On Metric Dimension of Graphs and Their Complements

Linda Eroh[1], **Cong X. Kang**[2] and **Eunjeong Yi**[3]
[1] University of Wisconsin Oshkosh, Oshkosh, WI 54901, USA
[2,3]Texas A&M University at Galveston, Galveston, TX 77553, USA
[1]*eroh@uwosh.edu*; [2]*kangc@tamug.edu*; [3]*yie@tamug.edu*

December 30, 2011

Abstract

The *metric dimension* of a graph G, denoted by $\dim(G)$, is the minimum number of vertices such that all vertices are uniquely determined by their distances to the chosen vertices. For a graph G and its complement $\overline{G}$, each of order $n \geq 4$ and connected, we show that $2 \leq \dim(G) + \dim(\overline{G}) \leq 2(n-3)$. It is readily seen that $\dim(G) + \dim(\overline{G}) = 2$ if and only if $n = 4$. We characterize graphs satisfying $\dim(G)+\dim(\overline{G}) = 2(n-3)$ when G is a tree or a unicyclic graph.

Key Words: distance, resolving set, metric dimension, metric dimension of the complement of a graph, tree, unicyclic graph
2000 Mathematics Subject Classification: 05C12

1 Introduction

Let $G = (V(G), E(G))$ be a finite, simple, undirected, connected graph of order $|V(G)| = n \geq 2$ and size $|E(G)|$. For $W \subseteq V(G)$, we denote by $\langle W \rangle$ the subgraph of G induced by W. The *degree* of a vertex $v \in V(G)$ is the the number of edges incident to the vertex v; an *end-vertex* is a vertex of degree one, and a *support vertex* is a vertex that is adjacent to an end-vertex. We denote by $\Delta(G)$ the maximum degree of a graph G. We denote by K_n, C_n, and P_n the complete graph, the cycle, and the path, respectively, on n vertices. The *distance* between vertices $v, w \in V(G)$, denoted by $d_G(v, w)$, is the length of the shortest path between v and w; we omit G when ambiguity is not a concern. The *complement* of G, denote by $\overline{G}$, is the graph such that $V(\overline{G}) = V(G)$, $E(G) \cup E(\overline{G}) = E(K_n)$, and $E(G) \cap E(\overline{G}) = \emptyset$. A vertex $x \in V(G)$ *resolves* a pair of vertices $v, w \in V(G)$

if $d(v,x) \neq d(w,x)$. A set of vertices $S \subseteq V(G)$ *resolves* G if every pair of distinct vertices of G is resolved by some vertex in S; then S is called a *resolving set* of G. For an ordered set $S = \{u_1, u_2, \ldots, u_k\} \subseteq V(G)$ of distinct vertices, the *metric code* (or *code*, for short) of $v \in V(G)$ with respect to S is the k-vector $(d(v,u_1), d(v,u_2), \ldots, d(v,u_k))$; it is denoted by $code_S(v)$, and we drop S if the meaning is clear in context. The *metric dimension* of G, denoted by $\dim(G)$, is the minimum cardinality over all resolving sets of G. For other terminologies in graph theory, refer to [3].

Slater [14, 15] introduced the concept of a resolving set for a connected graph under the term *locating set*. He referred to a minimum resolving set as a *reference set*, and the cardinality of a minimum resolving set as the *location number* of a graph. Independently, Harary and Melter in [8] studied these concepts under the term *metric dimension*. Since metric dimension is suggestive of the dimension of a vector space in linear algebra, sometimes a minimum resolving set of G is called a *basis* for G. Metric dimension as a graph parameter has numerous applications, among them are robot navigation [11], sonar [14], combinatorial optimization [13], and pharmaceutical chemistry [4]. In [7], it is noted that determining the metric dimension of a graph is an NP-hard problem. Metric dimension has been heavily studied; for a survey, see [6]. For more on metric dimension in graphs, see [2], [4], [5], [7], [8], [9], [11], [12], [14], [15].

The problem of characterizing connected graphs G of order n for which $\dim(G) = n-3$ is posed in [1]. An anonymous referee pointed out that our result (stated in the following paragraph) when G is a tree is a special case of Theorem 2.14 of the paper [9], "Extremal Graph Theory for Metric Dimension and Diameter"; see Remark 3.4. Also, it came to our attention after an earlier draft of this paper that, in March of 2011, the paper [10], "Characterization of n-Vertex Graphs with Metric Dimension $n-3$", was posted on the arXiv. Though this present paper may be regarded as dealing with a very special case of the $\dim(G) = n-3$ problem, our approach here is different from that taken in [9] or [10]. Further, our characterization here is simple and explicit, and it is thus of independent interest.

In this paper, we study metric dimension of graphs and their complements. For connected graphs G and $\overline{G}$ of order $n \geq 4$, we show that $2 \leq \dim(G) + \dim(\overline{G}) \leq 2(n-3)$. We show that $\dim(G) + \dim(\overline{G}) = 2$ if and only if $n = 4$. We further characterize graphs satisfying $\dim(G) + \dim(\overline{G}) = 2(n-3)$ when G is a tree or a unicyclic graph.

2 Bounds for $\dim(G) + \dim(\overline{G})$

We recall some results obtained in [4].

Theorem 2.1. *[4] Let G be a connected graph of order $n \geq 2$. Then*

(a) $\dim(G) = 1$ *if and only if* $G = P_n$,

(b) $\dim(G) = n-1$ *if and only if* $G = K_n$,

(c) for $n \geq 4$, $\dim(G) = n-2$ if and only if $G = K_{s,t}$ $(s,t \geq 1)$, $G = K_s + \overline{K_t}$ $(s \geq 1, t \geq 2)$, or $G = K_s + (K_1 \cup K_t)$ $(s,t \geq 1)$; here, $A + B$ denotes the graph obtained from the disjoint union of graphs A and B by joining every vertex of A with every vertex of B.

Note that if $\dim(G) = n-2$ then $\overline{G}$ is disconnected. So, we have the following

Theorem 2.2. *Let G and $\overline{G}$ be connected graphs of order $n \geq 4$. Then*

$$2 \leq \dim(G) + \dim(\overline{G}) \leq 2(n-3),$$

and both bounds are sharp. Moreover, $\dim(G) + \dim(\overline{G}) = 2$ if and only if $n = 4$.

Proof. Let G and $\overline{G}$ be connected graphs of order $n \geq 4$. Then $1 \leq \dim(G), \dim(\overline{G}) \leq n-3$, and thus the bounds follow. If $G = \overline{G} = P_4$ (the only possible decomposition when $n = 4$, see Figure 1), then $\dim(G) = \dim(\overline{G}) = 1$, achieving both the upper and the lower bounds of Theorem 2.2. If $G = \overline{G} = C_5$ (see Figure 2), then $\dim(G) + \dim(\overline{G})$ achieves the upper bound of Theorem 2.2. Next, we show that $\dim(G) + \dim(\overline{G}) = 2$ if and only if $n = 4$. Notice that

$$\dim(G) + \dim(\overline{G}) = 2 \iff \dim(G) = \dim(\overline{G}) = 1$$
$$\iff G \cong \overline{G} \cong P_n \text{ by Theorem 2.1 (a)} \iff |E(G)| = |E(\overline{G})| = n-1.$$

Now, $|E(K_n)| = \frac{n(n-1)}{2} = 2(n-1)$, which implies that $n = 4$. □

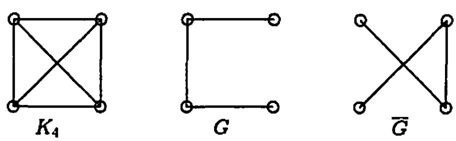

Figure 1: A graph and its complement that achieve the upper and lower bounds of Theorem 2.2

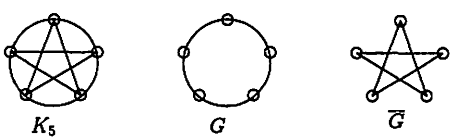

Figure 2: A graph and its complement that achieve the upper bound of Theorem 2.2

Remark 2.3. *We stress that, in order for both G and $\overline{G}$ to be connected and of order n, $\dim(G) + \dim(\overline{G}) = 2(n-3)$ is equivalent to $\dim(G) = n-3 = \dim(\overline{G})$.*

3 Characterization of $\dim(G)+\dim(\overline{G})=2(n-3)$ when G is a tree

The following definitions are stated in [4]. Fix a graph G. A vertex of degree at least three is called a *major vertex*. An end-vertex u is called *a terminal vertex of a major vertex* v if $d(u,v) < d(u,w)$ for every other major vertex w. The *terminal degree* of a major vertex v is the number of terminal vertices of v. A major vertex v is an *exterior major vertex* if it has positive terminal degree. Let $\sigma(G)$ denote the sum of terminal degrees of all major vertices of G, and let $ex(G)$ denote the number of exterior major vertices of G. Two vertices $u, v \in V(G)$ are called *twins* if $N(u) \setminus \{v\} = N(v) \setminus \{u\}$, where $N(u)$ is the set of all vertices adjacent to u in G; notice that for any set S with $S \cap \{u,v\} = \emptyset$, $code_S(u) = code_S(v)$.

Theorem 3.1. *[4] If T is a tree that is not a path, then* $\dim(T) = \sigma(T) - ex(T)$.

Next we state the following

Observation 3.2. *Let G and $\overline{G}$ be connected graphs of order $n \geq 4$.*

(a) If G is P_n, then $\dim(G) + \dim(\overline{G}) = 2(n-3)$ *if and only if* $n = 4$.

(b) If G is C_n, then $\dim(G) + \dim(\overline{G}) = 2(n-3)$ *if and only if* $n = 5$.

Theorem 3.3. *Let G and $\overline{G}$ be connected graphs of order $n \geq 5$. Also, let G be a tree. Then* $\dim(G)+\dim(\overline{G}) = 2(n-3)$ *if and only if G is a tree with $ex(G) = 1$, $\sigma(G) = n-2$, and one support vertex of degree two.*

Proof. ($\Longrightarrow$) By Observation 3.2 (a), if G is P_n for $n \geq 5$, then $\dim(G)+\dim(\overline{G}) < 2(n-3)$. Let G be a tree that is not a path such that $\dim(G) = n-3$. If $ex(G) \geq 2$, then $\sigma(G) \leq n-2$. By Theorem 3.1, $\dim(G) \leq n-4$. So, $ex(G) \leq 1$. Since G is not a path, $ex(G) = 1$ and $\sigma(G) \leq n-1$. If $\sigma(G) = n-1$, then $\Delta(G) = n-1$, but then $\overline{G}$ is disconnected. So, $\sigma(G) \leq n-2$. To achieve $\dim(G) = n-3$, we have $\sigma(G) = n-2$. So, we have only one vertex, say s, in G that is neither an end-vertex nor a major exterior vertex; hence, s is a support vertex of degree two.

($\Longleftarrow$) Let $V(G) = \{v, s, \ell_1, \ell_2, \ldots, \ell_{n-2}\}$ such that v is the major exterior vertex, s is the support vertex of degree two, and $\ell_1, \ell_2, \ldots, \ell_{n-2}$ are end-vertices of G with $s\ell_1 \in E(G)$ (see Figure 3). By Theorem 3.1, we have $\dim(G) = n-3$. Next, we consider $\dim(\overline{G})$. We denote by $\overline{S}$ a resolving set for $\overline{G}$. Since no vertex in $\{v, s, \ell_1\}$ distinguishes any two vertices in $\{\ell_j \mid 2 \leq j \leq n-2\}$ and that $\langle\{\ell_2, \ell_3, \ldots, \ell_{n-2}\}\rangle \cong K_{n-3}$ (i.e., any two vertices in $\{\ell_j \mid 2 \leq j \leq n-2\}$ are twins), $(n-4)$ vertices of $\{\ell_j \mid 2 \leq j \leq n-2\}$ must belong to $\overline{S}$. Without loss of generality, let $\overline{S_0} = \{\ell_2, \ell_3, \ldots, \ell_{n-3}\} \subseteq S$. Since $code_{\overline{S}_0}(\ell_1) = code_{\overline{S}_0}(s)$ and $d_{\overline{G}}(\ell_1, \ell_{n-2}) = d_{\overline{G}}(s, \ell_{n-2})$, one vertex in $\{v, s, \ell_1\}$ must belong to $\overline{S}$. Thus, $|\overline{S}| \geq n-3$. On the other hand, one can readily check that $\{\ell_i \mid 1 \leq i \leq n-3\}$ forms a resolving set for $\overline{G}$, and thus $\dim(\overline{G}) \leq n-3$. Therefore, if G is a tree as described in this theorem, $\dim(G) = \dim(\overline{G}) = n-3$. □

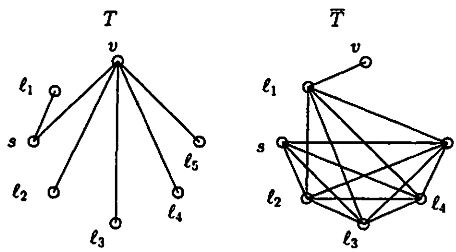

Figure 3: A tree T of order $n = 7$ satisfying $\dim(T) + \dim(\overline{T}) = 2(n-3)$

Remark 3.4. *Theorem 2.14 of [9] contains a characterization for a connected graph of order n and diameter D achieving metric dimension $n-D$. As mentioned in the introduction, the anonymous referee pointed out that the family of trees described in Theorem 3.3 has been implicitly shown in Theorem 2.14 of [9]. First, one notices that a tree of order n and diameter $D = 2$ must be $K_{1,n-1}$ for $n \geq 3$, which has metric dimension $n-2$. It thus suffices to let $D = 3$ and let G be a tree in Theorem 2.14 of [9], in order to obtain our result in Theorem 3.3. See [9] for details.*

4 Characterization of $\dim(G)+\dim(\overline{G})=2(n-3)$ when G is a unicyclic graph

The *cycle rank* of a graph G, denoted by $r(G)$, is defined as $|E(G)| - |V(G)| + 1$. For a tree T, $r(T) = 0$. If a graph G has $r(G) = 1$, we call it a *unicyclic* graph. By $T + e$, we shall mean a unicyclic graph obtained from a tree T by attaching a new edge e.

Theorem 4.1. *[12] If T is a tree of order at least three and e is an edge of $\overline{T}$, then*

$$\dim(T) - 2 \leq \dim(T + e) \leq \dim(T) + 1.$$

Now, we state our main characterization theorem whose proof relies upon results which will follow the main theorem.

Theorem 4.2. *Let G and $\overline{G}$ be connected graphs of order $n \geq 5$. Further, let G be a unicyclic graph. Then $\dim(G) + \dim(\overline{G}) = 2(n-3)$ if and only if*

(i) $n = 5$, *or*

(ii) $n \geq 6$ *and G is isomorphic to H, where* $V(H) = \{v, s, \ell_1, \ell_2, \ldots, \ell_{n-2}\}$ *and* $E(H) = \{v\ell_i \mid 2 \leq i \leq n-2\} \cup \{vs, s\ell_1, \ell_1\ell_2\}$.

Proof. Let G and $\overline{G}$ be connected graphs of order $n \geq 5$. Then $\Delta(G) \leq n-2$ and $\Delta(\overline{G}) \leq n-2$. Let S be a resolving set for G and let $\overline{S}$ be a resolving set for $\overline{G}$.

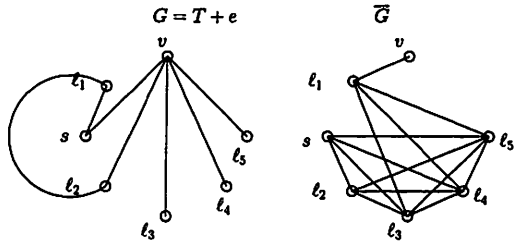

Figure 4: Unicyclic graph $G = T + e$ of order $n = 7$ satisfying $\dim(G) = \dim(\overline{G}) = n - 3$

($\Longleftarrow$) By Lemma 4.3, if $n = 5$, $\dim(G) = \dim(\overline{G}) = 2$, and hence $\dim(G) + \dim(\overline{G}) = 4 = 2(n-3)$. Next, we consider $n \geq 6$. Suppose that $G \cong H$ is the graph described in (ii); see Figure 4; then it is also the graph (B) in Figure 7. For the proof of $\dim(H) = \dim(\overline{H}) = n-3$, see Case 1 in the proof of Proposition 4.7.

($\Longrightarrow$) Write $G = T + e$, where T is a tree on $n \geq 5$ vertices and $e \in E(\overline{T})$, and suppose $n \neq 5$; i.e., $n \geq 6$.

Case 1. $ex(T) \geq 3$: In this case, $\sigma(T) \leq n - 3$, and thus $\dim(T) \leq n - 6$ by Theorem 3.1. By Theorem 4.1, $\dim(T+e) \leq n-5$, and thus $\dim(G) + \dim(\overline{G}) < 2(n-3)$.

Case 2. $ex(T) = 2$: By Proposition 4.5, $\dim(G) + \dim(\overline{G}) < 2(n-3)$.

Case 3. $ex(T) = 1$: By Proposition 4.7, if $\dim(G) + \dim(\overline{G}) = 2(n-3)$, then G is isomorphic to H, as specified in the statement of the present theorem.

Case 4. $ex(T) = 0$: That is, $T = P_n$ for $n \geq 5$. By Lemma 4.4, if G is $P_n + e$, then $\dim(G) + \dim(\overline{G}) = 2(n-3)$ if and only if $n = 5$. □

Lemma 4.3. *Let G and $\overline{G}$ be connected graphs of order 5. If G is a unicyclic graph, then* $\dim(G) = 2 = \dim(\overline{G})$.

Proof. Since G is not a path and since $\overline{G}$ needs to be connected, by the classification Theorem 2.1, $\dim(G) = 2$. Since $\overline{G}$ (the complement of a unicyclic graph in K_5) can not be a path and its complement (namely G) is connected, again by Theorem 2.1, $\dim(\overline{G}) = 2$. □

Lemma 4.4. *Let G and $\overline{G}$ be connected graphs of order $n \geq 5$. Additionally, let $G = P_n + e$. Then* $\dim(G) + \dim(\overline{G}) = 2(n-3)$ *if and only if $n = 5$.*

Proof. ($\Longleftarrow$) This is implied by Lemma 4.3.

($\Longrightarrow$) First, recall that the connectedness of G and $\overline{G}$ makes the condition $\dim(G) + \dim(\overline{G}) = 2(n-3)$ equivalent to the condition $\dim(G) = n - 3 = \dim(\overline{G})$. For

$n \geq 5$, let $P_n \subseteq G$ be a *fixed* path with the vertex set $\{v_i \mid 1 \leq i \leq n\}$ and the edge set $\{v_i v_{i+1} \mid 1 \leq i \leq n-1\}$. Since G contains a cycle, $\dim(G) \geq 2$. We consider three cases.

Case 1. G is a cycle: The metric dimension of any cycle is 2, and so $n = 5$. Now apply Lemma 4.3.

Case 2. G has one end-vertex: Let $e = E(G) \setminus E(P_n)$. We may, without loss of generality, assume that $e = v_1 v_j$ for some j, where $3 \leq j \leq n-1$ (see Figure 5). One can easily check that $S = \{v_{j-1}, v_n\}$ is a resolving set for G, and thus $\dim(G) = 2$ and $n = 5$. Again, apply Lemma 4.3.

Case 3. G has two end-vertices: In this case, the set $S = \{v_1, v_n\}$ is a resolving set for G. To see this, let the extra edge $e = v_i v_j$, where $i < j$ and $j - i \geq 2$ (see Figure 5). S will distinguish all vertices of $G' = G - C$ from each other; here C is the cycle $v_i, v_{i+1}, \ldots, v_{j-1}, v_j, v_i$. Note that S also distinguishes G' from C. Now, C is resolved by v_i and v_j, which are adjacent in C, and so it is also resolved by S; i.e., $(d(x, v_1), d(x, v_n)) = (d(x, v_i) + i - 1, d(x, v_j) + n - j)$, where $x \in V(C)$. Thus, we have $\dim(G) = 2$ and $n = 5$; so, apply Lemma 4.3 again.

In every case, we see that $\dim(G) + \dim(\overline{G}) = 2(n-3)$ implies $n = 5$. □

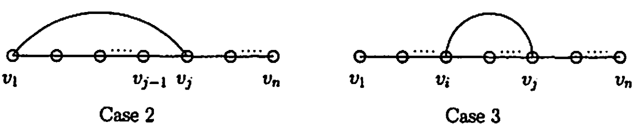

Figure 5: Unicyclic graphs $G = P_n + e$ of order $n \geq 5$ with $\dim(G) = 2$

Proposition 4.5. *Let G and $\overline{G}$ be connected graphs of order n. Also, let $G = T+e$, with $e \in E(\overline{T})$ and $ex(T) = 2$ (notice then $n \geq 6$). Then* $\dim(G)+\dim(\overline{G}) < 2(n-3)$.

Proof. Notice that $ex(T) = 2$ implies that the number of major vertices must be two, as well. Since $\sigma(T) \leq n-2$, $\dim(T) \leq n-4$ by Theorem 3.1. By Theorem 4.1, $\dim(T+e) \leq n-3$. In order for $G = T+e$ to satisfy $\dim(G) = n-3$, we must have $\sigma(T) = n-2$: this means that there are $(n-2)$ end-vertices. We will show that $\dim(G) < n-3$, and hence $\dim(G)+\dim(\overline{G}) < 2(n-3)$. We denote by v_1 and v_2 the two major exterior vertices of T, and let $N(v_1) \setminus \{v_2\} = \{\ell_i \mid 1 \leq i \leq t\}$, where $t \geq 2$; also let $N(v_2) \setminus \{v_1\} = \{\ell_j \mid t+1 \leq j \leq n-2\}$, where $n \geq 6$. There are only three potentially distinct cases to consider: 1) $e = \ell_1 \ell_2$; 2) $e = \ell_t \ell_{t+1}$; 3) $e = \ell_t v_2$; see Figure 6. In each case, one readily checks that the set $S = \{\ell_2, \ldots, \ell_t, \ell_{t+1}, \ldots, \ell_{n-3}\}$ forms a resolving set of cardinality $n-4$. Therefore, $\dim(G) < n-3$ in each case, and hence $\dim(G) + \dim(\overline{G}) < 2(n-3)$. □

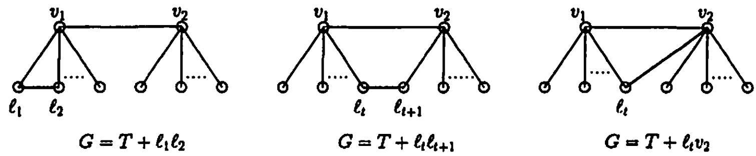

Figure 6: Unicyclic graphs $G = T + e$ with $ex(T) = 2$

The upper bound in the following theorem allows us to immediately see that $\dim(G) < n - 3$ in some cases. The *diameter*, *diam*(G), of a graph G is given by $\max\{d(u, v) \mid u, v \in V(G)\}$.

Theorem 4.6. *[4] If G is a connected graph of order $n \geq 2$ and diameter d, then*

$$f(n, d) \leq dim(G) \leq n - d,$$

where $f(n, d)$ is the least positive integer k for which $k + d^k \geq n$.

Proposition 4.7. *Let G and $\overline{G}$ be connected graphs of order n. Also, let $G = T + e$, with $e \in E(\overline{T})$ and $ex(T) = 1$ (it follows that $n \geq 5$ since, for $\overline{G}$ to be connected, $\Delta(T) \leq n-2$). If $\dim(G) + \dim(\overline{G}) = 2(n-3)$, then G is isomorphic to H, where $V(H) = \{v, s, \ell_1, \ell_2, \ldots, \ell_{n-2}\}$ and $E(H) = \{v\ell_i \mid 2 \leq i \leq n-2\} \cup \{vs, s\ell_1, \ell_1\ell_2\}$.*

Proof. Notice that $ex(T) = 1$ implies that the number of major vertices must be one. Since $\Delta(G) \leq n - 2$, $\sigma(T) \leq n - 2$ and $\dim(T) \leq n - 3$ by Theorem 3.1. In order for $G = T + e$ to satisfy $\dim(G) = n - 3$, we must have $\sigma(T) = n - 2$ or $\sigma(T) = n - 3$, by Theorem 4.1.

Case 1. $\sigma(T) = n - 2$: Let v be the major exterior vertex, let s be the support vertex of degree two, and let $\ell_1, \ell_2, \ldots, \ell_{n-2}$ be the end-vertices of a tree T such that $s\ell_1 \in E(T)$. Since $\dim(G) + \dim(\overline{G}) = 4 = 2(n - 3)$ for $n = 5$ by Lemma 4.3, we consider for $n \geq 6$. If $e = v\ell_1$, then $\deg_G(v) = n - 1$ and $\overline{G}$ would be disconnected. There are three other graphs $G = T + e$ up to isomorphism for consideration: (A) $e = s\ell_2$, (B) $e = \ell_1\ell_2$, (C) $e = \ell_2\ell_3$; see Figure 7. In cases (A) and (C), one can readily check that the set $S = \{\ell_1, \ell_3, \ell_4, \ldots, \ell_{n-3}\}$ forms a resolving set for G of cardinality $n - 4$, and thus $\dim(G) < n - 3$.

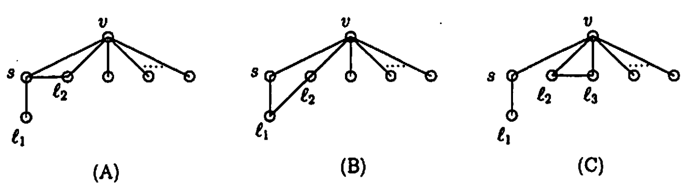

Figure 7: Unicyclic graphs $G = T + e$ with $ex(T) = 1$ and $\sigma(T) = n - 2$

Next, we will show that $\dim(G) = \dim(\overline{G}) = n-3$ in case (B). Since no vertex in $\{v, s, \ell_1, \ell_2\}$ distinguishes any two vertices in $\{\ell_i \mid 3 \leq i \leq n-2\}$ and that any two vertices in $\{\ell_i \mid 3 \leq i \leq n-2\}$ are twins, at least $(n-5)$ vertices of $\{\ell_i \mid 3 \leq i \leq n-2\}$ must belong to any resolving set S for G; without loss of generality, let $S_0 = \{\ell_i \mid 3 \leq i \leq n-3\} \subseteq S$. Since $code_{S_0}(s) = code_{S_0}(\ell_2) = code_{S_0}(\ell_{n-2})$, noting that s and ℓ_2 are twins, s or ℓ_2 must belong to S; let us say $\ell_2 \in S$. Let $S_1 = \{\ell_i \mid 2 \leq i \leq n-3\} \subseteq S$. Since $code_{S_1}(s) = code_{S_1}(\ell_{n-2})$, at least a vertex in $V(G) \setminus S_1$ must belong to S, and thus $|S| \geq n-3$. On the other hand, we have $\dim(G) \leq n-3$ by Theorem 2.2, and thus $\dim(G) = n-3$ for $n \geq 6$.

Now, it remains to show that $\dim(\overline{G}) = n-3$ for $n \geq 6$ (see Figure 4). Since no vertex in $\{v, s, \ell_1, \ell_2\}$ distinguishes any two vertices in $\{\ell_i \mid 3 \leq i \leq n-2\}$ and that $\langle\{\ell_3, \ell_4, \ldots, \ell_{n-2}\}\rangle \cong K_{n-4}$ in $\overline{G}$, at least $(n-5)$ vertices of $\{\ell_i \mid 3 \leq i \leq n-2\}$ must belong to any resolving set $\overline{S}$ for $\overline{G}$, say $\overline{S_0} = \{\ell_i \mid 3 \leq i \leq n-3\} \subseteq \overline{S}$. Since $code_{\overline{S_0}}(s) = code_{\overline{S_0}}(\ell_1) = code_{\overline{S_0}}(\ell_2) = code_{\overline{S_0}}(\ell_{n-2})$, at least one vertex in $\{v, s, \ell_1, \ell_2, \ell_{n-2}\}$ must belong to $\overline{S}$. If $v \in \overline{S}$ or $\ell_1 \in \overline{S}$, then s and ℓ_2 have the same code; if $s \in \overline{S}$, then ℓ_2 and ℓ_{n-2} have the same code; if $\ell_2 \in \overline{S}$, then s and ℓ_{n-2} have the same code; if $\ell_{n-2} \in \overline{S}$, then s, ℓ_1, and ℓ_2 have the same code. So, at least two vertices of $V(\overline{G}) \setminus \overline{S_0}$ must belong to $\overline{S}$, and hence $|\overline{S}| \geq n-3$. On the other hand, we have $\dim(\overline{G}) \leq n-3$ by Theorem 2.2, and thus $\dim(\overline{G}) = n-3$ for $n \geq 6$. Therefore, if G is isomorphic to $H \cong T + \ell_1\ell_2$, then $\dim(G) = \dim(\overline{G}) = n-3$ for $n \geq 6$.

Case 2. $\sigma(T) = n-3$: In this case, $n \geq 6$. Since $\dim(T) = n-4$ by Theorem 3.1, we have $\dim(G) = \dim(T+e) \leq n-3$ for $n \geq 6$ by Theorem 4.1. Since $ex(T) = 1$ and $\sigma(T) = n-3$, either T has two support vertices of degree two or, of the two vertices of degree two, T has only one support vertex.

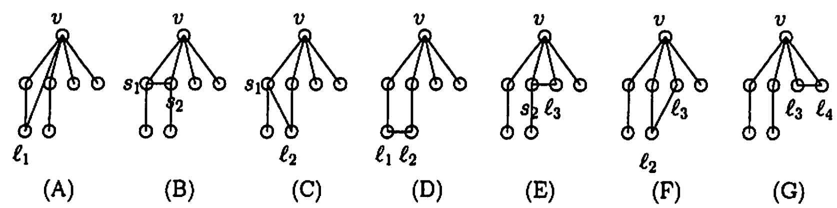

Figure 8: Unicyclic graphs $G = T + e$ of order 7 such that $ex(T) = 1$, $\sigma(T) = n-3$, and two support vertices of degree two in T

Subcase 2.1. T has two support vertices of degree two: We will show that $\dim(G) < n-3$, and hence $\dim(G) + \dim(\overline{G}) < 2(n-3)$ for $n \geq 6$. Let v be the major exterior vertex, let s_1 and s_2 be the support vertices of degree two, and let $\ell_1, \ell_2, \ldots, \ell_{n-3}$ be the end-vertices of a tree T such that $s_1\ell_1$, $s_2\ell_2 \in E(T)$. There exist seven not-obviously-isomorphic unicyclic graphs $G = T + e$: (A) $e = v\ell_1$, (B) $e = s_1s_2$, (C) $e = s_1\ell_2$, (D) $e = \ell_1\ell_2$, (E) $e = s_2\ell_3$, (F) $e = \ell_2\ell_3$, (G)

$e = \ell_3\ell_4$; see Figure 8. In cases (E), (F), and (G), we have $d_G(\ell_1, \ell_2) = 4$ and hence $diam(G) > 3$; so, $\dim(G) < n-3$ by Theorem 4.6. In cases (A), (B), (C), and (D), one can readily check that the set $S = \{\ell_1, \ell_2, \ldots, \ell_{n-4}\}$ forms a resolving set for G of cardinality $n-4$, and hence $\dim(G) < n-3$. Therefore, $\dim(G) < n-3$ in each case, and hence $\dim(G) + \dim(\overline{G}) < 2(n-3)$ for $n \geq 6$.

Subcase 2.2. T has a unique support vertex of degree two: Let v be the major exterior vertex, let s be the support vertex of degree two, let s' be the vertex of degree two that is adjacent to v and s, and let $\ell_1, \ell_2, \ldots, \ell_{n-3}$ be the end-vertices of a tree T such that $s\ell_1 \in E(T)$. There exist at most seven unicyclic graphs $G = T+e$ up to isomorphism: (A) $e = vs$, (B) $e = v\ell_1$, (C) $e = s'\ell_1$, (D) $e = s'\ell_2$, (E) $e = s\ell_2$, (F) $e = \ell_1\ell_2$, (G) $e = \ell_2\ell_3$; see Figure 9.

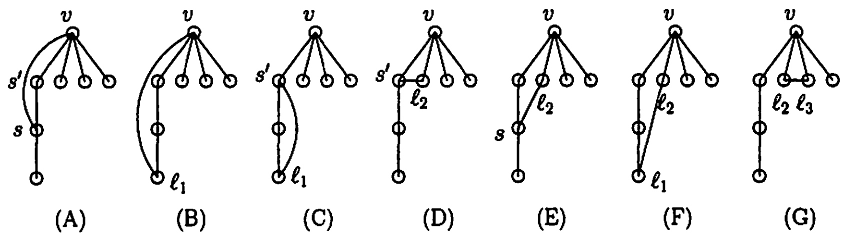

Figure 9: Unicyclic graphs $G = T + e$ of order 7 such that $ex(T) = 1$, $\sigma(T) = n-3$, and one support vertex of degree two in T

Among the seven graphs, we note that (B) of Figure 9 (the graph $G = T + v\ell_1$) is isomorphic to the graph $G = T + \ell_1\ell_2 \cong H$ in Case 1 (see (B) of Figure 7), where $\dim(H) + \dim(\overline{H}) = 2(n-3)$ for $n \geq 6$. For the other six cases, we will show that $\dim(G) < n-3$ for $n \geq 6$. In cases (D), (E), and (G) of Figure 9, we have $d_G(\ell_1, \ell_3) = 4$ and hence $diam(G) > 3$; thus, $\dim(G) < n-3$ by Theorem 4.6. In cases (A), (C), and (F) of Figure 9, one can readily check that the set $S = \{\ell_1, \ell_2, \ldots, \ell_{n-4}\}$ forms a resolving set for G of cardinality $n-4$, and hence $\dim(G) < n-3$. □

Acknowledgement. The authors thank the referee for some helpful comments and suggestions – especially for bringing to their attention Theorem 2.14 of [9].

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