On edge-3-equitability of $\overline{K}_n$–union of gears.

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Abstract

A k-edge labeling of a graph G is a function f from the vertex set $E(G)$ to the set of integers $\{0, \ldots, k-1\}$. Such a labeling induces a labeling f on the vertex set $V(G)$ by defining $f(v) = \sum f(e)$, where the summation is taken over all the edges incident on the vertex v and the value is reduced modulo k. Cahit calls this labeling edge-k-equitable if f assigns the labels $\{0, \ldots, k-1\}$ equitably to the vertices as well as edges.

If $G_1, \ldots, G_T$ is a family of graphs having a graph H as an induced subgraph, then by H-union G of this family we mean the graph obtained by identifying all the corresponding vertices as well as edges of the copies of H in $G_1, \ldots, G_T$.

In this paper we prove that $\overline{K}_n$-union of gears is edge-3-equitable.

1 INTRODUCTION

All the graphs we consider are simple and without loops. For a graph G, by $V(G)$ and $E(G)$, we mean the vertex set and the edge set of the graph G.

Let $G_1, G_2, \ldots, G_T$ be a family of graphs. Let $H_i, 1 \leq i \leq T$, be an induced subgraph of $G_i, 1 \leq i \leq T$, such that each H_i is isomorphic to a fixed graph H. If $v \in V(H)$, the vertex corresponding to it in H_i is denoted by v^i. Similarly if $e \in E(H)$, the edge corresponding to it in H_i is denoted by e^i.

Definition 1: By the **H-union** G of the family $G_1, G_2, \ldots, G_T$, we mean the graph obtained by identifying $v^1, v^2, \ldots v^T$ for every $v \in V(H)$ and identifying $e^1, e^2, \ldots, e^T$ for every $e \in E(H)$.

If $H = K_1$, the H-union is called the one point union. Similarly, if $H = K_2$, the H-union is called one edge union.

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By a k-edge labeling of a graph G we mean a map $f : E(G) \to \{0, 1, \ldots, k-1\}$. A k-edge labeling f induces a labeling, also denoted by f, on the vertex set $V(G)$ of G by defining $f(x) := \sum f(e)$, where the summation is taken over all the edges incident on the vertex x and the value is reduced modulo k. For a k-edge labeling f, by $v_f(j)$ (respectively $e_f(j)$), we mean the number of vertices (respectively edges) which are assigned the label j by the labeling f. These are called the vertex numbers, (respectively the edge numbers) of f. By $v_f(0, 1, \ldots k-1)$ and $e_f(0, 1, \ldots, k-1)$ we mean the k-tuples $(v_f(0), v_f(1), \ldots, v_f(k-1))$ and $(e_f(0), e_f(1), \ldots, e_f(k-1))$.

Definition 2: A k-edge labeling f is said to be **edge-k-equitable** if $|v_f(i) - v_f(j)| \leq 1, |e_f(i) - e_f(j)| \leq 1$ for all $1 \leq i, j \leq k-1$.

This concept was introduced in the year 2000, by Cahit and Yilmaz[1]. They chose to call such labelings E_k-cordial labelings. For a detailed history of such labelings we refer to Gallian[3].

Definition 3: The **gear** graph $\mathbf{G}_n$ is defined as follow:
$V(G_n) = \{v_0, v_1, \ldots, v_{2n}\}$ and
$E(G_n) = \{v_0 v_{2i-1} \mid 1 \leq i \leq n\} \bigcup \{v_i v_{i+1} \mid 1 \leq i \leq 2n-1\} \bigcup \{v_{2n} v_1\}$.

The vertex v_0 is called the apex of the gear G_n. The edge $v_0 v_{2i-1}$ is denoted by $c_i, 1 \leq i \leq n$ and is called a spoke. The edge $v_i v_{i+1}$ is denoted by $e_i, 1 \leq i \leq 2n$, where by v_{2n+1} we mean v_1. These edges are referred to as the cyclic edges. The vertices $\{v_2, v_4, \ldots, v_{2n}\}$, which are not joined to the apex are called the gear tips. We note that the subgraph induced by the set of all the gear tips is $\overline{K}_n$. A gear G_n is said to be of type $i, 1 \leq i \leq 3$ if $n = 3\alpha + i$.

Definition 4: Let $H_1, H_2, \ldots, H_T$ be k copies of the gear graph G_n. The vertices of H_j are denoted by $\{v_{j,0}, v_{j,1}, \ldots, v_{j,2n}\}$. A $\overline{K}_n$-union $G_{n,T}$ is obtained by identifying the gear tips of $H_i, 1 \leq i \leq T$ in cyclic order. $G_{n,T}$ is refered to as the T-tuple gear.

If $T = 2$, then $G_{n,T}$ is called a double gear, If $T = 3$, then $G_{n,T}$ is called a triple gear and so on. If f is an edge labeling of a T-tuple gear $G_{n,T}$, then let $v_f^1(i)$ (respectively $v_f^2(i)$) be the number of non-tip vertices (respectively gear tips), which are assigned the label i by f. By $v_f^1(0, 1, \ldots, k-1)$ we mean the k-tuple $(v_f^1(0), \ldots v_f^1(k-1))$. Similarly, by $v_f^2(0, 1, \ldots, k-1)$ we mean the k-tuple $(v_f^2(0), \ldots v_f^2(k-1))$.

In this paper we show that a T-tuple gear $G_{n,T}$ is edge-3-equitable for arbitrary natural numbers n, T.

Methodology

Method 1: we first take triple gear $G_{n,3}$ involving three copies of a gear G_n. If G is such a graph then $|V(G)| = 4n + 3$ and $|E(G)| = 9n$. The number of non-identified vertices is $3n + 3$. We then produce an edge-3-equitable labeling f for G in such a way that $v_f^1(0,1,2) = (n+1, n+1, n+1)$. The label number differences occur only in the identified vertices. These vertices are called $w_2, w_4, \dots, w_{2n}$.

After that, we take $3t$ gears and form t copies $K_1, K_2, \dots, K_t$ of triple gears. In K_α the identified gear tips are called $w_2^\alpha, w_4^\alpha, \dots, w_{2n}^\alpha$. For some value of $r, w_{2r}^1, w_{2r}^2, \dots, w_{2r}^t$ are identified. This value $2r$ is called the pivotal value. For K_j the vertices $w_2^i, w_4^i, \dots, w_{2r}^i, \dots w_{2n}^i$ are identified with their corresponding tip vertices $w_2^j, w_4^j, \dots, w_{2r}^j, \dots, w_{2n}^j$ in this order or in the reverse order $\dots, w_{2r+4}^j, w_{2r+2}^j, w_{2r}^j, w_{2r-2}^j, w_{2r-4}^j, \dots$. Same technique is also applied to $\overline{K}_n$ - union of some multiple gears with a set of other gears.

Example 1: Suppose we have six triple gears $K_1, K_2, \dots, K_6$. We decide the pivotal value to be 2 and we decide that for K_3 and K_6 we are going to reverse the order of identification around the pivotal value. The identification is done according to the following table:

w_2^1	w_4^1	w_6^1	$\dots$	w_{2n}^1
w_2^2	w_4^2	w_6^2	$\dots$	w_{2n}^2
w_2^3	w_{2n}^3	w_{2n-2}^3	$\dots$	w_4^3
w_2^4	w_4^4	w_6^4	$\dots$	w_{2n}^4
w_2^5	w_4^5	w_6^5	$\dots$	w_{2n}^5
w_2^6	w_{2n}^6	w_{2n-2}^6	$\dots$	w_4^6

Method 2: While identifying gear tips of several gears, in some of them we perform left (respectively right) shift.

Example Suppose we have two gears G_1, G_2 with some labelings θ_1, θ_2 with the gear tips $v_2^i, v_4^i, \dots, v_{2n}^i, i = 1, 2$. For the left shift, we identify $\{v_2^1, v_4^2\}, \{v_4^1, v_6^2\}, \{v_6^1, v_8^2\}, \{v_8^1, v_{10}^2\}, \ldots$. For the right shift we identify $\{v_2^1, v_{2n}^2\}, \{v_4^1, v_2^2\}, \{v_6^1, v_4^2\}, \{v_8^1, v_6^2\}, \ldots$.

2 LABELINGS OF GEARS

In this section we define various labelings of the gears, some of which are edge-3-equitable.

Gears of Type 1: $n = 3x + 4$.

Labeling $\mathbf{f_1}$: One defines the labeling f_1 as follows:

$$\begin{aligned} f_1(e_i) &= 1, i \equiv 1,2 \mod 6, & f_1(c_i) &= 2, i \equiv 0 \mod 3, \\ &= 0, i \equiv 3,4 \mod 6, & &= 0, i \equiv 1 \mod 3, \\ &= 2, i \equiv 5,0 \mod 6, & &= 1, i \equiv 2 \mod 3 \\ &\text{for } 1 \leq i \leq 6x & &\text{for } 1 \leq i \leq 3x \end{aligned}$$

$f_1(e_{6x+1}) = 2, f_1(e_{6x+2}) = 1, f_1(e_{6x+3}) = 2, f_1(e_{6x+4}) = 0, f_1(e_{6x+5}) = 0, f_1(e_{6x+6}) = 2, f_1(e_{6x+7}) = 1, f_1(e_{6x+8}) = 1$, In addition, $f_1(c_{3x+1}) = 0, f_1(c_{3x+2}) = 2, f_1(c_{3x+3}) = 0, f_1(c_{3x+4}) = 1$. It can be checked that $e_{f_1}(0,1,2) = (3x+4, 3x+4, 3x+4), v_{f_1}^1 = (x+1, x+2, x+2), v_{f_1}^2 = (x+1, x, x+3)$. This is not edge-3-equitable labeling though it labels the edges equitably. The gear tip vertices $v_2, v_4, \ldots, v_{2n}$ get the labels $2,0,1,2,0,1\ldots,2,0,1,0,2,2,2$.

Labeling $\mathbf{g_1}$: One defines the labeling g_1 as follows:

$$\begin{aligned} g_1(e_i) &= 0, i \equiv 1,2 \mod 6, & g_1(c_i) &= 2, i \equiv 0 \mod 3, \\ &= 1, i \equiv 3,4 \mod 6, & &= 1, i \equiv 1 \mod 3, \\ &= 2, i \equiv 5,0 \mod 6 & &= 0, i \equiv 2 \mod 3 \\ &\text{for } 1 \leq i \leq 6x & &\text{for } 1 \leq i \leq 3x \end{aligned}$$

$g_1(e_{6x+1}) = 2, g_1(e_{6x+2}) = 1, g_1(e_{6x+3}) = 1, g_1(e_{6x+4}) = 1, g_1(e_{6x+5}) = 0, g_1(e_{6x+6}) = 0, g_1(e_{6x+7}) = 1, g_1(e_{6x+8}) = 2$, In addition, $g_1(c_{3x+1}) = 2, g_1(c_{3x+2}) = 0, g_1(c_{3x+3}) = 0, g_1(c_{3x+4}) = 2$. It can be checked that $e_{g_1}(0,1,2) = (3x+4, 3x+4, 3x+4), v_{g_1}^1 = (x+2, x+2, x+1), v_{g_1}^2 = (x+3, x, x+1)$. This is not edge-3-equitable labeling though it labels the edges equitably. The gear tip vertices $v_2, v_4, \ldots, v_{2n}$ get the labels $0,2,1,0,2,1\ldots,0,2,1,0,2,0,0$.

Labeling $\mathbf{h_1}$: One defines the labeling h_1 as follows:

$$h_1(e_i) \begin{array}{l} = 0, i \equiv 1,2 \mod 6, \\ = 1, i \equiv 3,4 \mod 6, \\ = 2, i \equiv 5,0 \mod 6 \\ \text{for } 1 \leq i \leq 6x \end{array} \quad h_1(c_i) \begin{array}{l} = 2, i \equiv 0 \mod 3, \\ = 1, i \equiv 1 \mod 3, \\ = 0, i \equiv 2 \mod 3 \\ \text{for } 1 \leq i \leq 3x \end{array}$$

$h_1(e_{6x+1}) = 1, h_1(e_{6x+2}) = 2, h_1(e_{6x+3}) = 1, h_1(e_{6x+4}) = 0, h_1(e_{6x+5}) = 2, h_1(e_{6x+6}) = 0, h_1(e_{6x+7}) = 2, h_1(e_{6x+8}) = 2$, In addition, $h_1(c_{3x+1}) = 1, h_1(c_{3x+2}) = 0, h_1(c_{3x+3}) = 0, h_1(c_{3x+4}) = 1$. It can be checked that $e_{h_1}(0,1,2) = (3x+4, 3x+4, 3x+4), v^1_{h_1} = (x+2, x+1, x+2), v^2_{h_1} = (x+1, x+2, x+1)$. This is an edge-3-equitable labeling. The gear tip vertices $v_2, v_4, \ldots, v_{2n}$ get the labels $0,2,1,0,2,1\ldots,0,2,1,0,1,2,1$.

Labeling $\mathbf{k_1}$: One defines the labeling k_1 as follows:

$$k_1(e_i) \begin{array}{l} = 0, i \equiv 1,2 \mod 6, \\ = 1, i \equiv 3,4 \mod 6, \\ = 2, i \equiv 5,0 \mod 6 \\ \text{for } 1 \leq i \leq 6x \end{array} \quad k_1(c_i) \begin{array}{l} = 2, i \equiv 0 \mod 3, \\ = 1, i \equiv 1 \mod 3, \\ = 0, i \equiv 2 \mod 3 \\ \text{for } 1 \leq i \leq 3x \end{array}$$

$k_1(e_{6x+1}) = 2, k_1(e_{6x+2}) = 1, k_1(e_{6x+3}) = 1, k_1(e_{6x+4}) = 2, k_1(e_{6x+5}) = 1, k_1(e_{6x+6}) = 0, k_1(e_{6x+7}) = 1, k_1(e_{6x+8}) = 0$, In addition, $k_1(c_{3x+1}) = 2, k_1(c_{3x+2}) = 0, k_1(c_{3x+3}) = 0, k_1(c_{3x+4}) = 2$. It can be checked that $e_{k_1}(0,1,2) = (3x+4, 3x+4, 3x+4), v^1_{k_1} = (x+2, x+2, x+1), v^2_{k_1} = (x+2, x+2, x)$. This is not edge-3-equitable labeling though it labels the edges equitably. The gear tip vertices $v_2, v_4, \ldots, v_{2n}$ get the labels $0,2,1,0,2,1,\ldots,0,2,1,0,0,1,1$.

Labeling $\mathbf{l_1}$: One defines the labeling l_1 as follows:

$$l_1(e_i) \begin{array}{l} = 2, i \equiv 1,2 \mod 6, \\ = 1, i \equiv 3,4 \mod 6, \\ = 0, i \equiv 5,0 \mod 6 \\ \text{for } 1 \leq i \leq 6x \end{array} \quad l_1(c_i) \begin{array}{l} = 1, i \equiv 0 \mod 3, \\ = 2, i \equiv 1 \mod 3, \\ = 0, i \equiv 2 \mod 3 \\ \text{for } 1 \leq i \leq 3x \end{array}$$

$l_1(e_{6x+1}) = 1, l_1(e_{6x+2}) = 1, l_1(e_{6x+3}) = 2, l_1(e_{6x+4}) = 0, l_1(e_{6x+5}) = 2, l_1(e_{6x+6}) = 0, l_1(e_{6x+7}) = 1, l_1(e_{6x+8}) = 0$, In addition, $l_1(c_{3x+1}) = 0, l_1(c_{3x+2}) = 2, l_1(c_{3x+3}) = 1, l_1(c_{3x+4}) = 2$. One can see that $e_{l_1}(0,1,2) = (3x+4, 3x+4, 3x+4), v^1_{l_1} = (x+2, x+1, x+2), v^2_{l_1} = (x, x+1, x+3)$. This is not edge-3-equitable labeling though it labels the edges equitably. The gear tip vertices $v_2, v_4, \ldots, v_{2n}$ get the labels $1,2,0,1,2,0,\ldots,1,2,0,2,2,2,1$.

Labeling $\mathbf{p_1}$: One defines the labeling p_1 as follows:

$$p_1(e_i) \begin{array}{l} = 0, i \equiv 1,2 \mod 6, \\ = 1, i \equiv 3,4 \mod 6, \\ = 2, i \equiv 5,6 \mod 6 \\ \text{for } 1 \le i \le 6x \end{array} \quad p_1(c_i) \begin{array}{l} = 0, i \equiv 1 \mod 3, \\ = 2, i \equiv 2 \mod 3, \\ = 1, i \equiv 3 \mod 3 \\ \text{for } 1 \le i \le 3x \end{array}$$

$p_1(e_{6x+1}) = 1, p_1(e_{6x+2}) = 1, p_1(e_{6x+3}) = 0, p_1(e_{6x+4}) = 0, p_1(e_{6x+5}) = 2, p_1(e_{6x+6}) = 1, p_1(e_6x+7) = 1, p_1(e_{6x+8}) = 2$. The remaining 4 spokes are labeled as $p_1(c_{3x+1}) = p_1(c_{3x+4}) = 2, p_1(c_{3x+2}) = p_1(c_{3x+3}) = 0$. The sequence of labels gear tips is $p_1 : 0,2,1,0,2,1,\ldots,0,2,1,2,0,0,0$. One can check that $e_{p_1}(0,1,2) = (3x+4, 3x+4, 3x+4), v^1_{p_1}(0,1,2) = (x, x+3, x+2), v^2_{p_1}(0,1,2) = (x+3, x, x+1)$. Thus, $v_{p_1}(0,1,2) = (2x+3, 2x+3, 2x+3)$, That is, p_1 is an edge-3-equitable labeling.

Gears of Type 2 $n = 3x + 5$.

Labeling $\mathbf{f_2}$: One defines the labeling f_2 as follows:

$$f_2(e_i) \begin{array}{l} = 1, i \equiv 1,2 \mod 6, \\ = 0, i \equiv 3,4 \mod 6, \\ = 2, i \equiv 5,0 \mod 6 \\ \text{for } 1 \le i \le 2(n-5) \end{array} \quad f_2(c_i) \begin{array}{l} = 0, i \equiv 0 \mod 3, \\ = 1, i \equiv 1 \mod 3, \\ = 2, i \equiv 2 \mod 3 \\ \text{for } 1 \le i \le n-5 \end{array}$$

Finally, $f_2(e_{2n-9}) = 2, f_2(e_{2n-8}) = 2, f_2(e_{2n-7} = 0, f_2(e_{2n-6}) = 1, f_2(e_{2n-5}) = 0, f_2(e_{2n-4}) = 1, f_2(e_{2n-3}) = 2, f_2(e_{2n-2}) = 1, f_2(e_{2n-1}) = 0, f_2(e_{2n}) = 0, f_2(c_{n-4}) = 1, f_2(c_{n-3}) = 2, f_2(c_{n-2}) = 0, f_2(c_{n-1}) = 1, f_2(c_n) = 2$. It can be checked that $v^1_{f_2}(0,1,2) = (x+2, x+2, x+2), v^2_{f_2}(0,1,2) = (x+2, x+3, x), e_{f_2}(0,1,2) = (n,n,n)$. The gear tips get the labels $2,0,1,2,0,1,\ldots,2,0,1,1,1,1,0,0$.

Labeling $\mathbf{g_2}$: One defines the labeling g_2 as follows:

$$g_2(e_i) \begin{array}{l} = 1, i \equiv 1,2 \mod 6, \\ = 0, i \equiv 3,4 \mod 6, \\ = 2, i \equiv 5,0 \mod 6 \\ \text{for } 1 \le i \le 2(n-5) \end{array} \quad g_2(c_i) \begin{array}{l} = 0, i \equiv 0 \mod 3, \\ = 1, i \equiv 1 \mod 3, \\ = 2, i \equiv 2 \mod 3 \\ \text{for } 1 \le i \le n-5 \end{array}$$

Finally, $g_2(e_{2n-9}) = 0, g_2(e_{2n-8}) = 1, g_2(e_{2n-7}) = 2, g_2(e_{2n-6}) = 0, g_2(e_{2n-5}) = 2, g_2(e_{2n-4}) = 0, g_2(e_{2n-3}) = 2, g_2(e_{2n-2}) = 0, g_2(e_{2n-1}) = 1, g_2(e_{2n}) = 0, g_2(c_{n-4}) = 1, g_2(c_{n-3}) = 1, g_2(c_{n-2}) = 2, g_2(c_{n-1}) = 2, g_2(c_n) = 1$. It can be checked that $v^1_{g_2}(0,1,2) = (x+1, x+3, x+2), v^2_{g_2}(0,1,2) = (x, x+2, x+3), e_{g_2}(0,1,2) = (n,n,n)$. The gear tips get the labels $2,0,1,2,0,1,\ldots,2,0,1,1,2,2,2,1$.

Labeling $\mathbf{h_2}$: One defines the labeling h_2 as follows:

$$h_2(e_i) \quad \begin{aligned} &= 1, i \equiv 1,2 \mod 6,\\ &= 0, i \equiv 3,4 \mod 6,\\ &= 2, i \equiv 5,0 \mod 6\\ &\text{for } 1 \le i \le 2n-10 \end{aligned} \qquad h_2(c_i) \quad \begin{aligned} &= 2, i \equiv 0 \mod 3,\\ &= 0, i \equiv 1, \mod 3,\\ &= 1, i \equiv 2 \mod 3\\ &\text{for } 1 \le i \le n-5 \end{aligned}$$

Finally, $h_2(e_{2n-9}) = 2, h_2(e_{2n-8}) = 1, h_2(e_{2n-7}) = 2, h_2(e_{2n-6}) = 0, h_2(e_{2n-5}) = 0, h_2(e_{2n-4}) = 2, h_2(e_{2n-3}) = 1, h_2(e_{2n-2}) = 1, h_2(e_{2n-1}) = 0, h_2(e_{2n}) = 0, h_2(c_{n-4}) = 1, h_2(c_{n-3}) = 2, h_2(c_{n-2}) = 0, h_2(c_{n-1}) = 1, h_2(c_n) = 2$. It can be checked that $v^1_{h_2}(0,1,2) = (x+2, x+2, x+2), v^2_{h_2}(0,1,2) = (x+2, x, x+3), e_{g_2}(0,1,2) = (n,n,n)$. The gear tips get the labels $2,0,1,2,0,1,\ldots,2,0,1,0,2,2,2,0$.

Labeling $\mathbf{k_2}$: One defines the labeling k_2 as follows:

$$k_2(e_i) \quad \begin{aligned} &= 0, i \equiv 1,2 \mod 6,\\ &= 1, i \equiv 3,4 \mod 6,\\ &= 2, i \equiv 5,0 \mod 6\\ &\text{for } 1 \le i \le 2(n-5) \end{aligned} \qquad k_2(c_i) \quad \begin{aligned} &= 2, i \equiv 0 \mod 3,\\ &= 1, i \equiv 1, \mod 3,\\ &= 0, i \equiv 2 \mod 3\\ &\text{for } 1 \le i \le n-5 \end{aligned}$$

Finally, $k_2(e_{2n-9}) = 2, k_2(e_{2n-8}) = 1, k_2(e_{2n-7}) = 1, k_2(e_{2n-6}) = 1, k_2(e_{2n-5}) = 0, k_2(e_{2n-4}) = 0, k_2(e_{2n-3}) = 1, k_2(e_{2n-2}) = 2, k_2(e_{2n-1}) = 1, k_2(e_{2n}) = 0, k_2(c_{n-4}) = 2, k_2(c_{n-3}) = 0, k_2(c_{n-2}) = 0, k_2(c_{n-1}) = 2, k_2(c_n) = 2$. It can be checked that $v^1_{k_2}(0,1,2) = (x+2, x+2, x+2), v^2_{k_2}(0,1,2) = (x+3, x+1, x+1), e_{k_2}(0,1,2) = (n,n,n)$. The gear tips get the labels $0,2,1,0,2,1,\ldots,0,2,1,0,2,0,0,1$.

Labeling $\mathbf{l_2}$: One defines the labeling l_2 as follows:

$$l_2(e_i) \quad \begin{aligned} &= 0, i \equiv 1,2 \mod 6,\\ &= 1, i \equiv 3,4 \mod 6,\\ &= 2, i \equiv 5,0 \mod 6\\ &\text{for } 1 \le i \le 2n-10 \end{aligned} \qquad l_2(c_i) \quad \begin{aligned} &= 2, i \equiv 0 \mod 3,\\ &= 1, i \equiv 1 \mod 3,\\ &= 0, i \equiv 2 \mod 3\\ &\text{for } 1 \le i \le n-5 \end{aligned}$$

Finally, $l_2(e_{2n-9}) = 2, l_2(e_{2n-8}) = 2, l_2(e_{2n-7}) = 1, l_2(e_{2n-6}) = 0, l_2(e_{2n-5}) = 1, l_2(e_{2n-4}) = 0, l_2(e_{2n-3}) = 2, l_2(e_{2n-2}) = 0, l_2(e_{2n-1}) = 1, l_2(e_{2n}) = 0, l_2(c_{n-4}) = 1, l_2(c_{n-3}) = 2, l_2(c_{n-2}) = 0, l_2(c_{n-1}) = 1, l_2(c_n) = 2$. One can see that $e_{l_2}(0,1,2) = (n,n,n), v^2_{l_2}(0,1,2) = (x, x+4, x+1), v^1_{l_2}(0,1,2) = (x+2, x+2, x+2)$. The labels of gear tips are
$0,2,1,0,2,1,\ldots,0,2,1,1,1,1,2,1$.

Labeling $\mathbf{p_2}$: One defines the labeling p_2 as follows:

$$p_2(e_i) \begin{cases} = 0, i \equiv 1,2 \bmod 6, \\ = 1, i \equiv 3,4 \bmod 6, \\ = 2, i \equiv 5,0 \bmod 6 \\ \text{for } 1 \le i \le 2n-10 \end{cases} \qquad p_2(c_i) \begin{cases} = 0, i \equiv 0 \bmod 3, \\ = 2, i \equiv 1 \bmod 3, \\ = 1, i \equiv 2 \bmod 3 \\ \text{for } 1 \le i \le n-5 \end{cases}$$

Finally, $p_2(e_{2n-9}) = 0, p_2(e_{2n-8}) = 0, p_2(e_{2n-7}) = 0, p_2(e_{2n-6}) = 1, p_2(e_{2n-5}) = 1, p_2(e_{2n-4}) = 1, p_2(e_{2n-3}) = 2, p_2(e_{2n-2}) = 2, p_2(e_{2n-1}) = 0, p_2(e_{2n}) = 2, p_2(c_{n-4}) = 2, p_2(c_{n-3}) = 1, p_2(c_{n-2}) = 0, p_2(c_{n-1}) = 2, p_2(c_n) = 1$. It can be checked that $v^1_{p_2}(0,1,2) = (x+2, x+2, x+2), v^2_{p_2}(0,1,2) = (x+1, x+2, x+2), e_{p_2}(0,1,2) = (n,n,n)$. This is an edge-3-equitable labeling. The gear tips get the labels $0,2,1,0,2,1,\ldots,0,2,1,0,1,2,1,2$.

Gears of Type 3 Let $n = 3x$.

Labeling $\mathbf{f_3}$: One defines the labeling f_3 as follows:

$$f_3(e_i) \begin{cases} = 0, i \equiv 1,2 \bmod 6, \\ = 1, i \equiv 3,4 \bmod 6, \\ = 2, i \equiv 5,0 \bmod 6 \\ \text{for } 1 \le i \le 2n \end{cases} \qquad f_3(c_i) \begin{cases} = 1, i \equiv 0 \bmod 3, \\ = 0, i \equiv 1 \bmod 3, \\ = 2, i \equiv 2 \bmod 3 \\ \text{for } 1 \le i \le n-3 \end{cases}$$

Finally, $f_3(c_{n-2}) = 2, f_3(c_{n-1}) = 2, f_3(c_n) = 1$. It can be checked that $e_{f_3}(0,1,2) = (3x-1, 3x.3x+1), v^1_{f_3} = (x, x+1, x), v^2_{f_3} = (x,x,x)$. This is not edge-3-equitable labeling. The gear tip vertices $v_2, v_4, \ldots, v_{2n}$ get the labels $0,2,1,0,2,1\ldots$.

Labeling $\mathbf{g_3}$: One defines the labeling g_3 as follows:

$$g_3(e_i) \begin{cases} = 0, i \equiv 1,5 \bmod 6, \\ = 2, i \equiv 2,4 \bmod 6, \\ = 1, i \equiv 3,0 \bmod 6 \\ \text{for } 1 \le i \le 2n-1 \end{cases} \qquad g_3(c_i) \begin{cases} = 0, i \equiv 0 \bmod 3, \\ = 2, i \equiv 1 \bmod 3, \\ = 1, i \equiv 2 \bmod 3 \\ \text{for } 1 \le i \le n-3 \end{cases}$$

Finally, $g_3(e_{2n}) = 0$, $g_3(c_{n-2}) = 2$, $g_3(c_{n-1}) = 0$, $g_3(c_n) = 1$. It can be checked that $e_{g_3}(0,1,2) = (3x+1, 3x-1, 3x), v^1_{g_3} = (x+2, x-1, x), v^2_{g_3} = (x+1, x-1, x)$. This is not edge-3-equitable labeling. The gear tip vertices $v_2, v_4, \ldots, v_{2n}$ get the labels $2,0,1,2,0,1\ldots 2,0,1,2,0,0$.

Labeling $\mathbf{h_3}$: One defines the labeling h_3 as follows:

$$h_3(e_i) \begin{cases} = 0, i \equiv 1,5 \bmod 6, \\ = 1, i \equiv 2,4 \bmod 6, \\ = 2, i \equiv 3,0 \bmod 6 \\ \text{for } 1 \le i \le 2n-1 \end{cases} \qquad h_3(c_i) \begin{cases} = 1, i \equiv 0 \bmod 3, \\ = 2, i \equiv 1 \bmod 3, \\ = 0, i \equiv 2 \bmod 3 \\ \text{for } 1 \le i \le n-3 \end{cases}$$

Finally, $h_3(e_{2n}) = 0$, $h_3(c_{n-2}) = 2$, $h_3(c_{n-1}) = 1$, $h_3(c_n) = 1$. It can be checked that $e_{h_3}(0,1,2) = (3x, 3x+1, 3x-1), v^1_{h_3} = (x-1, x+1, x+1)$, $v^2_{h_3} = (x+1, x, x-1)$. This is not edge-3-equitable labeling though it labels the vertices equitably. The gear tip vertices $v_2, v_4, \ldots, v_{2n}$ get the labels $1,0,2,1,0,2,\ldots 1,0,2,1,0,0$.

Labeling $\mathbf{k_3}$: One defines the labeling k_3 as follows:

$$k_3(e_i) \begin{cases} = 0, i \equiv 1,2 \bmod 6, \\ = 1, i \equiv 3,4 \bmod 6, \\ = 2, i \equiv 5,0 \bmod 6 \\ \text{for } 1 \le i \le 2n \end{cases} \qquad k_3(c_i) \begin{cases} = 0, i \equiv 1 \bmod 3, \\ = 2, i \equiv 2 \bmod 3, \\ = 1, i \equiv 3 \bmod 3 \\ \text{for } 1 \le i \le n \end{cases}$$

One can see that $k_3(v_0) = 0$ and $v^1_{k_3}(0,1,2) = (x+1, x, x), v^2_{k_3}(0,1,2) = (x,x,x), e_{k_3}(0,1,2) = (3x, 3x, 3x)$. Hence K_3 is an edge-3-equitable labeling. The gear tips get the labels $0,2,1,0,2,1,\ldots,0,2,1$.

Labeling $\mathbf{l_3}$: One defines the labeling l_3 as follows:

$$l_3(e_i) \begin{cases} = 0, i \equiv 1,2 \bmod 6, \\ = 1, i \equiv 3,4 \bmod 6, \\ = 2, i \equiv 5,0 \bmod 6 \\ \text{for } 1 \le i \le 2n \end{cases} \qquad l_3(c_i) \begin{cases} = 0, i \equiv 0 \bmod 3, \\ = 2, i \equiv 1 \bmod 3, \\ = 1, i \equiv 2 \bmod 3 \\ \text{for } 1 \le i \le n-1 \end{cases}$$

$l_3(c_n) = 1$. One can see that $l_3(v_0) = 0$. Thus, $v^1_{l_3}(0,1,2) = (x-1, x+2, x), v^2_{l_3}(0,1,2) = (x,x,x), e_{l_3}(0,1,2) = (3x-1, 3x+1, 3x)$. Hence K_3 is an edge-3-equitable labeling. The gear tips get the labels $0,2,1,0,2,1,\ldots,0,2,1$.

Labeling $\mathbf{p_3}$: One defines the labeling p_3 as follows:

$$p_3(e_i) \begin{array}{l} = 0, i \equiv 1,2 \bmod 6, \\ = 1, i \equiv 3,4 \bmod 6, \\ = 2, i \equiv 5,0 \bmod 6 \\ \text{for } 1 \leq i \leq 2n \end{array} \qquad p_3(c_i) \begin{array}{l} = 1, i \equiv 0 \bmod 3, \\ = 0, i \equiv 1 \bmod 3, \\ = 2, i \equiv 2 \bmod 3 \\ \text{for } 1 \leq i \leq n-1 \end{array}$$

Finally, $p_3(c_n) = 0$. It can be checked that $e_{p_3}(0,1,2) = (3x+1, 3x-1, 3x), v_{f_3}^1 = (x+1, x-1, x+1), v_{f_3}^2 = (x, x, x)$. This is not edge-3-equitable labeling though it labels the vertices equitably. The gear tip vertices $v_2, v_4, \ldots, v_{2n}$ get the labels $0, 2, 1, 0, 2, 1 \ldots$.

3 Labelings of $\overline{\mathbf{K}}_{\mathbf{n}}$ - union of gears

Through out this section we take copies $H_1, \cdots H_k$ of gear G_n and take G to be $\overline{K}_n$ -union of them. The final labeling constructed is called $f_{k,i}$, where i indicates the type of the gears under consideration.

Proposition: G is edge-3-equitable for $k = 1, 2, 3$.
Proof: For $k = 1$ we have the edge-3-equitable labelings p_1, p_2, k_3, for gears of type 1, 2, 3 respectively.

Case 1: $k = 2$. The following table explains which labelings are assigned to the individual gears. Since the $|E(G)|$ is a multiple of three and the labelings constructed are edge-3-equitable the edge numbers are not mentioned in this table.

gear (type)	labeling	sequence of gear tips
$H_1(1)$	g_1	$0, 2, 1, \cdots, 0, 2, 0, 0$
$H_2(1)$	p_1	$0, 2, 1, \cdots, 2, 0, 0, 0$
$G(1)$	$f_{2,1}$	$0, 1, 2, \cdots, 2, 2, 0, 0$
$H_1(2)$	f_2	$2, 0, 1, \cdots, 1, 1, 1, 0, 0$
$H_2(2)$	g_2	$2, 0, 1, \cdots, 1, 2, 2, 2, 1$
$G(2)$	$f_{2,2}$	$1, 0, 2, \cdots, 2, 0, 0, 2, 1$
$H_1(3)$	l_3	$0, 2, 1, \cdots, 0, 2, 1$
$H_2(3)$	p_3	$0, 2, 1, \cdots, 0, 2, 1$
$G(3)$	$f_{2,3}$	$0, 1, 2, \cdots, 0, 1, 2$

When we add the entries of the last column we see that the vertex numbers are as shown in the following table.

gear (type)	$v^1(0,1,2)$	$v^2(0,1,2)$
$H_1(1)$	$(x+2, x+2, x+1)$	$(x+3, x, x+1)$
$H_2(1)$	$(x, x+3, x+2)$	$(x+3, x, x+1)$
$G(1)$	$(2x+2, 2x+5, 2x+3)$	$(x+2, x, x+2)$
$H_1(2)$	$(x+2, x+2, x+2)$	$(x+2, x+3, x)$
$H_2(2)$	$(x+1, x+3, x+2)$	$(x, x+2, x+3)$
$G(2)$	$(2x+3, 2x+5, 2x+4)$	$(x+2, x+1, x+2)$
$H_1(3)$	$(x-1, x+2, x)$	(x, x, x)
$H_2(3)$	$(x+1, x-1, x+1)$	(x, x, x)
$G(3)$	$(2x, 2x+1, 2x+1)$	(x, x, x)

The sums in the last two columns show that $f_{2,i}$ is edge-3-equitable for each $i = 1, 2, 3$.

Case 2: $k = 3$. Again we give the assignment of labelings in the form of a table.

gear (type)	labeling	sequence of gear tips
$H_1(1)$	f_1	$2, 0, 1, \cdots, 0, 2, 2, 2$
$H_2(1)$	g_1	$0, 2, 1, \cdots, 0, 2, 0, 0$
H_3 (1)	h_1	$0, 2, 1, \cdots, 0, 1, 2, 1$
G (1)	$f(3,1)$	$2, 1, 0, \cdots, 0, 2, 1, 0$
H_1 (2)	h_2	$2, 0, 1, \cdots, 0, 2, 2, 2, 0$
H_2 (2)	k_2	$0, 2, 1, \cdots, 0, 2, 0, 0, 1$
H_3 (1)	l_2	$0, 2, 1, \cdots, 1, 1, 1, 2, 1$
G (2)	$f(3,2)$	$2, 1, 0, \cdots, 1, 2, 0, 1, 2$
H_1 (3)	f_3	$0, 2, 1, \cdots, 0, 2, 1$
H_2 (3)	g_3	$2, 0, 1, \cdots, 2, 0, 0$
H_3 (3)	h_3	$1, 0, 2, \cdots, 1, 0, 0,$
G (3)	$f(3,3)$	$0, 2, 1, \cdots, 0, 2, 1$

Again when we add the entries of the last column we see that the vertex numbers are as shown in the following table.

gear (type)	$v^1(0,1,2)$	$v^2(0,1,2)$
H_1 (1)	$(x+1,x+2,x+2)$	$(x+1,x,x+3)$
H_2 (1)	$(x+2,x+2,x+1)$	$(x+3,x,x+1)$
H_3 (1)	$(x+2,x+1,x+2)$	$(x+1,x+2,x+1)$
$G(1)$	$(3x+5,3x+5,3x+5)$	$(x+2,x+1,x+1)$
$H_1(2)$	$(x+2,x+2,x+2)$	$(x+2,x,x+3)$
$H_2(2)$	$(x+2,x+2,x+2)$	$(x+3,x+1,x+1)$
$H_3(1)$	$(x+2,x+2,x+2)$	$(x,x+4,x+1)$
$G(2)$	$(3x+6,3x+6,3x+6)$	$(x+1,x+2,x+2)$
$H_1(3)$	$(x,x+1,x)$	(x,x,x)
$H_2(3)$	$(x+2,x-1,x)$	$(x+1,x-1,x)$
$H_3(3)$	$(x-1,x+1,x+1)$	$(x+1,x,x-1)$
$G(3)$	$(3x+1,3x+1,3x+1)$	(x,x,x)

The sum of the respective last columns of bothe tables shows that $f_{2,i}$ and $f_{3,i}$ are edge-3-equitable for each $i=1,2,3$. •

Proposition: $\overline{K}_n$ -union of $3t$ gears is edge-3-equitable for all $t>1$.
Proof: As before we will present the assignment of labelings in a table. Since $f_{3,i}$ labels non-tip vertices equitably, only labels of gear tips are menioned in these tables. Let $H_1, H_2, \cdots, H_{3t}$ be copies on the gear G_n. First we form triple gears $L_1, \cdots, L_t$ using $H_1, \cdots, H_{3t}$.

Case 1: The gears are all of type 1. Let $n=3x+4$. We assign the labeling to $L_1, L_2, \ldots$ according to the following table:

gear	labeling	sequence of gear tips	$v^2(0,1,2)$
L_1	$f_{3,1}$	$2,1,0,2,1,0,2,1,0,\cdots,0,2,1,0$	$(x+2,x+1,x+1)$
L_2	$f_{3,1}$	$2,1,0,2,1,0,2,1,0,\cdots,0,2,1,0$	$(x+2,x+1,x+1)$
L_{2j+1}	$f_{3,1}$ reversed	$2,0,1,2,0,0,1,2,0,\cdots,1,2,0,1$ pivotal 2	$(x+2,x+1,x+1)$
L_{2j}	$f_{3,1}$	$2,1,0,2,1,0,2,1,0,\cdots,0,2,1,0$	$(x+2,x+1,x+1)$

The last two rows represent the assignment for all $j \geq 1$ and $j>1$ respectively. The resulting labeling is called $f_{3t,1}$. One can check that after identifying the gear tips we get the sequences of labels as shown in the following table. Since

t	Labels of gear tips in $f_{3t,1}$
$t \equiv 1 \bmod 6$	$2,1,0,2,1,0,2,1,0,\ldots,2,1,0,0,2,1,0.$
$t \equiv 2 \bmod 6$	$1,2,0,1,2,0,1,2,0,\ldots,1,2,0,0,1,2,0.$
$t \equiv 3 \bmod 6$	$0,2,1,0,2,0,2,1,0,\ldots,2,1,0,1,0,2,1.$
$t \equiv 4 \bmod 6$	$2,0,1,2,0,0,1,2,0,\ldots,1,2,0,1,2,0,1.$
$t \equiv 5 \bmod 6$	$1,0,2,1,0,0,2,1,0,\ldots,2,1,0,2,1,0,2.$
$t \equiv 6 \bmod 6$	$0,1,2,0,1,0,1,2,0,\ldots,1,2,0,2,0,1,2.$

Thus in all the cases, one has $v^2_{f_{3t,1}}(0,1,2) = (x+2, x+1, x+1)$, that is, $v_{f_{3t,1}}(0,1,2) = (3xt + 5t + x + 2, 3xt + 5t + x + 1, 3xt + 5t + x + 1)$. and $e_{f_{3t,1}}(0,1,2) = (nt, nt, nt)$. Thus, the combined gears of $3t$ gears is edge-3-equitable.

Case 2: All the gears are of type 2. Let $n = 3x+5$. We assign the labeling to $L_1, L_2, \ldots$ according to the following table:

gear	labeling	sequence of gear tips	$v^2(0,1,2)$
L_1	$f_{3,2}$	$2,1,0,2,1,0,2,1,0,\cdots,1,2,0,1,2$	$(x+1, x+2, x+2)$
L_2	$f_{3,2}$	$2,1,0,2,1,0,2,1,0,\cdots,1,2,0,1,2$	$(x+1, x+2, x+2)$
L_{2j+1}	$f_{3,2}$ reverse reversed	$0,1,2,0,1,2,0,1,2,\cdots,2,1,0,2,1$ pivotal $2n-4$	$(x+1, x+2, x+2)$
L_{2j}	$f_{3,2}$	$2,1,0,2,1,0,2,1,0,\cdots,1,2,0,1,2$	$(x+1, x+2, x+2)$

The resulting labeling is called $f_{3t,2}$. Clearly, $e_{f_{3t,2}}(0,1,2) = (3nt, 3nt, 3nt)$, $v^1_{f_{3t,2}}(0,1,2) = (9xt + 6t, 9xt + 6t, 9xt + 6t)$. The sequence of the labels of the identified gear tips is given in the following table:

t	Labels of gear tips in $f_{3t,2}$
$t \equiv 1 \bmod 6$	$2,1,0,2,1,0,2,1,0,\ldots,2,1,0,1,2,0,1,2$
$t \equiv 2 \bmod 6$	$1,2,0,1,2,0,1,2,0,\ldots,1,2,0,2,1,0,2,1$
$t \equiv 3 \bmod 6$	$1,0,2,1,0,2,1,0,2,\ldots,1,0,2,1,2,0,1,2.$
$t \equiv 4 \bmod 6$	$0,1,2,0,1,2,0,1,2,\ldots,0,1,2,2,1,0,2,1.$
$t \equiv 5 \bmod 6$	$0,2,1,0,2,1,0,2,1,\ldots,0,2,1,1,2,0,1,2.$
$t \equiv 6 \bmod 6$	$2,0,1,2,0,1,2,0,1,\ldots,2,0,1,2,1,0,2,1.$

In all the cases $v^2_{f_{3t,2}}(0,1,2) = (x+1, x+2, x+2)$. Thus $f_{3t,2}$ is edge-3-equitable.

Case 3: All the gears are of type 3. Let $n = 3x$. We assign the labeling to $L_1, L_2, \ldots$ according to the following table:

gear	labeling	sequence of gear tips	$v^2(0,1,2)$
L_1	$f_{3,3}$	$0,2,1,0,2,1,\cdots,0,2,1,$	(x,x,x)
L_2	$f_{3,3}$	$0,2,1,0,2,1,\cdots,0,2,1,$	(x,x,x)
L_{2j+1}	$f_{3,3}$ reversed, pivotal 2	$0,1,2,0,1,2,\cdots,0,1,2,$	$(x,x,x44)$
L_{2j}	$f_{3,3}$	$0,2,1,0,2,1,\cdots,0,2,1,$	(x,x,x)

Call the resulting labeling $f_{3t,3}$. The sequence of labels of the identified gear tips is ine the following table.

t	Labels of gear tips in $f_{3t,1}$	$v^2_{f_{3t,1}}(0,1,2)$
even	$0,1,2,0,1,2,\ldots,0,1,2.$	(x,x,x)
odd	$0,2,1,0,2,1,\ldots,0,2,1.$	(x,x,x)

One can check that $e_{f_{3t,3}}(0,1,2) = (9xt, 9xt, 9xt)$ and $v^1_{f_{3t,3}}(0,1,2) = (3xt+t, 3xt+t, 3xt+t)$. and $v^2_{f_{3t,3}}(0,1,2) = (x,x,x)$

This shows that $f_{3t,i}$ is edge-3-equitable for each $1 \leq i \leq 3$. •

Proposition: $\overline{K}_n$ -union of $3t+1$ gears is edge-3-equitable for all $t \geq 1$.
Proof: As before we divide the proof in three cases. We first take $\overline{K}_n$ -union of $3t$ copies of the gear G_n and assign it the labeling $f_{3t,i}$ constructed in the previous proposition. For the remaining copy of G_n we assign the labels according the tables given in each case. The labeling constructed is called $f_{3t+1,i}, 1 \leq i \leq 3$. Suppose $t \equiv r \bmod 6, 1 \leq r \leq 6$. Let $\Delta = t(n+1)$.

Case 1: All gears are of type 1. Let $n = 3x+4$. For the last copy H_{3t+1} we assign a different labeling depending on the value of r. The sum of the last columns in the following table show that $f_{3t+1,1}$ is edge-3-equitable.

Case 2: All gears are of type 2. Let $n = 3t+5$. As before, for the last copy of H_{3t+1} of G_n we assign a labeling as shown in the following table. The resulting labeling is called $f_{3t+1,2}$. Since the label numbers for $f_{3t+1,2}$ are same for all the cases, they are given only at the end. Here $\Delta = 3xt + 6t$. Clearly, $e_{f_{3t+1,2}}(0,1,2) = (n(3t+1), n(3t+1), n(3t+1)), v^1_{f_{3t+1,2}}(0,1,2) = (3xt+6t+x+2, 3xt+6t+x+2, 3xt+6t+x+2), v^2_{f_{3t+2,2}}(0,1,2) = (x+1, x+2, x+2)$, that is, G is edge-3-equitable.

r	Labeling	Sequence of gear tips	$v^1(0,1,2)$	$v^2(0,1,2)$
1	$f_{3t,1}$:	$2,1,0,2,1,0,2,1,0,\ldots,2,1,0,0,2,1,0.$	$(3xt+5t, 3xt+5t, 3xt+5t)$	$(x+2, x+1, x+1)$
	k_1 :	$0,2,1,0,2,1,0,2,1,\ldots,0,2,1,0,0,1,1.$	$(x+2, x+2, x+1)$	$(x+2, x+2, x)$
	$f_{3t+1,1}$:	$2,0,1,2,0,1,2,0,1,\ldots,2,0,1,0,2,2,1.$	$(3xt+5t+x+2, 3xt+5t+x+2, 3xt+5t+x+1)$	$(x+1, x+1, x+2)$
2	$f_{3t,1}$:	$1,2,0,1,2,0,1,2,0,\ldots,1,2,0,0,1,2,0.$	$(3xt+5t, 3xt+5t, 3xt+5t)$	$(x+2, x+1, x+1)$
	l_1 :	$1,2,0,1,2,0,1,2,0,\ldots,1,2,0,2,2,2,1.$	$(x+2, x+1, x+2)$	$(x, x+1, x+3)$
	$f_{3t+1,1}$:	$2,1,0,2,1,0,2,1,0,\ldots,2,1,0,2,0,1,1.$	$(3xt+5t+x+2, 3xt+5t+x+1, 3xt+5t+x+2)$	$(x+1, x+2, x+1)$
3	$f_{3t,1}$:	$0,2,1,0,2,0,2,1,0,\ldots,2,1,0,1,0,2,1.$	$(3xt+5t, 3xt+5t, 3xt+5t)$	$(x+2, x+1, x+1)$
	k_1 :	$1,0,2,1,0,2,1,0,2,\ldots,1,0,0,1,1,0,2.$	$(x+2, x+2, x+1)$	$(x+2, x+2, x)$
	double	left shift		
	$f_{3t+1,1}$:	$1,2,0,1,2,2,0,1,2,\ldots,0,1,0,2,1,2,0.$	$(3xt+5t+x+2, 3xt+5t+x+2, 3xt+5t+x+1)$	$(x+1, x+1, x+2)$
4	$f_{3t,1}$:	$2,0,1,2,0,0,1,2,0,\ldots,1,2,0,1,2,0,1.$	$(3xt+5t, 3xt+5t, 3xt+5t)$	$(x+2, x+1, x+1)$
	k_1 :	$0,1,1,0,0,1,2,0,1,\ldots,2,0,1,2,0,1,2.$	$(x+2, x+2, x+1)$	$(x+2, x+2, x)$
	reverse	pivotal 2		
	$f_{3t+1,1}$:	$2,1,2,2,0,1,0,2,1,\ldots,0,2,1,0,2,1,0.$	$(3xt+5t+x+2, 3xt+5t+x+2, 3xt+5t+x+1)$	$(x+1, x+1, x+2)$
5	$f_{3t,1}$:	$1,0,2,1,0,0,2,1,0,\ldots,2,1,0,2,1,0,2..$	$(3xt+5t, 3xt+5t, 3xt+5t)$	$(x+2, x+1, x+1)$
	f_1 :	$2,2,2,2,0,1,0,2,1,\ldots,0,2,1,0,2,1,0.$	$(x+1, x+2, x+2)$	$(x+1, x, x+3)$
	reverse	pivotal 2		
	$f_{3t+1,1}$:	$0,2,1,0,0,1,2,0,1,\ldots,2,0,1,2,0,1,2.$	$(3xt+5t+x+1, 3xt+5t+x+2, 3xt+5t+x+2)$	$(x+2, x+1, x+1)$
6	$f_{3t,1}$:	$0,1,2,0,1,0,1,2,0,\ldots,1,2,0,2,0,1,2.$	$(3xt+5t, 3xt+5t, 3xt+5t)$	$(x+2, x+1, x+1)$
	l_1 :	$2,1,1,2,0,1,2,0,1,\ldots,2,0,1,2,0,2,2.$	$(x+2, x+1, x+2)$	$(x, x+1, x+3)$
	double	right shift		
	$f_{3t+1,1}$:	$2,2,0,2,1,1,0,2,1,\ldots,0,2,1,1,0,0,1.$	$(3xt+5t+x+2, 3xt+5t+x+1, 3xt+5t+x+2)$	$(x+1, x+2, x+1)$

Table for the case 1

r	Labeling	Sequence of gear tips	$v^1(0,1,2)$	$v^2(0,1,2)$
1	$f_{3t,2}$:	$2,1,0,2,1,0,\ldots,1,2,0,1,2.$	(Δ,Δ,Δ)	$(x+1,x+2,x+2)$
	p_2 :	$0,2,1,0,2,1,\ldots,0,1,2,1,2.$	$(x+2,x+2,x+2)$	$(x+1,x+2,x+2)$
	$f_{3t+1,2}$:	$2,0,1,2,0,1,\ldots,1,0,2,2,1.$		
2	$f_{3t,2}$:	$1,2,0,1,2,0,\ldots,2,1,0,2,1.$	(Δ,Δ,Δ)	$(x+1,x+2,x+2)$
	h_2 :	$2,0,1,2,0,1,\ldots,0,2,2,2,0.$	$(x+2,x+2,x+2)$	$(x+2,x,x+3)$
	$f_{3t+1,2}$:	$2,0,1,2,0,1,\ldots,2,0,2,1,1.$		
3	$f_{3t,2}$:	$1,0,2,1,0,2,\ldots,1,2,0,1,2.$	(Δ,Δ,Δ)	$(x+1,x+2,x+2)$
	h_2 :	$1,0,2,1,0,2,\ldots,0,2,2,2,0.$	$(x+2,x+2,x+2)$	$(x+2,x,x+3)$
	reverse	pivotal $2n-4$		
	$f_{3t+1,2}$:	$2,0,1,2,0,1,\ldots,1,1,2,0,2.$		
4	$f_{3t,2}$:	$0,1,2,0,1,2,\ldots,2,1,0,2,1.$	(Δ,Δ,Δ)	$(x+1,x+2,x+2)$
	f_2 :	$2,0,1,2,0,1,\ldots,1,1,1,0,0.$	$(x+2,x+2,x+2)$	$(x+2,x+3,x)$
	$f_{3t+1,2}$:	$2,1,0,2,1,0,\ldots,0,2,1,2,1.$		
5	$f_{3t,2}$:	$0,2,1,0,2,1,\ldots,1,2,0,1,2.$	(Δ,Δ,Δ)	$(x+1,x+2,x+2)$
	p_2 :	$0,2,1,0,2,1,\ldots,0,1,2,1,2.$	$(x+2,x+2,x+2)$	$(x+1,x+2,x+2)$
	$f_{3t+1,2}$:	$0,1,2,0,1,2,\ldots,1,0,2,2,1.$		
6	$f_{3t,2}$:	$2,0,1,2,0,1,\ldots,2,1,0,2,1.$	(Δ,Δ,Δ)	$(x+1,x+2,x+2)$
	f_2 :	$2,0,1,2,0,1,\ldots,1,1,1,0,0.$	$(x+2,x+2,x+2)$	$(x+2,x+3,x)$
	$f_{3t+1,2}$:	$1,0,2,1,0,2,\ldots,0,2,1,2,1.$		

Table for the case 2

Case 3: All the gears are of type 3 and $n = 3x$. For this case we consider t odd and t even separately and the last gear H_{3t+1} is labeled accordingly. With $\Delta = 3xt + t$ we have,

t	Labeling	Sequence of gear tips	$v^1(0,1,2)$	$v^2(0,1,2)$
odd	$f_{3t,3}$:	$0,2,1,0,2,1,\ldots,0,2,1$	(Δ,Δ,Δ)	(x,x,x)
	k_3 :	$0,2,1,0,2,1,\ldots,0,2,1$	$(x+1,x,x)$	(x,x,x)
	$f_{3t+1,3}$	$0,1,2,0,1,2,\ldots,0,1,2$	$(\Delta+1,\Delta,\Delta)$ $+(x,x,x)$	(x,x,x)
even	$f_{3t,3}$:	$0,1,2,0,1,2,\ldots,0,1,2$	$(\Delta,\Delta,\Delta))$	(x,x,x)
	k_3 :	$0,1,2,0,1,2,\ldots,0,1,2$	$(x+1,x,x)$	(x,x,x)
	reverse	pivotal 2		
	$f_{3t+1,3}$	$0,2,1,0,2,1,\ldots,0,2,1.$	$(\Delta+1,\Delta,\Delta)$ $+(x,x,x,)$	(x,x,x)

This finishes all the cases and shows that $f_{3t+1,i}$ is edge-3-equitable for all $1 \leq i \leq 3$. •

Proposition: $\overline{K}_n$ -union of $3t+2$ gears is edge-3-equitable for all $t \geq 1$.
Proof: As before we divide the proof in three cases. We first take $\overline{K}_n$ - union of $3t$ copies of the gear G_n and assign it the labeling $f_{3t,i}$ constructed in the previous proposition. For the remaining copies of G_n we assign the labels according the tables given in each case. The labeling constructed is called $f_{3t+2,i}, 1 \leq i \leq 3$. Suppose $t \equiv r \bmod 6, 1 \leq r \leq 6$. As before $\Delta = t(n+1)$.

r	Labeling	Sequence of gear tips
1	$f_{3t,1}$:	$2,1,0,2,1,0,2,1,0,\ldots,2,1,0,0,2,1,0$
	k_1 :	$0,2,1,0,2,1,0,2,1,\ldots,0,2,1,0,0,1,1.$
	l_1 :	$1,2,0,1,2,0,1,2,0,\ldots,1,2,0,2,2,2,1$
	$f_{3t+2,1}$:	$0,2,1,0,2,1,0,2,1,\ldots,0,2,1,2,1,1,2.$
2	$f_{3t,1}$:	$1,2,0,1,2,0,1,2,0,\ldots,1,2,0,0,1,2,0.$
	l_1 :	$1,2,0,1,2,0,1,2,0,\ldots,1,2,0,2,2,2,1.$
	g_1 :	$0,2,1,0,2,1,0,2,1,\ldots,0,2,1,0,2,0,0.$
	$f_{3t+2,1}$:	$2,0,1,2,0,1,2,0,1,\ldots,2,0,1,2,2,1,1.$

t	Labeling	Sequence of gear tips
3	$f_{3t,1}$:	$0,2,1,0,2,0,2,1,0,\ldots,2,1,0,1,0,2,1.$
	k_1 :	$1,0,2,1,0,2,1,0,2,\ldots,1,0,0,1,1,0,2.$
	double	left shift
	f_1	$2,0,1,2,0,1,2,0,1,\ldots,2,0,1,0,2,2,2.$
	$f_{3t+2,1}$:	$0,2,1,0,2,0,2,1,0,\ldots,2,1,1,2,0,1,2$
4	$f_{3t,1}$:	$2,0,1,2,0,0,1,2,0,\ldots,1,2,0,1,2,0,1.$
	k_1 :	$0,1,1,0,0,1,2,0,1,\ldots,2,0,1,2,0,1,2.$
	reverse	pivotal 2
	l_1	$1,1,2,2,2,0,2,1,0,\ldots,2,1,0,2,1,0,2.$
	reverse	pivotal 2
	$f_{3t+2,1}$:	$0,2,1,1,2,1,2,0,1,\ldots,2,0,1,2,0,1,2$
5	$f_{3t,1}$:	$1,0,2,1,0,0,2,1,0,\ldots,2,1,0,2,1,0,2..$
	f_1 :	$2,2,2,2,0,1,0,2,1,\ldots,0,2,1,0,2,1,0.$
	reverse	pivotal 2
	g_1	$0,0,0,2,0,1,2,0,1,\ldots,2,0,1,2,0,1,2.$
	reverse	pivotal 2
	$f_{3t+2,2}$:	$0,2,1,2,0,2,1,0,2,\ldots,1,0,2,1,0,2,1.$
6	$f_{3t,1}$:	$0,1,2,0,1,0,1,2,0,\ldots,1,2,0,2,0,1,2.$
	l_1 :	$2,1,1,2,0,1,2,0,1,\ldots,2,0,1,2,0,2,2.$
	Double	right shift
	h_1	$2,1,0,2,1,0,2,1,0,\ldots,2,1,0,1,2,1,0.$
		left shift
	$f_{3t+2,1}$:	$1,0,0,1,2,1,2,0,1,\ldots,2,0,1,2,2,1,1.$

For each case when we add the entries of the respective labelings we see that the vertex numbers are as shown in the following table.

r	Labeling	$v^1(0,1,2)$	$v^2(0,1,2)$
1	$f_{3t,1}$:	(Δ,Δ,Δ)	$(x+2,x+1,x+1)$
	k_1 :	$(x+2,x+2,x+1)$	$(x+2,x+2,x)$
	l_1 :	$(x+2,x+1,x+2)$	$(x,x+1,x+3)$
	$f_{3t+2,1}$:	$(\Delta,\Delta,\Delta)+$ $(2x+4,2x+3,2x+3)$	$(x,x+2,x+2)$
2	$f_{3t,1}$:	(Δ,Δ,Δ)	$(x+2,x+1,x+1)$
	l_1 :	$(x+2,x+1,x+2)$	$(x,x+1,x+3)$
	g_1 :	$(x+2,x+2,x+1)$	$(x+3,x,x+1)$
	$f_{3t+2,1}$:	$(\Delta,\Delta,\Delta)+$ $(2x+4,2x+3,x+3)$	$(x,x+2,x+2)$

t	Labeling	$v^1(0,1,2)$	$v^2(0,1,2)$
3	$f_{3t,1}$: k_1 : double f_1 $f_{3t+2,1}$:	(Δ,Δ,Δ) $(x+2,x+2,x+1)$ left shift $(x+1,x+2,x+2)$ $(\Delta,\Delta,\Delta)+$ $(2x+3,2x+4,2x+3)$	$(x+2,x+1,x+1)$ $(x+2,x+2,x)$ $(x+1,x,x+3)$ $(x+1,x+1,x+2)$
4	$f_{3t,1}$: k_1 : reverse l_1 reverse $f_{3t+2,1}$:	(Δ,Δ,Δ) $(x+2,x+2,x+1)$ pivotal 2 $(x+2,x+1,x+2)$ pivotal 2 $(\Delta,\Delta,\Delta)+$ $(2x+4,2x+3,2x+3)$	$(x+2,x+1,x+1)$ $(x+2,x+2,x)$ $(x,x+1,x+3)$ $(x,x+2,x+2)$
5	$f_{3t,1}$: f_1 : reverse g_1 reverse $f_{3t+2,2}$:	(Δ,Δ,Δ) $(x+1,x+2,x+2)$ pivotal 2 $(x+2,x+2,x+1)$ pivotal 2 $(\Delta,\Delta,\Delta)+$ $(2x+3,2x+4,2x+3)$	$(x+2,x+1,x+1)$ $(x+1,x,x+3)$ $(x+3,x,x+1)$ $(x+1,x+1,x+2)$
6	$f_{3t,1}$: l_1 : Double h_1 $f_{3t+2,1}$:	(Δ,Δ,Δ) $(x+2,x+1,x+2)$ right shift $(x+2,x+1,x+2)$ left shift $(\Delta,\Delta,\Delta)+$ $(2x+4,2x+2,2x+4)$	$(x+2,x+1,x+1)$ $(x,x+1,x+3)$ $(x+1,x+2,x+1)$ $(x,x+3,x+1)$

Case 2: All the gears are type 2 and $n = 3x+5$. The first $3t$ gears get the labeling $f_{3t,2}$ and the last two gears are labeled as per the following table. Here the value of Δ is $3xt+6t$.

r	Labeling	Sequence of gear tips
1	$f_{3t,2}$:	$2,1,0,2,1,0,\ldots,1,2,0,1,2.$
	f_2 :	$2,0,1,2,0,1,\ldots,1,1,1,0,0.$
	p_2 :	$0,2,1,0,2,1,\ldots,0,1,2,1,2.$
	$f_{3t+2,2}$:	$1,0,2,1,0,2,\ldots,2,1,0,2,1.$
2	$f_{3t,2}$:	$1,2,0,1,2,0,\ldots,2,1,0,2,1.$
	f_2 :	$2,0,1,2,0,1,\ldots,1,1,1,0,0.$
	l_2 :	$0,2,1,0,2,1,\ldots,1,1,1,2,1.$
	$f_{3t+2,2}$:	$0,1,2,0,1,2,\ldots,1,0,2,1,2.$
3	$f_{3t,2}$:	$1,0,2,1,0,2,\ldots,1,2,0,1,2.$
	f_2 :	$2,0,1,2,0,1,\ldots,1,1,1,0,0.$
	p_2 :	$0,2,1,0,2,1,\ldots,0,1,2,1,2.$
	$f_{3t+2,2}$:	$0,2,1,0,2,1,\ldots,2,1,0,2,1.$
4	$f_{3t,2}$:	$0,1,2,0,1,2,\ldots,2,1,0,2,1.$
	f_2 :	$2,0,1,2,0,1,\ldots,1,1,1,0,0.$
	l_2 :	$0,2,1,0,2,1,\ldots,1,1,1,2,1.$
	$f_{3t+2,2}$:	$2,0,1,2,0,1,\ldots,1,0,2,1,2.$
5	$f_{3t,2}$:	$0,2,1,0,2,1,\ldots,1,2,0,1,2.$
	p_2 :	$0,2,1,0,2,1,\ldots,0,1,2,1,2.$
	k_2 :	$1,2,0,1,2,0,\ldots,1,0,0,2,0.$
	$f_{3t+2,2}$:	$1,0,2,1,0,2,\ldots,2,0,2,1,1.$
6	$f_{3t,2}$:	$2,0,1,2,0,1,\ldots,2,1,0,2,1.$
	f_2 :	$2,0,1,2,0,1,\ldots,1,1,1,0,0.$
	l_2 :	$0,2,1,0,2,1,\ldots,1,1,1,2,1.$
	$f_{3t+2,2}$:	$1,2,0,1,2,0,\ldots,1,0,2,1,2.$

After calculating the labels of the sequence for $f_{3t+2,i}$ we see that the vertex numbers of these labelings are as shown in the following table.

r	Labeling	$v^1(0,1,2)$	$v^2(0,1,2)$
1	$f_{3t,2}$:	(Δ,Δ,Δ)	$(x+1,x+2,x+2)$
	f_2 :	$(x+2,x+2,x+2)$	$(x+2,x+3,x)$
	p_2 :	$(x+2,x+2,x+2)$	$(x,x+2,x+2)$
	$f_{3t+2,2}$:	$(\Delta+2x+4,\Delta+2x+4,\Delta+2x+4)$	$(x+1,x+2,x+1)$
2	$f_{3t,2}$:	(Δ,Δ,Δ)	$(x+1,x+2,x+2)$
	f_2 :	$(x+2,x+2,x+2)$	$(x+2,x+3,x)$
	l_2 :	$(x+2,x+2,x+2)$	$(x,x+4,x+1)$
	$f_{3t+2,2}$:	$(\Delta+2x+4,\Delta+2x+4,\Delta+2x+4)$	$(x+1,x+2,x+2)$
3	$f_{3t,2}$:	(Δ,Δ,Δ)	$(x+1,x+2,x+2)$
	f_2 :	$(x+2,x+2,x+2)$	$(x+2,x+3,x)$
	p_2 :	$(x+2,x+2,x+2)$	$(x+1,x+2,x+2)$
	$f_{3t+2,2}$:	$(\Delta+2x+4,\Delta+2x+4,\Delta+2x+4)$	$(x+1,x+2,x+2)$
4	$f_{3t,2}$:	(Δ,Δ,Δ)	$(x+1,x+2,x+2)$
	f_2 :	$(x+2,x+2,x+2)$	$(x+2,x+3,x)$—
	l_2 :	$(x+2,x+2,x+2)$	$(x,x+4,x+1)$
	$f_{3t+2,2}$:	$(\Delta+2x+4,\Delta+2x+4,\Delta+2x+4)$	$(x+1,x+2,x+2)$
5	$f_{3t,2}$:	(Δ,Δ,Δ)	$(x+1,x+2,x+2)$
	p_2 :	$(x+2,x+2,x+2)$	$(x+1,x+2,x+2)$
	k_2 :	$(x+2,x+2,x+2)$	$(x+3,x+1,x+1)$
	$f_{3t+2,2}$:	$(\Delta+2x+4,\Delta+2x+4,\Delta+2x+4)$	$(x+1,x+2,x+2)$
6	$f_{3t,2}$:	(Δ,Δ,Δ)	$(x+1,x+2,x+2)$
	f_2 :	$(x+2,x+2,x+2)$	$(x+2,x+3,x)$
	l_2 :	$(x+2,x+2,x+2)$	$(x,x+4,x+1)$
	$f_{3t+2,2}$:	$(\Delta+2x+4,\Delta+2x+4,\Delta+2x+4)$	$(x+1,x+2,x+2)$

Finally we come to the last case.

Case 3: All gears are of type 3 and $n = 3x$. For the first $3t$ gears we assign the labeling $f_{3t,3}$ and for the remaining two gears we assign the labelings as per the following table. Here $\Delta = 3xt + t$. For all the labelings in both the cases $v^2(0,1,2) = (x,x,x)$.

t	Labeling	Sequence of gear tips	$v^1(0,1,2)$
odd	$f_{3t,3}$:	$0,2,1,0,2,1,\ldots,0,2,1$	(Δ,Δ,Δ)
	l_3 :	$0,2,1\ldots,0,2,1$	$(x-1,x+2,x)$
	p_3 :	$0,1,2\ldots,0,1,2$	$(x+1,x-1,x+1)$
	reverse	pivotal 2	
	$f_{3t+2,3}$	$0,2,1,\ldots,0,2,1.$	$(\Delta+2x,\Delta+2x+1,\Delta+2x+1)$
even	$f_{3t,3}$:	$0,1,2,0,1,2,\ldots,0,1,2$	(Δ,Δ,Δ)
	l_3 :	$0,2,1\ldots,0,2,1$	$(x-1,x+2,x)$
	p_3 :	$0,1,2\ldots,0,1,2$	$(x+1,x-1,x+1)$
	reverse	pivotal 2	
	$f_{3t+2,3}$	$0,1,2,\ldots,0,1,2.$	$(\Delta+2x,\Delta+2x+1,\Delta+2x+1)$

This shows that $f_{3t_2,i}$ is edge-3-equitable for all $1 \leq i \leq 3$. •

Some Questions:

1: Can one identify the gear tips in any order (not necessarily cyclic) and still get edge-3-equitable labelings of $\overline{K}_n$-union of gears?
2: If $G_{n_1},\ldots,G_{n_T}$ are gears with $n_1 \leq n_2, \leq \ldots, \leq n_T$, can we take $\overline{K}_{n_1}$-union of them and still find an edge-3-equitable labeling of them?

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