On Chromatic Transversal Domination in Graphs

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Abstract

Let $G = (V, E)$ be a graph with chromatic number k. A dominating set D of G is called a chromatic transversal dominating set (ctd-set) if D intersects every color class of any k-coloring of G. The minimum cardinality of a ctd-set of G is called the chromatic transversal domination number of G and is denoted by $\gamma_{ct}(G)$. In this paper we obtain sharp upper and lower bounds for γ_{ct} for the Mycielskian $\mu(G)$ and the shadow graph $Sh(G)$ of any graph G. We also prove that for any $c \geq 2$, the decision problem corresponding to γ_{ct} is NP-hard for graphs with $\chi(G) = c$.

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1 Introduction

By a graph $G = (V, E)$, we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [3].

Graph coloring and domination are two major areas in graph theory that have been well studied. An excellent treatment of fundamentals of domination is given in the book by Haynes et al. [4] and survey papers on several advanced topics are given in the book edited by Haynes et al. [5].

Let $G = (V, E)$ be a graph. A subset S of V is called a dominating set of G if every vertex in $V - S$ is adjacent to a vertex in S. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G. A proper coloring of a graph G is an assignment of colors to the vertices of G in such a way that no two adjacent vertices receive the same color. The chromatic number $\chi(G)$ is the minimum number of colors required for a proper coloring of G. A graph G is said to be χ-critical if $\chi(G-v) = \chi(G)-1$ for all $v \in V$.

Benedict et al. [1] introduced the concept of chromatic transversal domination, which combines the concept of domination and coloring. A dominating set D of G is called a chromatic transversal dominating set (ctd-set) if D intersects every color class of any k-coloring of G, where $\chi(G) = k$. The minimum cardinality of a ctd-set of G is called the chromatic transversal domination number of G and is denoted by $\gamma_{ct}(G)$. A chromatic transversal dominating set of cardinality γ_{ct} is called a γ_{ct}-set of G. In this paper we present further results on chromatic transversal domination.

We need the following definitions and theorems.

Definition 1.1. *For a graph $G = (V, E)$, the* Mycielskian *of G is the graph $\mu(G)$ with vertex set $V \cup V' \cup \{u\}$, where $V' = \{x' : x \in V\}$ and is disjoint from V, and edge set $E' = E \cup \{xy', yx' : xy \in E\} \cup \{x'u : x' \in V'\}$. The vertices x and x' are called twins of each other and u is called the root of $\mu(G)$. Also the graph $\mu(G) - u$ is called the shadow graph of G and is denoted by $Sh(G)$.*

Theorem 1.2. [3] *For any graph G, $\chi(\mu(G)) = \chi(G) + 1$ and $\omega(\mu(G)) = \omega(G)$.*

Observation 1.3. *For any graph G, $\chi(G) = \chi(Sh(G))$.*

Definition 1.4. *Let $G_1, G_2, \ldots, G_k$ be set of k-graphs. Then the graph $G_1 + G_2 + \cdots + G_k$ is obtained from $G_1, G_2, \ldots, G_k$ by joining every vertex of G_i with every vertex of G_j, whenever $i \neq j$.*

Definition 1.5. *The corona of two graphs G_1 and G_2 is defined to be the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 where the i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2.*

Theorem 1.6. [1] *Let G be a connected bipartite graph with bipartition (X, Y), where $|X| \leq |Y|$ and $n \geq 3$. Then $\gamma_{ct}(G) = \gamma(G) + 1$ if and only if every vertex in X has at least two pendant neighbors.*

Theorem 1.7. [1] *Let G be a connected graph of order n. Then $\gamma_{ct}(G) = n$ if and only if G is χ-critical.*

2 Basic Results

Let $G = (V, E)$ be a graph with $\chi(G) = k$ and let S be a ctd-set of G with $|S| = \gamma_{ct}$. Since S is a dominating set and intersects every color class of any k-coloring of G, it follows that $\gamma_{ct}(G) \geq max\{\chi(G), \gamma(G)\}$.

Example 2.1. *For the graph $G = H \circ K_1$, where H is any graph of order $n \geq 2$, we have $\gamma_{ct}(G) = \gamma(G) = n$.*

Example 2.2. *For any split graph G with clique number $\omega(G)$, we have $\gamma_{ct}(G) = \chi(G) = \omega(G)$.*

Example 2.3. *For the complete multipartite graph $G = K_{a_1, a_2, \ldots, a_k}$, we have $\gamma_{ct}(G) = \chi(G) = k$.*

We start with the following simple lemma which is very useful in finding lower bounds for γ_{ct}.

Lemma 2.4. *Let $\mathcal{F}$ be a family of disjoint independent sets in G such that each $F \in \mathcal{F}$ is a color class of a χ-coloring of G. Then $\gamma_{ct}(G) \geq |\mathcal{F}|$.*

Proof. Let S be a ctd-set of G with $|S| = \gamma_{ct}$. Since $S \cap F \neq \emptyset$ for all $F \in \mathcal{F}$, it follows that $\gamma_{ct}(G) = |S| \geq |\mathcal{F}|$. □

Proposition 2.5. *If G is a uniquely colorable graph, then $\gamma_{ct}(G) \leq \gamma(G) + \chi(G) - 1$ and the bound is sharp.*

Proof. Let $\{V_1, V_2, \ldots, V_k\}$ be the unique k-coloring of G where $\chi(G) = k$. Let S be a γ-set of G. We may assume without loss of generality that $S \cap V_1 \neq \emptyset$. Now choose $v_i \in V_i$, $2 \leq i \leq k$. Then $S \cup \{v_2, v_3, \ldots, v_k\}$ is a ctd-set of G and hence $\gamma_{ct}(G) \leq |S| + k - 1 = \gamma(G) + \chi(G) - 1$.

Further for any two integers n, $m \geq 2$, $G = K_n + \overline{K}_m$ is a split graph with $\chi(G) = \gamma_{ct}(G) = n + 1$ and $\gamma(G) = 1$ and hence $\gamma_{ct}(G) = \gamma(G) + \chi(G) - 1$. □

Corollary 2.6. *Let G be a uniquely colorable graph with $\gamma_{ct}(G) = \gamma(G) + \chi(G) - 1$. Then $\gamma(G) = i(G)$, where $i(G)$ is the independent domination number of G. Further every $i(G)$-set of G is a color class of the unique χ-coloring of G.*

Corollary 2.7. *Let G be a connected bipartite graph. Then $\gamma_{ct}(G) = \gamma(G)$ or $\gamma(G) + 1$.*

Problem 2.8. *Characterize uniquely colorable graphs G for which $\gamma_{ct}(G) = \gamma(G) + \chi(G) - 1$.*

Proposition 2.9. *Let $G_1, G_2, \ldots, G_s$ be uniquely colorable graphs and let $G = G_1 + G_2 + \cdots + G_s$. Then $\gamma_{ct}(G) = \chi(G) = \sum_{i=1}^{s} \chi(G_i)$.*

Proof. Let $C = \{V_1, V_2, \ldots, V_k\}$ be the unique k-coloring of G where $k = \sum_{i=1}^{s} \chi(G_i)$. Now choose $v_i \in V_i$, $1 \le i \le k$. Clearly $S = \{v_1, v_2, \ldots, v_k\}$ is a ctd-set of G and hence $\gamma_{ct}(G) \le |S| = \sum_{i=1}^{s} \chi(G_i) = \chi(G)$. Since $\gamma_{ct}(G) \ge \chi(G)$, the result follows. □

Theorem 2.10. *For any graph G, $\gamma_{ct}(G\Box K_2) \le |V(G)|$. Further if G is χ-critical, then equality holds.*

Proof. Since $\chi(G\Box K_2) = \chi(G)$, it follows that $V(G)$ is a ctd-set of $G\Box K_2$ and hence $\gamma_{ct}(G\Box K_2) \le |V(G)|$. Now suppose G is χ-critical. Let $V(G\Box K_2) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$ where $\langle \{v_1, v_2, \ldots, v_n\} \rangle$ and $\langle \{u_1, u_2, \ldots, u_n\} \rangle$ are both isomorphic to G and $v_iu_i \in E(G\Box K_2)$. Let $\mathcal{F} = \{A_1, A_2, \ldots, A_n\}$ where $A_i = \{v_i, u_{i+1}\}$ is a family of disjoint independent sets in G. Since G is χ-critical, it follows that $\chi(\langle V(G\Box K_2) - A_i \rangle) = \chi(G\Box K_2) - 1$ and hence every A_i is a color class of a χ-coloring of $G\Box K_2$. Hence it follows from Lemma 2.4 that $\gamma_{ct}(G) \ge |V(G)|$. Thus $\gamma_{ct}(G) = |V(G)|$. □

Corollary 2.11. *For the graph $G = C_n\Box K_2$,*

$$\gamma_{ct}(G) = \begin{cases} \gamma(G) & \text{if } n \text{ is even} \\ n & \text{otherwise.} \end{cases}$$

Proof. If n is even, then G is bipartite and hence it follows from Corollary 2.7 and Theorem 1.6 that $\gamma_{ct}(G) = \gamma(G)$. If n is odd, then C_n is χ-critical and hence it follows from Theorem 2.10 that $\gamma_{ct}(G) = n$. □

Problem 2.12. *If $\gamma_{ct}(G\Box K_2) = |V(G)|$, then is G χ-critical ?*

Theorem 2.13. *Let G be a graph having a unique cut vertex v and let $B_1, B_2, \ldots, B_k$ be the blocks of G such that every block B_i is a cycle, $|B_i| = n_i$, at least one n_i is odd and $n_1 = min\{n_i : n_i \text{ is odd}\}$. Then $\gamma_{ct}(G) = n_1 + \sum_{j=2}^{k} \lceil \frac{n_j - 3}{3} \rceil$.*

Proof. Clearly $\chi(G) = 3$. Let S_1 be a γ-set of $\langle V - N[V(B_1)] \rangle$. Since the induced subgraph $\langle V - N[V(B_1)] \rangle$ is isomorphic to the union of paths, it follows that $|S_1| = \sum_{j=2}^{k} \lceil \frac{n_j-3}{3} \rceil$. Clearly $S = S_1 \cup V(B_1)$ is a ctd-set of G and hence $\gamma_{ct}(G) \le n_1 + \sum_{j=2}^{k} \lceil \frac{n_j-3}{3} \rceil$. Now, let W be a γ_{ct}-set of G. Since $G - v$ is bipartite, it follows that $v \in W$. We claim that there exists an odd cycle C_i such that $V(C_i) \subseteq W$. If not for every odd cycle C_j in G, we choose $x_j \in V(C_j) - W$ and let D denote the set of all these vertices x_j. Clearly D is independent and $\chi(G-D) = 2$. Hence there exists a 3-coloring of G such that $\{D\}$ is a color class. Since $W \cap D = \emptyset$, it follows that W is not a ctd-set of G, which is a contradiction. Hence we may assume $V(B_1) \subseteq W$. Also since W is a dominating set of G, it follows that $\gamma_{ct}(G) \geq n_1 + \sum_{j=2}^{k} \lceil \frac{n_j-3}{3} \rceil$.

Hence $\gamma_{ct}(G) = n_1 + \sum_{j=2}^{k} \lceil \frac{n_j-3}{3} \rceil$. □

3 Bounds on γ_{ct} for $\mu(G)$ and $Sh(G)$

Theorem 3.1. *Let G be a graph with $\chi(G) \geq 2$. Then $2\chi(G) + 1 \leq \gamma_{ct}(\mu(G)) \leq 2\gamma_{ct}(G) + 1$.*

Proof. Let $C = \{V_1, V_2, \ldots, V_k\}$ be a k-coloring of G where $\chi(G) = k$. Let $\mathcal{F} = \{V_1, V_2, \ldots, V_k, V_1', V_2', \ldots, V_k', \{u\}\}$ where for any subset S of $V(G)$, $S' = \{v' : v \in S\}$. We now claim that for each $F \in \mathcal{F}$, there exists a $(k+1)$-coloring of $\mu(G)$ having F as a color class. If $F = V_i$, let $C_i = (C - \{V_j\}) \cup \{V_j \cup \{u\}\} \cup \{V'\}$ where $j \neq i$. If $F = V_i'$, let $C_i' = \{V_1 \cup V_1', V_2 \cup V_2', \ldots, V_i \cup \{u\}, V_{i+1} \cup V_{i+1}', \ldots, V_k \cup V_k'\} \cup \{V_i'\}$. If $F = \{u\}$, let $C_u = \{V_i \cup V_i' : 1 \leq i \leq k\} \cup \{u\}$. In all the above cases we get a $(k+1)$-coloring of $\mu(G)$ having F as a color class. Hence it follows from Lemma 2.4 that $\gamma_{ct}(\mu(G)) \geq |\mathcal{F}| = 2k+1 = 2\chi(G)+1$. Now, let S_1 be a γ_{ct}-set of G. Let $S = S_1 \cup S_1' \cup \{u\}$. We claim that S is a ctd-set of $\mu(G)$. Clearly S is a dominating set of $\mu(G)$. Let $C = \{V_1, V_2, \ldots, V_{k+1}\}$ be a $(k+1)$-coloring of $\mu(G)$ where $k = \chi(G)$. It is enough to show that $S \cap V_i \neq \emptyset$ for all $i = 1, 2, \ldots, k+1$. If $V_1 = \{u\}$, then $\{V_i \cap V(G) : 2 \leq i \leq k+1\}$ is a k-coloring of G. Since S_1 is γ_{ct}-set of G, it follows that $S_1 \cap V_i \neq \emptyset$ for all $i = 2, 3, \ldots, k+1$. Hence $S \cap V_i \neq \emptyset$ for all $i = 2, 3, \ldots, k+1$. Suppose $\{u\} \notin C$. Let $u \in V_1$. We now recolor each vertex of $V_1 \cap V(G)$ with the color of its twin. This gives a χ-coloring C_1 of $\mu(G)$ such that $\{u\} \in C_1$. Let $C_1 = \{W_1 = \{u\}, W_2, \ldots, W_{k+1}\}$. It is clear that $S \cap W_i \cap V(G) \neq \emptyset$ for all $i = 2, 3, \ldots, k+1$. Let $w \in S \cap W_i \cap V(G)$. If $w \in V_i$, then $S \cap V_i \neq \emptyset$. Otherwise $w' \in V_i$ and hence $w' \in S$. Hence $S \cap V_i \neq \emptyset$ for all $i = 1, 2, \ldots, k+1$. Thus $\gamma_{ct}(\mu(G)) \leq |S| = 2\gamma_{ct}(G) + 1$. □

For any graph G with $\chi(G) = \gamma_{ct}(G)$, we have

$$\gamma_{ct}(\mu(G)) = 2\chi(G) + 1 = 2\gamma_{ct}(G) + 1 \qquad (1)$$

which shows that the bounds given in Theorem 3.1 are sharp. In particular (1) holds for split graphs and complete multipartite graphs. The following theorem gives another family of graphs for which the upper bound in Theorem 3.1 is attained.

Theorem 3.2. *Let G be a χ-critical graph of order n. Then $\gamma_{ct}(\mu(G)) = 2\gamma_{ct}(G) + 1$.*

Proof. Let $\chi(G) = k$. We first prove that $\mu(G)$ is χ-critical, by showing that for every $x \in V(\mu(G))$, there exists a $(k+1)$-coloring $\mathcal{C}$ of $\mu(G)$ such that $\{x\} \in \mathcal{C}$. Suppose $x = v \in V(G)$. Since G is χ-critical, it follows that there exists a χ-coloring $\mathcal{C}_1$ of G such that $\{v\} \in \mathcal{C}_1$. Let $\mathcal{C}_1 = \{V_1 = \{v\}, V_2, \ldots, V_k\}$. Then $\mathcal{C} = \{\{v\}, V_2 \cup \{u\}, V_3, \ldots, V_k\} \cup \{V'\}$ is a $(k+1)$-coloring of $\mu(G)$. Suppose $x = v' \in V'$. Let v be the twin of v' in G. Let $\mathcal{C}_1 = \{V_1 = \{v\}, V_2, \ldots, V_k\}$ be a k-coloring of G. Then $\mathcal{C}' = \{V_1 \cup \{u\}, V_2 \cup V_2', V_3 \cup V_3', \cdots, V_k \cup V_k'\} \cup \{v'\}$ is a $(k+1)$-coloring of $\mu(G)$. If $x = u$, then $\mathcal{C} = \{V_1 \cup V_1', V_2 \cup V_2', \ldots, V_k \cup V_k'\} \cup \{u\}$ is a $(k+1)$-coloring of $\mu(G)$. Thus $\mu(G)$ is χ-critical and it follows from Theorem 1.7 that $\gamma_{ct}(\mu(G)) = 2n + 1 = 2\gamma_{ct}(G) + 1$. □

Problem 3.3. *Characterize graphs G for which $\gamma_{ct}(\mu(G)) = 2\gamma_{ct}(G)+1$.*

Problem 3.4. *Characterize graphs G for which $\gamma_{ct}(\mu(G)) = 2\chi(G) + 1$.*

Theorem 3.5. *Let G be a graph and let $Sh(G)$ be the shadow graph of G. Then $\gamma_{ct}(G) \leq \gamma_{ct}(Sh(G)) \leq \gamma_{ct}(G) + \gamma(G)$. Further $\gamma_{ct}(G) = \gamma_{ct}(Sh(G))$ if and only if there exists a γ_{ct}-set S of G such that $\langle S \rangle$ has no isolates.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $V(Sh(G)) = V(G) \cup \{v_1', v_2', \ldots, v_n'\}$. Let S_1 be a γ_{ct}-set of $Sh(G)$. Let $S = (S_1 \cap V(G)) \cup \{v_i : v_i' \in S_1\}$. Clearly $|S| \leq |S_1|$. We claim that S is a ctd-set of G. Clearly S is a dominating set of G. Let $\mathcal{C} = \{V_1, V_2, \ldots, V_k\}$ be a k-coloring of G where $\chi(G) = k$. Then $\mathcal{C}_1 = \{W_1 = V_1 \cup U_1, W_2 = V_2 \cup U_2, \ldots, W_k = V_k \cup U_k\}$ where $U_i = \{v_i' : v_i \in V_i\}$, for $1 \leq i \leq k$ is a k-coloring of $Sh(G)$. Since S_1 is γ_{ct}-set of $Sh(G)$, it follows that $S_1 \cap W_i \neq \emptyset$ for all i. Hence $S \cap V_i \neq \emptyset$ for all i. Thus S is a ctd-set of G and $\gamma_{ct}(G) \leq \gamma_{ct}(Sh(G))$. Since $\chi(G) = \chi(Sh(G))$, the upper bound follows trivially. Now, suppose $\gamma_{ct}(G) = \gamma_{ct}(Sh(G))$. Then $|S_1| = |S|$. Hence it follows that at most one of v_i, v_i' is in S_1 for each i, $1 \leq i \leq n$. Now, suppose $v_i \in S_1 \cap V(G)$. Then $v_i \in S$ and hence $v_i' \notin S_1$. Since S_1 is a dominating set of $Sh(G)$, it follows that $N_G(v_i) \cap S \neq \emptyset$, so that $\langle S \rangle$ has no isolated vertices. Conversely, suppose there exists a γ_{ct}-set D of G such that $\langle D \rangle$ has no isolates. Then $N_G(v_i) \cap D \neq \emptyset$ for all i and hence D is a dominating set of $Sh(G)$. Also since $\chi(G) = \chi(Sh(G))$, it follows that D is a ctd-set of $Sh(G)$ and hence $\gamma_{ct}(G) = \gamma_{ct}(Sh(G))$. □

Problem 3.6. *Characterize graphs G for which $\gamma_{ct}(Sh(G)) = \gamma_{ct}(G) + \gamma(G)$.*

4 Complexity Results

In this section we prove that for any $c \geq 2$, the decision problem corresponding to γ_{ct} is NP-hard for graphs with $\chi(G) = c$.

3SAT
INSTANCE: Collection $\mathcal{C} = \{C_1, C_2, \ldots, C_r\}$ of clauses on a finite set $U = \{u_1, u_2, \ldots, u_s\}$ of variables such that $|C_i| = 3$ for $1 \leq i \leq r$.
QUESTION: Is there a truth assignment for U that satisfies all the clauses in $\mathcal{C}$?

CTD
INSTANCE: A graph $G = (V, E)$ and a positive integer k.
QUESTION: Is $\gamma_{ct}(G) \leq k$?

Theorem 4.1. *For any integer $c \geq 2$, CTD is NP-hard for graphs with $\chi(G) = c$.*

Proof. Case i. $c \geq 3$.

The proof is by reduction from 3-SAT. Let $U = \{u_1, u_2, \ldots, u_s\}$ and $\mathcal{C} = \{C_1, C_2, \ldots, C_r\}$ be any instance of 3-SAT. We construct an instance of CTD as follows. For each variable $u_i \in U$, we take a triangle T_i with vertices u_i, u_i' and x_i. For each clause C_j, we take a copy of $K_c \circ \overline{K_2}$, where $V(K_c) = \{x_{j1}, x_{j2}, \ldots, x_{jc}\}$ in which exactly one pendant edge is subdivided. We label the corresponding pendant vertex as c_j. (The graph when $c = 4$ is shown in Figure 1)

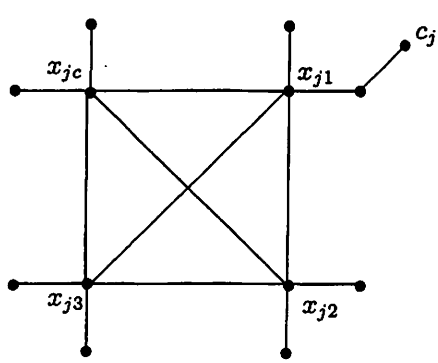

Figure 1

The graph G is obtained by joining each vertex with label c_j to the three vertices whose labels are the literals in C_j. Clearly $\chi(G) = c$. Let $k = cr + s$. We claim that the instance $(U, \mathcal{C})$ of 3-SAT has a satisfying truth assignment if and only if G has a ctd-set S with $|S| \leq cr + s$. Suppose $(U, \mathcal{C})$ has a satisfying truth assignment. Let T denote the set of all literals u_i or u_i'

having the value true. Clearly $|T| = s$. Let $S = T \cup (\bigcup_{j=1}^{r} \{x_{j1}, x_{j2}, \ldots, x_{jc}\})$. Clearly $|S| = cr + s$ and S is a ctd-set of G. Conversely, let S be a ctd-set of G with $|S| \leq k = cr + s$. Clearly S contains the cr vertices $x_{j1}, x_{j2}, \ldots, x_{jc}, 1 \leq j \leq r$ and S contains exactly one vertex of each of the triangles T_i. If $x_i \in S$, then we may replace x_i either by u_i or u'_i and hence we may assume that S contains exactly one of u_i, u'_i for each $i, 1 \leq i \leq s$. Now for each variable u_i, we assign u_i the value True if $u_i \in S$ and the value False otherwise. Clearly this is a satisfying truth assignment for the instance (U, C) of 3-SAT.

Case ii. $c = 2$.

In this case G is a bipartite graph. It follows from Corollary 2.7 that $\gamma_{ct}(G) = \gamma(G)$ or $\gamma(G) + 1$. Further it follows from Theorem 1.6 that whether $\gamma_{ct}(G) = \gamma(G) + 1$ or not can be determined in linear time. Since the domination problem is NP-hard for bipartite graphs [2], it follows that CTD is also NP-hard for bipartite graphs. □

5 Conclusion

For any graph theoretic parameter the study of the effect of removal of a vertex or an edge on the parameter is an important problem. In particular the study of the effect on $\gamma_{ct}(G)$ when a vertex or an edge is deleted is a promising research area and results this direction will be reported in a subsequent paper.

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