The first nontrivial three color upper domination Ramsey number is 13[†]

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Abstract. The upper domination Ramsey number $u(3,3,3)$ is the smallest integer n such that every 3-coloring of the edges of complete graph K_n contains a monochromatic graph G with $\Gamma(\overline{G}) \geq 3$, where $\Gamma(\overline{G})$ is the maximum order over all the minimal dominating sets of the complement of G. In this note, with the help of computers, we determine that $U(3,3,3) = 13$, which improves the results that $13 \leq U(3,3,3) \leq 14$ provided by Michael A. Henning and Ortrud R. Oellermann.

1 Introduction

Let G be a graph without multiple edges or loops. The set of vertices and edges of G is denoted by $V(G)$ and $E(G)$ respectively, the complementary graph of G is denoted by $\overline{G}$. Let $v \in V(G)$, the open neighborhood is $N_G(v) = \{u \in V | uv \in E(G)\}$ and the closed neighborhood of v is $N_G[v] = \{v\} \cup N_G(v)$. For $S \in V(G)$, the open neighborhood of S is $N_G(S) = \bigcup_{v \in S} N(v)$, and the closed neighborhood of S is $N_G[S] = N_G(S) \cup S$. A

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set $S \in V(G)$ is a dominating set if for any $v \notin S$ is adjacent to a vertex in S, i.e, $N_G[S] = V(G)$. The independence number $\beta(G)$ of G is the maximum cardinality among the independent sets of vertices of G. The upper domination number $\Gamma(G)$ is the maximum cardinality of a minimal dominating set of G.

For positive integers $k_1, k_2, \cdots, k_r$, the Ramsey number $R(k_1, k_2, \cdots, k_r)$ is defined as the least positive integer n such that every r-coloring of the edges of complete graph K_n contains a complete graph of order k_i with color i for some $1 \leq i \leq r$.

For positive integers $k_1, k_2, \cdots, k_r$, The upper domination Ramsey number $u(k_1, k_2, \cdots, k_r)$ is defined as the smallest n such that for every r-edge coloring $G_1, G_2, \cdots, G_r$ of K_n, there is at least one $i \in \{1, 2, \cdots, r\}$ for which $\Gamma(G_i) \geq k_i$.

A complete graph G with edges colored with r colors $\{1, 2, \cdots, r\}$ is called a $(k_1, k_2, \cdots, k_r)$-R-graph if G does not contain a complete graph with color i for some $i \in \{1, 2, \cdots, r\}$. A complete graph G with edges colored with r colors $\{1, 2, \cdots, r\}$ is called a $(k_1, k_2, \cdots, k_r)$-U-graph if G does not contain a subgraph H with color i satisfying $\Gamma(\overline{H}) \geq k_i$ for some $i \in \{1, 2, \cdots, r\}$. It is easy to see that

Lemma 1 *if G is a $(k_1, k_2, \cdots, k_r)$-U-graph of order n, then $u(k_1, k_2, \cdots, k_r) \geq n + 1$.*

The upper domination Ramsey numbers were researched recent years. They were mainly studied by M. A. Henning and O. R. Oellermann [2, 3]. Until now, very few exact values were determined. The known two color upper domination Ramsey numbers were shown in Theorem 1 (see [2]).

Theorem 1 *$u(3,3) = 6$, $u(3,4) = 8$, $u(3,5) = 12$, $u(3,6) = 15$, $u(3,7) = 6$.*

There are no known nontrivial three color upper domination Ramsey numbers. In [2], by a very complicated proof, it is shown that

Theorem 2 *$13 \leq u(3,3,3) \leq 14$.*

In this note, we give a computer assisted proof showing that $u(3,3,3) = 13$.

2 Computation of $u(3,3,3)$

Before computation, we need some definitions.

Definition 1 *Let G_1, G_2, G_3 be graphs of order 13 and their vertices are all labeled with $\{1, 2, \cdots, 13\}$. The graph $K = G_1 \oplus G_2 \oplus G_3$ is defined* *as follows.* $V(K) = \{1, 2, \cdots, 13\}$*,* $(x, y) \in E(K)$ *iff there is exactly one* $i \in \{1, 2, 3\}$ *such that* $(x, y) \in G_i$ *for* $x, y \in V(K)$*. We call this operation* XOR union.

Definition 2 *Let* G *be a graph and* f *be a permutation on* $V(G)$*.* $f(G)$ *is defined as follows.* $V(f(G)) = V(G)$*,* $E(f(G)) = \{(f(x), f(y)|(x, y) \in E(G))\}$.

The relation between Ramsey numbers and upper domination Ramsey numbers are discussed in [2, 3], from which the following results hold.

Lemma 2 *Let* $k_1, k_2, \cdots, k_r$ *be positive integers,*
(1). $u(k_1, k_2, \cdots, k_r) \leq R(k_1, k_2, \cdots, k_r)$.
(2). A $(k_1, k_2, \cdots, k_r)$*-U-graph is a* $(k_1, k_2, \cdots, k_r)$*-R-graph.*

In order to obtain the upper bound 13 for $u(3, 3, 3)$, we need to establish some lemmas.

Suppose to the contrary that $u(3, 3, 3) \geq 14$, then there exists a $(3, 3, 3)$-U-graph of order 13. Then there exists a 3-coloring G_1, G_2, G_3 of the edges of K_{13} such that $\Gamma(\overline{G_1}) \leq 2$, $\Gamma(\overline{G_2}) \leq 2$ and $\Gamma(\overline{G_3}) \leq 2$. By Lemma 2, G is also a $(3, 3, 3)$-R-graph and G_1, G_2, G_3 contain no 3 independent set. By merging two colors in G, we can obtain a $(6, 3)$-R-graph.

Therefore, we can execute the following steps to determine whether there exists a $(3, 3, 3)$-U-graph of order 13. For convenience, we label the vertices of the considered graphs with $\{1, 2, \cdots, 13\}$. Firstly, enumerate all the $(6, 3)$-R-graphs. Secondly, for each triple H_1, H_2, H_3 of $(6, 3)$-R-graphs, search permutation f, g on $\{1, 2, \cdots, 13\}$ such that $K_{13} = H_1 \oplus f(H_2) \oplus g(H_3)$. If there are no such triple, then there does not exist a $(3, 3, 3)$-U-graph of order 13 and hence $u(3, 3, 3) = 13$.

By computer search, the following useful result is obtained, where all the $(6, 3; 13)$-R-graphs can be found in [4].

Lemma 3 *In all the* $(6, 3; 13)$*-R-graphs, there are* 2135 *graphs whose maximum cardinality over all the minimal dominating sets of its complement graph is less than 3.*

Although the number 13 is small, there are $(13!)^2 \times \binom{2135+3-1}{3}$ steps to perform by brute force search. Therefore, wee need to reduce the search space to a feasible level. Then main operation is to find f, g such that the *XOR union* of $G_1, f(G_2), g(G_3)$ forms K_{13} for three graphs G_1, G_2, G_3.

To implement this efficiently, we use the following Function 1. Even though Function 1 is simple, it is effective for this problem. The vertices of G_1, G_2 and G_3 are labeled with $\{1, 2, \cdots, 13\}$. Function 1 generates the permutations f, g of $V(G_2) = V(G_3) = \{1, 2, \cdots, 13\}$. When the condition of line 4 holds, we get permutations of f and g, then check if the resulting graphs is valid. In Function 1, the following results are ensured.

Function 1 bool MatchThreeGraphs(G_1, G_2, G_3, f, g,FLevel,GLevel)

```
if |E(G1)| + |E(G2)| + |E(G3)| ≠ n(n − 1)/2 then
  return false;
end if
if FLevel = n and GLevel = n then
  if G1 ⊕ f(G2) ⊕ g(G3) = Kn then
    return true;
  else
    return false;
  end if
end if
if FLevel = GLevel then
  if FLevel > 1 then
    if the edges in the graph induced by {0, 1, ···, FLevel − 2} in
    G1, G2, G3 are not valid, return false;
  end if
  for i = FLevel to n − 1 do
    swap f[FLevel] and f[i]
    if MatchThreeGraphs(G1, G2, G3, f, g, FLevel + 1, GLevel) then
      return true;
    end if
    swap f[FLevel] and f[i]
  end for
else
  if GLevel > 1 then
    if the edges in the graph induced by {0, 1, ···, FLevel − 2} in
    G1, G2, G3 are not valid, return false;
  end if
  for i = GLevel to n − 1 do
    swap g[GLevel] and g[i]
    if MatchThreeGraphs(G1, G2, G3, f, g, FLevel, GLevel + 1) then
      return true;
    end if
    swap g[GLevel] and g[i]
  end for
end if
return false;
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1. The sum of the edge number of G_1, G_2 and G_3 equals to the edge number of the complete graph of K_{13}. (see line 1 and line 2)

2. f and g are permutations of $V(G_2)$ and $V(G_3)$, respectively. Line 15-21 is the permutation of $V(G_2)$ and Line 26-32 is the permutation of $V(G_3)$.

3. For each edge $e = (i, j)$, $1 \le i < j \le 13$, e only lies in one of the graphs G_1, G_2, G_3. This can be restricted by line 12-14 and 23-25.

All the 2135 graphs in Lemma 3 are applied and each triple is carried out using Function 1. All the graphs are successfully processed on a computer with 2.66G Hz CPU and 2G memory within 12 hours. Before calling Function 1, we set $n = 13$, $f[i] = i$ and $g[i] = i$ for $1 \le i \le n$. Then we call the Function MatchThreeGraphs(G_1, G_2, G_3, f, g,0,0). If it returns true, then G_1, G_2, G_3 forms a valid triple. Otherwise they are not. The output shows that there exists no triple in all the 2135 graphs in Lemma 3, and thus we have

Theorem 3 $u(3,3,3) = 13$.

3 Discussion

The determination of upper domination Ramsey number is more difficult than that of Ramsey numbers. Until now, several values of $u(m, n)$ such as $u(4,4)$, $u(3,7)$ and $u(3,8)$ remain unresolved for positive integers m, n. Without computer search, they are very difficult to determine. For two color upper domination Ramsey number, the value of $u(m, n)$ can be computed by testing all the (m, n)-R-graphs. For multicolor upper domination Ramsey number such as $u(3,3,3)$, $u(3,3,4)$ and $u(3,3,3,3)$, we can consider the method in this paper.

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