Note on strict-double-bound numbers of paths, cycles, and wheels

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Abstract

For a poset $P = (X, \leq_P)$, the *strict-double-bound graph* (*sDB-graph*) of $P = (X, \leq_P)$ is the graph $\mathrm{sDB}(P)$ on X for which vertices u and v of $\mathrm{sDB}(P)$ are adjacent if and only if $u \neq v$ and there exist x and y in X distinct from u and v such that $x \leq u \leq y$ and $x \leq v \leq y$. The *strict-double-bound number* $\zeta(G)$ is defined as $\min\{n; G \cup N_n$ is a strict-double-bound graph $\}$, where N_n is the graph with n vertices and no edges.

In this paper we deal with strict-double-bound numbers of some graphs. For example, we obtain that $\zeta(P_n) = \left\lceil 2\sqrt{n-1} \right\rceil$ $(n \geq 2)$, $\zeta(C_n) = \left\lceil 2\sqrt{n} \right\rceil$ $(n \geq 4)$, $\zeta(W_n) = \left\lceil 2\sqrt{n-1} \right\rceil$ $(n \geq 5)$, and $\zeta(G + K_n) = \zeta(G)$ for a graph G with no isolated vertices.

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1 Introduction

In this paper we consider finite undirected simple graphs. Let $P = (X, \leq_P)$ be a poset and $x \in X$ an element of P. We put $U_P(x) = \{y \in X \ ; \ x \leq_P y\}$ and $L_P(x) = \{y \in X \ ; \ y \leq_P x\}$, and denoted by $\mathrm{Max}(P)$ the set of all maximal elements of P and $\mathrm{Min}(P)$ the set of all minimal elements of P.

McMorris and Zaslavsky [3] introduced a concept of double bound graphs. Diny [1] characterized double bound graphs. Scott [5] introduced a concept of strict-double-bound graphs. In [2] Era and Tsuchiya dealt with a concept of strict-double-bound graphs. In [4] Ogawa, Tagusari and Tsuchiya gave some results on strict-double-bound numbers.

For a poset $P = (X, \leq)$, the *strict-double-bound graph* (*sDB-graph*) of $P = (X, \leq)$ is the graph on X for which u and v are adjacent if and only if $u \neq v$ and there exist $x \in X$ and $y \in X$ distinct from u and v such that $x \leq u \leq y$ and $x \leq v \leq y$. We say that a graph G is a *strict-double-bound graph* if there exists a poset whose strict-double-bound graph is isomorphic to G. Then maximal elements and minimal elements of a poset P are isolated vertices of sDB(P). Thus almost connected graphs are not strict-double-bound graphs. So we introduce a concept of strict-double-bound numbers. The *strict-double-bound number* $\zeta(G)$ of G is defined as $\min\{n \ ; \ G \cup N_n$ is a strict-double-bound graph $\}$, where N_n is the graph with n vertices and no edges.

Ogawa, Tagusari, Tsuchiya [4] and Scott [5] obtained the following results on strict-double-bound numbers.

A *clique* in a graph G is the vertex set of a maximal complete subgraph of G. A family $\mathcal{Q} = \{Q_1, Q_2, \ldots, Q_n\}$ of cliques of G is an *edge clique cover* of G if and only if for each edge $uv \in E(G)$, there exists $Q_i \in \mathcal{Q}$ such that $u, v \in Q_i$. In some cases we allow that an edge clique cover contains empty sets. The following results give an upper bound and a lower bound of strict-double-bound numbers.

Theorem 1.1 (Scott [5] 1987) *For a connected graph G and its minimum edge clique cover $\mathcal{Q}$,* $\left\lceil 2\sqrt{|\mathcal{Q}|} \right\rceil \leq \zeta(G) \leq |\mathcal{Q}| + 1$. □

Theorem 1.2 (Ogawa, Tagusari and Tsuchiya [4]) *Let G be a graph with a minimum edge clique cover $\mathcal{Q}(G) = \{Q_1, Q_2, ..., Q_l\}$ such that there exists non-maximal clique $Q \neq \emptyset$ satisfying the following conditions:*
(1) for $Q_i, Q_j \in \mathcal{Q}(G)$, $Q_i \cap Q_j = Q$ and,
(2) for each $Q_i \in \mathcal{Q}(G)$, $Q_i - Q \neq \emptyset$.
Then $\zeta(G) = \left\lceil 2\sqrt{|\mathcal{C}(G)|} \right\rceil$. □

Theorem 1.3 (Ogawa, Tagusari and Tsuchiya [4]) *For a star $K_{1,n}$,* $\zeta(K_{1,n}) = \left\lceil 2\sqrt{n} \right\rceil$. □

Ogawa, Tagusari and Tsuchiya [4] also gave an upper bound of strict-double-bound numbers on trees. In this paper we deal with other families of graphs, whose strict-double-bound numbers are the lower bound of Theorem 1.1.

2 Strict-double-bound numbers of paths.

In this section we consider strict-double-bound numbers of paths.

Proposition 2.1 *Let G be a graph with a minimum edge clique cover $\mathcal{Q}(G) = \{Q_1, Q_2, ..., Q_l\}$ such that (1) $Q_i \cap Q_{i+1} \neq \emptyset$ $(i = 1, 2, ..., l-1)$ and (2) $Q_i \cap Q_j = \emptyset$ (if $j \neq i-1, i+1$). Then $\zeta(G) = \lceil 2\sqrt{l} \rceil$.*

Proof. For $l \leq 3$, $\zeta(G) = \lceil 2\sqrt{l} \rceil$ by Theorem 1.1. So we assume that $l \geq 4$. Let m and n be integers such that $\lceil 2\sqrt{l} \rceil = m + n$ and $0 \leq n - m \leq 1$.

For a minimal edge clique cover $\mathcal{Q}(G)$, we get an edge clique cover $\mathcal{Q}'(G) = \{Q_{1,1}, ..., Q_{1,n}, Q_{2,1}, ..., Q_{2,n}, ..., Q_{m,1}, ..., Q_{m,n}\}$ such that

$$Q_{j,k} = \begin{cases} Q_{(j-1)n+k} & \text{if } 1 \leq j \leq m-1 \text{ and } 1 \leq k \leq n, \\ Q_{(j-1)n+k} & \text{if } j = m \text{ and } 1 \leq k \leq n - mn + l, \\ \emptyset & \text{if } j = m \text{ and } n - mn + l + 1 \leq k \leq n. \end{cases}$$

For an edge clique cover $\mathcal{Q}'(G)$, we construct a poset $P_{m,n}$ such that (1) $V(P_{m,n}) = V(G) \cup \{x_1, x_2, ..., x_m\} \cup \{y_1, y_2, ..., y_n\}$ and (2) for $v \in Q_{j,k} \in \mathcal{Q}'(G)$, $y_h \leq_{P_{m,n}} v \leq_{P_{m,n}} x_j$ $(h \equiv j - k + 1 (\text{mod } m))$. Then $\text{sDB}(P_{m,n}) = G \cup N_{m+n}$, $\zeta(G) = m + n = \lceil 2\sqrt{l} \rceil$ by Theorem 1.1. □

For a path $P_n (n \geq 2)$, each clique is K_2 and P_n is covered by $n-1$ cliques. So we obtain a following result by Proposition 2.1.

Corollary 2.2 *For a path P_n with $n \geq 2$, $\zeta(P_n) = \lceil 2\sqrt{n-1} \rceil$.* □

3 Strict-double-bound numbers of cycles

Next we consider strict-double-bound numbers of cycles.

Proposition 3.1 *Let G be a graph with a minimum edge clique cover $\mathcal{Q}(G) = \{Q_1, Q_2, ..., Q_l\}$ such that (1) $Q_i \cap Q_{i+1} \neq \emptyset$ $(i = 1, 2, ..., l-1)$, (2) $Q_l \cap Q_1 \neq \emptyset$, (3) $Q_i \cap Q_j = \emptyset$ (if $j \neq i-1, i+1$) and (4) $Q_l \cap Q_i = \emptyset$ (if $i \neq 1, l-1$). Then $\zeta(G) = \lceil 2\sqrt{l} \rceil$.*

Proof. For $l \leq 3$, $\zeta(G) = \lceil 2\sqrt{l} \rceil$ by Theorem 1.1. So we assume that $l \geq 4$. We consider two cases on $\lceil 2\sqrt{l} \rceil$ as follows:

Case 1. $\lceil 2\sqrt{l} \rceil$ is even.

Let $\lceil 2\sqrt{l} \rceil = 2n$ and $q = n^2 - l$. For a minimal edge clique cover $\mathcal{Q}(G)$, we get an edge clique cover $\mathcal{Q}'(G) = \{Q_{1,1}, ..., Q_{1,n}, Q_{2,1}, ..., Q_{2,n}, ..., Q_{n,1}, ..., Q_{n,n}\}$ such that

$$Q_{j,k} = \begin{cases} Q_{(j-1)(n-1)+k} & \text{if } 1 \leq j \leq q \text{ and } k = 1, \\ \emptyset & \text{if } 1 \leq j \leq q \text{ and } k = 2, \\ Q_{(j-1)(n-1)+k-1} & \text{if } 1 \leq j \leq q \text{ and } 3 \leq k \leq n, \\ Q_{(j-1)n+k-q} & \text{if } q+1 \leq j \leq n \text{ and } 1 \leq k \leq n. \end{cases}$$

For an edge clique cover $\mathcal{Q}'(G)$, we construct a poset $P_{n,n}$ such that (1) $V(P_{n,n}) = V(G) \cup \{x_1, x_2, ..., x_n\} \cup \{y_1, y_2, ..., y_n\}$ and (2) for $v \in Q_{j,k} \in \mathcal{Q}'(G)$, $y_h \leq_{P_{n,n}} v \leq_{P_{n,n}} x_j$ $(h \equiv j - k + 1 (\text{mod } n))$. Then $\text{sDB}(P_{n,n}) = G \cup N_{2n}$, $\zeta(G) = 2n = \lceil 2\sqrt{l} \rceil$ by Theorem 1.1.

Case 2. $\lceil 2\sqrt{l} \rceil$ is odd.

Let $\lceil 2\sqrt{l} \rceil = 2n - 1$ and $q = n^2 - n - l$. For a minimal edge clique cover $\mathcal{Q}(G)$, we get an edge clique cover $\mathcal{Q}'(G) = \{Q_{1,1}, ..., Q_{1,n}, Q_{2,1}, ..., Q_{2,n}, ..., Q_{n-1,1}, ..., Q_{n-1,n}\}$ such that

$$Q_{j,k} = \begin{cases} Q_{(j-1)n+k} & \text{if } 1 \leq j \leq n-2 \text{ and } 1 \leq k \leq n, \\ Q_{(j-1)n+k} & \text{if } j = n-1 \text{ and } 1 \leq k \leq n-q-1, \\ \emptyset & \text{if } j = n-1 \text{ and } n-q \leq k \leq n-1, \\ Q_l & \text{if } j = n-1 \text{ and } k = n. \end{cases}$$

For an edge clique cover $\mathcal{Q}'(G)$, we construct a poset $P_{n,n-1}$ such that (1) $V(P_{n,n-1}) = V(G) \cup \{x_1, x_2, ..., x_{n-1}\} \cup \{y_1, y_2, ..., y_n\}$, (2-1) for $v \in Q_{j,k} \in \mathcal{Q}'(G)$ $(j = 1, 2, ..., n-1$ and $k = 1, 2, ..., n-2)$, $y_h \leq_{P_{n,n-1}} v \leq_{P_{n,n-1}} x_j$ $(h \equiv j - k + 1 (\text{mod } (n-1)))$, (2-2) for $v \in Q_{j,n-1} \in \mathcal{Q}'(G)$ $(j = 1, 2, ..., n-1)$, $y_n \leq_{P_{n,n-1}} v \leq_{P_{n,n-1}} x_j$ and (2-3) for $v \in Q_{j,n} \in \mathcal{Q}'(G)$ $(j = 1, 2, ..., n-1)$, $y_h \leq_{P_{n,n-1}} v \leq_{P_{n,n-1}} x_j$ $(h \equiv j + 1 (\text{mod } (n-1)))$. Then $\text{sDB}(P_{n,n-1}) = G \cup N_{2n-1}$, $\zeta(G) = 2n - 1 = \lceil 2\sqrt{l} \rceil$ by Theorem 1.1. □

For a cycle $C_n (n \geq 4)$, each clique is K_2 and C_n is covered by n cliques. So we obtain a following result by Proposition 3.1.

Corollary 3.2 *For a cycle C_n with $n \geq 4$, $\zeta(C_n) = \lceil 2\sqrt{n} \rceil$.* □

4 Strict-double-bound numbers of the sum of graphs.

The *sum* $G + H$ of two graphs G and H is the graph with the vertex set $V(G+H) = V(G) \cup V(H)$ and the edge set $E(G+H) = E(G) \cup E(H) \cup \{uv; u \in V(G), v \in V(H)\}$.

Proposition 4.1 *For a graph G with no isolated vertices, $\zeta(G + K_n) = \zeta(G)$.*

Proof. For a graph G, let P_G be a poset such that $\mathrm{sDB}(P_G) \cong G \cup N_{\zeta(G)}$. For the poset P_G, we construct the poset P'_G such that (1) $V(P'_G) = V(P_G)$ and (2) $u \leq_{P'_G} v$ if (a) $u, v \in V(P_G)$ and $u \leq_{P_G} v$ or (b) $u \in \mathrm{Min}(P_G)$ and $v \in \mathrm{Max}(P_G)$. Then $\mathrm{sDB}(P'_G) = \mathrm{sDB}(P_G)$.

For the poset P'_G, we construct the poset P_H such that (1) $V(P_H) = V(P'_G) \cup V(K_n)$ and (2) $u \leq_{P_H} v$ if (a) $u, v \in V(P'_G)$ and $u \leq_{P'_G} v$, (b) $u \in V(K_n)$ and $v \in \mathrm{Max}(P'_G)$ or (c) $u \in \mathrm{Min}(P'_G)$ and $v \in V(K_n)$.

In P'_G each vertex of G has a maximal element as a strict upper bound and a minimal element as a strict lower bound since G has no isolated vertices. In P_H, each maximal element of P'_G is a strict upper bound of all vertices of K_n and each minimal element of P'_G is a strict lower bound of all vertices of K_n. So in $\mathrm{sDB}(P_H)$, each vertex of K_n is adjacent to all vertices of G, and any two vertices of K_n are adjacent. For each vertex v of G, $U_{P_H}(v) = U_{P'_G}(v)$ and $L_{P_H}(v) = L_{P'_G}(v)$. So for $u, v \in V(G)$, u is adjacent to v in $\mathrm{sDB}(P_H)$ if and only if u is adjacent to v in $\mathrm{sDB}(P'_G)$. Thus $\zeta(G + K_n) \leq \zeta(G)$.

Let P be a poset such that $m = |\mathrm{Max}(P) \cup \mathrm{Min}(P)| = \zeta(G + K_n)$ and $\mathrm{sDB}(P) \cong (G + K_n) \cup N_m$. Let P' be a poset which is obtained from P by deleting any vertices of K_n by P. Since $V(K_n) \subseteq V(P) - \mathrm{Max}(P) \cup \mathrm{Min}(P)$, $\mathrm{sDB}(P') \cong G \cup N_m$. Thus $\zeta(G) \leq m = \zeta(G + K_n)$.

Therefore $\zeta(G + K_n) = \zeta(G)$. □

A wheel W_n is the sum of C_{n-1} and K_1. So we obtain a following result by Proposition 4.1 and Corollary 3.2.

Corollary 4.2 *For a wheel W_n with $n(\geq 5)$ vertices, $\zeta(W_n) = \left\lceil 2\sqrt{n-1} \right\rceil$.*
□

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