldots, f_d\}$ of signed distance k-dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \leq 1$ for each $v \in V(G)$, is called a *signed distance k-dominating family* (of functions) on G. The maximum number of functions in a signed distance k-dominating family on G is the *signed distance k-domatic number* of G, denoted by $d_{k,s}(G)$. Note that $d_{1,s}(D)$ is the classical signed domatic number $d_s(D)$. In this paper we initiate the study of signed distance k-domatic numbers in graphs and we present some sharp upper bounds for $d_{k,s}(G)$.

Keywords: signed distance k-domatic number, signed distance k-dominating function, signed distance k-domination number
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1 Introduction

In this paper, k is a positive integer and G is a finite simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. For a vertex $v \in V(G)$, the *open k-neighborhood* $N_{k,G}(v)$ is the set $\{u \in V(G) \mid u \neq v \text{ and } d(u,v) \leq k\}$ and the *closed k-neighborhood* $N_{k,G}[v]$ is the set $N_{k,G}(v) \cup \{v\}$. The *open k-neighborhood* $N_{k,G}(S)$ of a set $S \subseteq V$ is the set $\bigcup_{v \in S} N_{k,G}(v)$, and the *closed neighborhood* $N_{k,G}[S]$ of S is the set $N_{k,G}(S) \cup S$. The *k-degree* of a vertex v is defined as $\deg_{k,G}(v) = |N_{k,G}(v)|$. The minimum and maximum k-degree of a graph G are denoted by $\delta_k(G)$ and $\Delta_k(G)$, respectively. If $\delta_k(G) = \Delta_k(G)$, then the graph G is called *distance-k-regular.* The *k-th power* G^k of a graph G is the graph with vertex set $V(G)$ where two different vertices u and v are adjacent if and only if the distance $d(u,v)$ is at most k in G. Now we observe that $N_{k,G}(v,G) = N_{1,G^k}(v) = N_{G^k}(v)$, $N_{k,G}[v] = N_{1,G^k}[v] = N_{G^k}[v]$, $\deg_{k,G}(v) = \deg_{1,G^k}(v) = \deg_{G^k}(v)$, $\delta_k(G) = \delta_1(G^k) = \delta(G^k)$ and $\Delta_k(G) = \Delta_1(G^k) = \Delta(G^k)$. Consult [9] for the notation and terminology which are not defined here.

For a real-valued function $f : V(G) \longrightarrow \mathbb{R}$, the weight of f is $w(f) = \sum_{v \in V} f(v)$. For $S \subseteq V$, we define $f(S) = \sum_{v \in S} f(v)$. So $w(f) = f(V)$. A *signed distance k-dominating function* (SDkD function) is a function $f : V(G) \to \{-1, 1\}$ satisfying $\sum_{u \in N_{k,G}[v]} f(u) \geq 1$ for every $v \in V(G)$. The minimum of the values of $\sum_{v \in V(G)} f(v)$ taken over all signed distance k-dominating functions f is called the *signed distance k-domination number* and is denoted by $\gamma_{k,s}(G)$. Then the function assigning $+1$ to every vertex of G is a SDkD function, called the function ϵ, of weight n. Thus $\gamma_{k,s}(G) \leq n$ for every graph of order n. Moreover, the weight of every SDkD function different from ϵ is at most $n-2$ and more generally, $\gamma_{k,s}(G) \equiv n \pmod 2$. Hence $\gamma_{k,s}(G) = n$ if and only if ϵ is the unique SDkD function of G. In the special case when $k = 1$, $\gamma_{k,s}(G)$ is the signed domination number investigated in [3] and has been studied by several authors (see for example [2, 4]). The signed distance 2-domination number of graphs was introduced by Zelinka [11] and the signed distance k-domination number of graphs was introduced by Xing et al. [10]. By these definitions, we easily obtain

$$\gamma_{k,s}(G) = \gamma_s(G^k). \tag{1}$$

A set $\{f_1, f_2, \ldots, f_d\}$ of signed distance k-dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \leq 1$ for each $v \in V(G)$, is called a *signed distance k-dominating family* on G. The maximum number of functions in a signed distance k-dominating family on G is the *signed distance k-domatic number* of G, denoted by $d_{k,s}(G)$. The signed distance k-domatic number is well-defined and $d_{k,s}(G) \geq 1$ for all graphs G, since the set consisting of any one SDkD function, for instance the function ϵ, forms a

SDkD family of G. A *$d_{k,s}$-family* of a graph G is a SDkD family containing $d_{k,s}(G)$ SDkD functions. The signed distance 1-domatic number $d_{1,s}(G)$ is the usual signed domatic number $d_s(G)$ which was introduced by Volkmann and Zelinka in [8] and has been studied by several authors (see for example [5, 6, 7]). Obviously,

$$d_{k,s}(G) = d_s(G^k). \tag{2}$$

Observation 1. Let G be a graph of order n. If $\gamma_{k,s}(G) = n$, then ϵ is the unique SDkD function of G and so $d_{k,s}(G) = 1$.

We first study basic properties and sharp upper bounds for the signed distance k-domatic number of a graph. Some of them generalize the result obtained for the signed domatic number.

In this paper we make use of the following results.

Proposition A. [3] Let G be a graph of order n. Then $\gamma_s(G) = n$ if and only if every nonisolated vertex of G is either an endvertex or adjacent to an endvertex.

Observation 2. If G is a graph of order n, then $\gamma_{k,s}(G) = n$ if and only if

1. $k = 1$ and each vertex of G is isolated, a leaf or a support vertex,
2. $k \geq 2$ and $G = rK_1 \cup sK_2$ for some nonnegative integers r and s.

Proof. If 1. or 2. hold, then obviously each SDkD function satisfies $f(x) = +1$ for all $x \in V(G)$. Therefore ϵ is the unique SkD function and $\gamma_{k,s}(G) = n$.

Conversely, assume that $\gamma_{k,s}(G) = n$. If $k = 1$, then the result follows from Proposition A. Let $k \geq 2$. If $\Delta(G) \geq 2$, then assume that $x_1x_2\dots x_m$ is a longest path in G. It is easy to see that the function $f : V(G) \to \{-1, 1\}$ defined by $f(x_2) = -1$ and $f(x) = 1$ otherwise is a signed distance k-dominating function of G which is a contradiction. Thus $\Delta(G) \leq 1$ and the result follows. This completes the proof. □

Proposition B. [8] The signed domatic number $d_s(G)$ of a graph G is an odd integer.

Proposition C. [8] If G is a graph, then $1 \leq d_s(G) \leq \delta(G) + 1$.

Proposition D. [7] Let G be a graph, and let v be a vertex of odd degree $\deg_G(v) = 2t + 1$ with an integer $t \geq 1$. Then $d_s(G) \leq t$ when t is odd and $d_s(G) \leq t + 1$ when t is even.

Proposition E. [10] Let $k \geq 1$ be an integer. For any integer $n \geq 2$, we have

$$\gamma_{k,s}(K_n) = \gamma_s(K_n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod 2 \\ 2 & \text{otherwise.} \end{cases} \tag{3}$$

Proposition F. [8] If $G = K_n$ is the complete graph of order n, then

$$d_s(K_n) = \begin{cases} n & \text{if } n \text{ is odd,} \\ p & \text{if } n = 2p \text{ and } p \text{ is odd} \\ p-1 & \text{if } n = 2p \text{ and } p \text{ is even.} \end{cases}$$

Since $N_{k,K_n}[v] = N[v]$ for each vertex $v \in V(K_n)$ and each positive integer k, each signed dominating function of K_n is a signed distance k-dominating function of K_n and vice versa. Using Theorem F, we obtain

$$d_{k,s}(K_n) = d_s(K_n) = \begin{cases} n & \text{if } n \text{ is odd,} \\ p & \text{if } n = 2p \text{ and } p \text{ is odd} \\ p-1 & \text{if } n = 2p \text{ and } p \text{ is even.} \end{cases} \quad (4)$$

More general, the following result is valid.

Observation 3. Let $k \geq 1$ be an integer, and let G be a graph of order n. If the diameter $\text{diam}(G) \leq k$, then $\gamma_{k,s}(G) = \gamma_s(K_n)$ and $d_{k,s}(G) = d_s(K_n)$.

Next result is immediate consequences of Observation 3, Propositions E and F.

Corollary 4. If $k \geq 2$ and G is a graph of order n with $\text{diam}(G) = 2$, then

$$\gamma_{k,s}(G) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even,} \end{cases}$$

and

$$d_{k,s}(G) = \begin{cases} n & \text{if } n \text{ is odd,} \\ p & \text{if } n = 2p \text{ and } p \text{ is odd} \\ p-1 & \text{if } n = 2p \text{ and } p \text{ is even.} \end{cases}$$

Corollary 5. Let $k \geq 2$ be an integer, and let G be a graph of order n. If $\text{diam}(G) \neq 3$, then $\gamma_{k,s}(G) = \gamma_s(K_n)$ and $d_{k,s}(G) = d_s(K_n)$ or $\gamma_{k,s}(\overline{G}) = \gamma_s(K_n)$ and $d_{k,s}(\overline{G}) = d_s(K_n)$.

Proof. If $\text{diam}(G) \leq 2$, then it follows from Observation 3 that $\gamma_{k,s}(G) = \gamma_s(K_n)$ and $d_{k,s}(G) = d_s(K_n)$. If $\text{diam}(G) \geq 3$, then the hypothesis $\text{diam}(G) \neq 3$ implies that $\text{diam}(G) \geq 4$. Now, according to a result of Bondy and Murty [1] (page 14), we deduce that $\text{diam}(\overline{G}) \leq 2$. Applying again Observation 3, we obtain $\gamma_{k,s}(\overline{G}) = \gamma_s(K_n)$ and $d_{k,s}(\overline{G}) = d_s(K_n)$. □

Corollary 6. If $k \geq 3$ is an integer and G a graph of order n, then $\gamma_{k,s}(G) = \gamma_s(K_n)$ and $d_{k,s}(G) = d_s(K_n)$ or $\gamma_{k,s}(\overline{G}) = \gamma_s(K_n)$ and $d_{k,s}(\overline{G}) = d_s(K_n)$.

2 Basic properties of the signed distance k-domatic number

In this section we present basic properties of $d_{k,s}(G)$ and sharp bounds on the signed distance k-domatic number of a graph.

Proposition 7. The signed distance k-domatic number of a graph is an odd integer.

Proof. According to the identity (2), we have $d_{k,s}(G) = d_s(G^k)$. In view of Proposition B, $d_s(G^k)$ is odd and thus $d_{k,s}(G)$ is odd, and the proof is complete. □

Theorem 8. If G is a graph of order n, then

$$1 \leq d_{k,s}(G) \leq \delta_k(G) + 1.$$

Moreover, if $d_{k,s}(G) = \delta_k(G)+1$, then for each function of any $d_{k,s}$−family $\{f_1, f_2, \cdots, f_d\}$ and for all vertices v of minimum k-degree $\delta_k(G)$, $\sum_{u\in N_{k,G}[v]} f_i(u) = 1$ and $\sum_{i=1}^{d} f_i(u) = 1$ for every $u \in N_{k,G}[v]$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a SDkD family of G such that $d = d_{k,s}(G)$ and let v be a vertex of minimum k-degree $\delta_k(G)$. Then $|N_{k,G}[v]| = \delta_{k,G}(v)+1$ and

$$\begin{aligned} 1 \leq d &= \sum_{i=1}^{d} 1 \\ &\leq \sum_{i=1}^{d} \sum_{u\in N_{k,G}[v]} f_i(u) \\ &= \sum_{u\in N_{k,G}[v]} \sum_{i=1}^{d} f_i(u) \\ &\leq \sum_{u\in N_{k,G}[v]} 1 \\ &= \delta_k(G) + 1. \end{aligned}$$

If $d_{k,s}(G) = \delta_k(G) + 1$, then the two inequalities occurring in the proof become equalities, which gives the two properties given in the statement. □

Theorem 9. Let $k \geq 1$ be an integer, and let G be a graph. If G contains a vertex v of odd k-degree $\deg_k(v, G) = 2t + 1$ with an integer $t \geq 1$, then $d_{k,s}(G) \leq t$ when t is odd and $d_{k,s}(G) \leq t + 1$ when t is even.

Proof. Since $\deg_{k,G}(v) = \deg_{G^k}(v) = 2t + 1$, Proposition D and (2) imply that $d_{k,s}(G) = d_s(G^k) \leq t$ when t is odd and $d_{k,s}(G) = d_s(G^k) \leq t + 1$ when t is even. □

Restricting our attention to graphs G of odd minimum k-degree, Theorem 9 leads to a considerable improvement of the upper bound of $d_{k,s}(G)$ given in Theorem 8

Corollary 10. If $k \geq 1$ is an integer, and G is a graph of odd minimum k-degree $\delta_k(G)$, then $d_{k,s}(G) \leq (\delta_k(G) - 1)/2$ when $\delta_k(G) \equiv 3 \pmod 4$ and $d_{k,s}(G) \leq (\delta_k(G) + 1)/2$ when $\delta_k(G) \equiv 1 \pmod 4$.

The equations in (4) show that the bounds in Theorem 8 and Corollary 10 are sharp.

Theorem 11. Let G be a graph of order n with signed distance k-domination number $\gamma_{k,s}(G)$ and signed distance k-domatic number $d_{k,s}(G)$. Then

$$\gamma_{k,s}(G) \cdot d_{k,s}(G) \leq n.$$

Moreover, if $\gamma_{k,s}(G) \cdot d_{k,s}(G) = n$, then for each $d_{k,s}$-family $\{f_1, f_2, \cdots, f_d\}$ on G, each function f_i is a $\gamma_{k,s}$-function and $\sum_{i=1}^{d} f_i(v) = 1$ for all $v \in V$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a SDkD family on G such that $d = d_{k,s}(G)$ and let $v \in V$. Then

$$\begin{aligned} d \cdot \gamma_{k,s}(G) &= \sum_{i=1}^{d} \gamma_{k,s}(G) \\ &\leq \sum_{i=1}^{d} \sum_{v \in V} f_i(v) \\ &= \sum_{v \in V} \sum_{i=1}^{d} f_i(v) \\ &\leq \sum_{v \in V} 1 \\ &= n. \end{aligned}$$

If $\gamma_{k,s}(G) \cdot d_{k,s}(G) = n$, then the two inequalities occurring in the proof become equalities. Hence for the $d_{k,s}$-family $\{f_1, f_2, \cdots, f_d\}$ on G and for each i, $\sum_{v \in V} f_i(v) = \gamma_{k,s}(G)$, thus each function f_i is a $\gamma_{k,s}$-function, and $\sum_{i=1}^{d} f_i(v) = 1$ for all v. □

Corollary 12 is a consequence of Theorems 11 and 7 and improves Observation 1.

Corollary 12. If $\gamma_{k,s}(G) > \frac{n}{3}$, then $d_{k,s}(G) = 1$.

The upper bound on the product $\gamma_{k,s}(G) \cdot d_{k,s}(G)$ leads to a bound on the sum.

Corollary 13. If G is a graph of order $n \geq 4$, then

$$\gamma_{k,s}(G) + d_{k,s}(G) \leq n + 1.$$

Equality $\gamma_{k,s}(G) + d_{k,s}(G) = n + 1$ occurs if and only if $d_{k,s}(G) = n$ and $\gamma_{k,s}(G) = 1$ or $d_{k,s}(G) = 1$ and $\gamma_{k,s}(G) = n$.

Proof. According to Theorem 11, we obtain

$$\gamma_{k,s}(G) + d_{k,s}(G) \leq \frac{n}{d_{k,s}(G)} + d_{k,s}(G). \tag{5}$$

The bound results from the facts that the function $g(x) = x + n/x$ is decreasing for $1 \leq x \leq \sqrt{n}$ and increasing for $\sqrt{n} \leq x \leq n$ and that $1 \leq d_{k,s}(G) \leq n$ by Theorem 8. Equality occurs if and only if $d_{k,s}(G) = n$ and $\gamma_{k,s}(G) = 1$ or $d_{k,s}(G) = 1$ and $\gamma_{k,s}(G) = n$. □

By Corollary 13, $\gamma_{k,s}(G) + d_{k,s}(G)$ can be equal to $n+1$ if $\gamma_{k,s}(G) = n$ or 1 or if $d_{k,s}(G) = n$ or 1. But if $1 < \gamma_{k,s}(G) < n$ or if $1 < d_{k,s}(G) < n$ or if $\min\{\gamma_{k,s}(G), d_{k,s}(G)\} > 1$, we can lower the upper bound $n+1$.

Corollary 14. Let G be a graph of order $n \geq 4$. If $2 \leq \gamma_{k,s}(G) \leq n-1$ or if $2 \leq d_{k,s}(G) \leq n-1$, then

$$\gamma_{k,s}(G) + d_{k,s}(G) \leq n - 1.$$

Proof. By Corollary 13, $\gamma_{k,s}(G) + d_{k,s}(G) < n+1$. The result follows from Theorem 7 and the fact that, as seen in the introduction, $\gamma_{k,s}(G) \equiv n \pmod 2$. □

Corollary 15. Let G be a graph of order n, and let $k \geq 1$ be an integer. If $\min\{\gamma_{k,s}(G), d_{k,s}(G)\} \geq a$, with $2 \leq a \leq \sqrt{n}$, then

$$\gamma_{k,s}(G) + d_{k,s}(G) \leq a + \frac{n}{a}.$$

Proof. Since $\min\{\gamma_{k,s}(G), d_{k,s}(G)\} \geq a \geq 2$, it follows from Theorem 11 that $2 \leq d_{k,s}(G) \leq \frac{n}{a}$. Applying the inequality (5), we obtain

$$\gamma_{k,s}(G) + d_{k,s}(G) \leq d_{k,s}(G) + \frac{n}{d_{k,s}(G)}.$$

The bound results from the facts that the function $g(x) = x + \frac{n}{x}$ is decreasing for $1 \leq x \leq \sqrt{n}$ and increasing for $\sqrt{n} \leq x \leq n$. □

References

[1] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, The Macmillan Press Ltd., London and Basingstoke (1976).

[2] E.J. Cockayne and C.M. Mynhardt, *On a generalisation of signed dominating functions of a graph*, Ars Combin. **43** (1996), 235–245.

[3] J. Dunbar, S.T. Hedetniemi, M.A. Henning, P.J. Slater, *Signed domination in graphs*, Graph Theory, Combinatorics, and Applications, Vol. 1, Wiley, New York, 1995, 311–322.

[4] O. Favaron, *Signed domination in regular graphs*, Discrete Math. **158** (1996), 287–293.

[5] D. Meierling, L. Volkmann and S. Zitzen, *The signed domatic number of some regular graphs*, Discrete Appl. Math. **157** (2009), 1905–1912.

[6] L. Volkmann, *Signed domatic numbers of the complete bipartite graphs*, Util. Math. **68** (2005), 71–77.

[7] L. Volkmann, *Some remarks on the signed domatic number of graphs with small minimum degree*, Applied Math. Lett. **22** (2009), 1166–1169.

[8] L. Volkmann and B. Zelinka, *Signed domatic number of a graph*, Discrete Appl. Math. **150** (2005), 261–267.

[9] D.B. West, *Introduction to Graph Theory*, Prentice-Hall, Inc, 2000.

[10] H. Xing, L. Sun and X. Chen, *On signed distance-k-domination in graphs*, Czechoslovak Math. J., **56**(131) (2006), 229–2383.

[11] B. Zelinka, *Signed 2-domination in caterpillars*, Math. Bohem. **129** (2004), 393–398.