Up-smooth samples of geometric variables

Arnold Knopfmacher
The John Knopfmacher Centre for Applicable Analysis and Number Theory, School of Mathematics
University of the Witwatersrand, Johannesburg, South Africa
Arnold.Knopfmacher@ wits.ac.za

Toufik Mansour
Department of Mathematics
University of Haifa, 31905 Haifa, Israel
toufik@math.haifa.ac.il

Abstract

We study samples $\Gamma = (\Gamma_1, \ldots, \Gamma_n)$ of length n where the letters Γ_i are independently generated according to the geometric distribution $\mathbb{P}(\Gamma_j = i) = pq^{i-1}$, for $1 \le j \le n$, with $p + q = 1$ and $0 < p < 1$. An *up-smooth sample* Γ is a sample such that $\Gamma_{i+1} - \Gamma_i \le 1$. We find generating functions for the probability that a sample of n geometric variables is up-smooth, with or without a specified first letter. We also extend the up-smooth results to words over an alphabet of k letters and to compositions of integers. In addition we study *smooth samples* Γ of geometric random variables, where the condition now is $|\Gamma_{i+1} - \Gamma_i| \le 1$.

1 Introduction

Let X denote a geometrically distributed random variable, i. e. $\mathbb{P}\{X = k\} = pq^{k-1}$ for $k \in \mathbb{N}$ and $q = 1 - p$. The combinatorics of n geometrically distributed independent random variables $X_1, \ldots, X_n$ has attracted recent interest, especially because of applications in computer science, for example see [1, 2, 7, 8]. In particular, Prodinger [8] studied the number of left-to-right maxima for samples $(\Gamma_1, , \ldots, \Gamma_n)$ with Γ_i independently generated according to the geometric distribution.

JCMCC **83** (2012), pp. 51-63

Consider now the sample $\Gamma = (\Gamma_1, \ldots, \Gamma_n)$ of length n where $\mathbb{P}(\Gamma_j = i) = pq^{i-1}$, for $1 \le j \le n$, with $p + q = 1$ and $0 < p < 1$. An *up-smooth sample* Γ is a sample such that $\Gamma_{i+1} - \Gamma_i \le 1$. For instance, when $n = 3$ we have the following samples starting with a 1: 111, 112, 121, 122 and 123 which occur with respective probabilities p^3, p^3q, p^3q, p^3q^2 and p^3q^3. Thus the probability that a sample of length 3 with first letter 1 is up-smooth is $p^3(1 + 2q + q^2 + q^3)$. Also the probability that a sample of length 3 with arbitrary first letter is up-smooth is

$$\sum_{i=1}^{\infty} pq^{i-1} \sum_{j=1}^{i+1} pq^{j-1} \sum_{k=1}^{j+1} pq^{k-1} = \frac{1 + 2q - q^3 - 2q^4 + q^6}{1 + 2q + 2q^2 + q^3}.$$

In this paper we find generating functions for the probability that a sample of n geometric variables is up-smooth, with or without a specified first letter. The up-smooth condition can be thought of as an extension of the idea of *smooth words*, where the corresponding restriction is $|\Gamma_{i+1} - \Gamma_i| \le 1$. The smooth restriction has been studied for words over an alphabet of k letters in [5], for set partitions in [6] and for compositions of integers in [4]. In Section 3 we extend the up-smooth results to words over an alphabet of k letters and to compositions of integers. Samples of geometric variables with the original smooth condition have not been previously considered, and these are studied in Section 4.

2 Results

Let $f_n(u)$ be the generating function related to n geometrically distributed random variables, with last part marked by the variable u, and the whole thing satisfying the up-smooth condition.

The adding-a-new-slice technique of Flajolet and Prodinger [3] works here: Let $f_n(u)$ be such that $[u^i]f_n(u)$ is the probability that a word of length n to be an up-smooth with rightmost letter equals i. Then $f_1(u) = \sum_{i\ge1} pq^{i-1}u^i = \frac{pu}{1-qu}$, and we have the replacement rule

$$u^i \longrightarrow pu + pqu^2 + \cdots + pq^i u^{i+1} = \frac{pu}{1-qu} - \frac{pqu^2}{1-qu}(qu)^i.$$

Consequently,

$$f_{n+1}(u) = f_n(1)\frac{pu}{1-qu} - \frac{pqu^2}{1-qu} f_n(qu). \tag{2.1}$$

Now define

$$F(z,u) := \sum_{n\geq 1} f_n(u) z^n.$$

Then

$$F(z,u) = \frac{zpu}{1-qu} + F(z,1)\frac{zpu}{1-qu} - \frac{pqu^2z}{1-qu}F(z,qu). \tag{2.2}$$

By iteration we get

$$F(z,u) = (1+F(z,1))\sum_{j\geq 1} \frac{z^j p^j (-1)^{j-1} u^{2j-1} q^{j(j-1)}}{\prod_{i=1}^{j}(1-q^i u)}, \tag{2.3}$$

which implies that

$$1 + F(z,1) = \frac{1}{1-\sum_{j\geq 1}\frac{z^j p^j (-1)^{j-1} q^{j(j-1)}}{(q;q)_j}}. \tag{2.4}$$

Now consider the case where the first part of the geometric word is i. Corresponding to the above, we will now use the notation $f_{n,i}$ and $F_i(z,u)$. Then $f_{1,i} = pq^{i-1}u^i$ and we find

$$F_i(z,u) = pq^{i-1}u^i z + F_i(z,1)\frac{zpu}{1-qu} - \frac{pqu^2z}{1-qu}F_i(z,qu). \tag{2.5}$$

By iteration we obtain

$$\begin{aligned} F_i(z,u) = F_i(z,1)\sum_{j\geq 1} &\frac{z^j p^j (-1)^{j-1} u^{2j-1} q^{j(j-1)}}{\prod_{k=1}^{j}(1-q^k u)} \\ +\sum_{j\geq 1} &\frac{z^j p^j (-1)^{j-1} u^{2j-2+i} q^{j^2-2j+ij}}{\prod_{k=1}^{j-1}(1-q^k u)}. \end{aligned} \tag{2.6}$$

From this we can state the following result.

Theorem 1 *We have*

$$F_i(z) := F_i(z,1) = \frac{\sum_{j\geq 1}\frac{z^j p^j (-1)^{j-1} q^{j^2-2j+ij}}{(q;q)_{j-1}}}{1-\sum_{j\geq 1}\frac{z^j p^j (-1)^{j-1} q^{j(j-1)}}{(q;q)_j}}. \tag{2.7}$$

In particular we have the following interesting identity concerning $F_1(z)$.

$$F_1(z) = \frac{\sum_{j\geq 1} \frac{z^j p^j (-1)^{j-1} q^{j(j-1)}}{(q;q)_{j-1}}}{1 - \sum_{j\geq 1} \frac{z^j p^j (-1)^{j-1} q^{j(j-1)}}{(q;q)_j}}$$

$$= \cfrac{pz}{-pz + \cfrac{1}{1 + \cfrac{pqz}{-pqz + \cfrac{1}{1 + \cfrac{pq^2 z}{-pq^2 z + \cfrac{1}{\ddots}}}}}}. \tag{2.8}$$

The continued fraction expression follows from the Lemma below.

Lemma 2 *The generating function $F_1(z)$ is given by*

$$\cfrac{pz}{-pz + \cfrac{1}{1 + \cfrac{pqz}{-pqz + \cfrac{1}{1 + \cfrac{pq^2 z}{-pq^2 z + \cfrac{1}{\ddots}}}}}}.$$

Proof Let π be any up-smooth geometric word over alphabet $\mathbb{N}$ with its first letter equal to 1. If in the word π the letter 1 occurs exactly m times, then π can be decomposed as $\pi = 1\pi^{(1)}1\pi^{(2)}\cdots 1\pi^{(m)}$, where $\pi^{(j)}$ is any up-smooth word over alphabet $\{2,3,\ldots\}$ with its first letter 2. Note that each up-smooth word $a_1a_2\cdots a_n$ over alphabet $\{2,3,\ldots\}$ with $a_1 = 2$ can be mapped to an up-smooth word $(a_1-1)(a_2-1)\cdots(a_n-1)$ over alphabet $\{1,2,3,\ldots\}$ with $a_1 - 1 = 1$, it follows that the generating function for the number of directed smooth word $a_1a_2\cdots a_n$ over alphabet $\{2,3,\ldots\}$ with $a_1 = 2$ is $F_1(qz)$. Therefore

$$F_1(z) = \sum_{m\geq 1} \left(pz(1+F_1(qz))\right)^m = \frac{pz(1+F_1(qz))}{1-pz(1+F_1(qz))},$$

which is equivalent to

$$F_1(z) = \cfrac{pz}{-pz + \cfrac{1}{1+F_1(qz)}}. \tag{2.9}$$

We iterate this equation an infinite number of times to complete the proof. □

2.1 Asymptotics

From Theorem 2.1, each $F_i(z)$ has a simple dominant pole at the least positive root of the denominator of $F(z) := F(z,1)$, which we denote by $\alpha \equiv \alpha_q$. So

$$F_i(z) \sim \frac{A_{i,q}}{1-\frac{z}{\alpha}}$$

with some constant $A_{i,q}$ given by

$$A_{i,q} = -\frac{1}{\alpha}\lim_{x\to\alpha}(x-\alpha)F_i(z), \tag{2.10}$$

from which

$$[z^n]F_i(z) \sim A_{i,q}\alpha^{-n}. \tag{2.11}$$

The next lemma will help us to find a relationship between the constants $A_i \equiv A_{i,q}$, for $i \geq 1$. Let $F_{i_1 i_2}(z)$ be the generating function for all geometric up-smooth samples $(\Gamma_1, \Gamma_2, \ldots, \Gamma_n)$ of length n such that $\Gamma_j = i_j$ for $j = 1, 2$.

Lemma 3 *The generating functions $F_i(z)$ satisfy the recurrence relation*

$$F_i(z) = pq^{i-1}z(1 + F_{i+1}(z) + F_i(z) + \cdots + F_1(z)),$$

for all $i \geq 1$.

Proof Each sample $\Gamma = (\Gamma_1, \Gamma_2, \ldots, \Gamma_n)$ of length n starting with i has a second letter j with $1 \leq j \leq i+1$, or else consists of the single letter i. Thus, by the definitions we have

$$F_i(z) = pq^{i-1}z + \sum_{j=1}^{i+1} F_{ij}(z) = pq^{i-1}z + pq^{i-1}z\sum_{j=1}^{i+1} F_j(z),$$

which completes the proof. □ By substituting the asymptotic expression (2.11) into Lemma 2.3 we find that for $i \geq 1$,

$$A_{i+1} = \frac{A_i\alpha^{-1}}{pq^{i-1}} - (A_1 + A_2 + \cdots + A_i). \tag{2.12}$$

For example, using (2.12) with $i = 1, 2, 3$ gives

$$\begin{aligned}
A_2 &= A_1 \left(\frac{1}{\alpha p} - 1 \right), \\
A_3 &= A_2 \left(\frac{1}{\alpha p q} - 1 \right) - A_1 \\
&= A_1 \left((\frac{1}{\alpha p} - 1)(\frac{1}{\alpha p q} - 1) - 1 \right), \\
A_4 &= A_3 \left(\frac{1}{\alpha p q^2} - 1 \right) - A_1 - A_2 \\
&= A_1 \left[\left(\frac{1}{\alpha p} - 1 \right) \left(\frac{1}{\alpha p q} - 1 \right) \left(\frac{1}{\alpha p q^2} - 1 \right) - \left(\frac{1}{\alpha p q^2} - 1 \right) - \frac{1}{\alpha p} \right].
\end{aligned}$$

By subtraction we can rewrite (2.12) as

$$A_{i+1} = \frac{1}{\alpha p q^{i-1}} (A_i - q A_{i-1}), \quad i \geq 1, \tag{2.13}$$

where we define $A_0 = \alpha p q^{-1} A_1$.

Proposition 3.1 *For all $i \geq 2$,*

$$\frac{A_i}{A_{i-1}} = (\alpha p q^{i-2})^{-1} - \cfrac{(\alpha p q^{i-3})^{-1}}{(\alpha p q^{i-3})^{-1} - \cfrac{(\alpha p q^{i-4})^{-1}}{\ddots \atop (\alpha p q^{i-4})^{-1} - \cfrac{}{(\alpha p q)^{-1} - \frac{(\alpha p)^{-1}}{(\alpha p)^{-1} - 1}}}}.$$

Proof From (2.13) we obtain that

$$\frac{A_i}{A_{i-1}} = \frac{1}{\alpha p q^{i-2}} - \frac{1}{\alpha p q^{i-3}} \frac{1}{\frac{A_{i-1}}{A_{i-2}}},$$

for all $i \geq 2$. Applying the above recurrence with using the initial condition that $A_1/A_0 = q/(\alpha p)$, we get the desired result. □

Note that the ratio $\frac{A_i}{A_{i-1}}$ is given by a finite continued fraction. Actually, if we rewrite (2.13) in different way as

$$\frac{A_{i-1}}{A_i} = \frac{1}{q} - \frac{\alpha p q^{i-1}}{\frac{A_i}{A_{i+1}}}.$$

Applying this infinite number of times we obtain that the ratio $\frac{A_{i-1}}{A_i}$ is given by infinite continued fraction as

$$\frac{A_{i-1}}{A_i} = q^{-1} - \cfrac{\alpha p q^{i-1}}{q^{-1} - \cfrac{\alpha p q^{i}}{q^{-1} - \cfrac{\alpha p q^{i+1}}{\ddots}}}.$$

Thus all the constants A_i can be expressed in terms of A_1 which can be numerically calculated for each q using (2.10), see Figure 1.

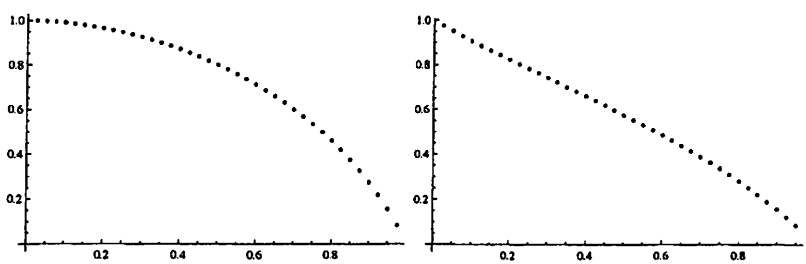

Figure 1: *The functions* α_q^{-1} *and* $A_{1,q}$ *for* $0 < q < 1$.

3 Extensions to words and compositions

In this section we present two extensions of the main results of the previous section, to up-smooth words over the alphabet $[k]$ and to compositions of n.

3.1 Up-smooth words

Let $W_k(y;i)$ be the generating function for the number up-smooth words $\pi_1\pi_2\cdots\pi_n$ of length n over alphabet $[k] = \{1, 2, \ldots, k\}$ such that $\pi_1 = i$. Define $W_k(y,z) = \sum_{i=1}^{k} W_k(y;i)z^{i-1}$.

Lemma 4 *For all* $k \geq 0$,

$$W_k(y,0) = \frac{U_{k-1}\left(\frac{1}{2\sqrt{y}}\right)}{U_{k+1}\left(\frac{1}{2\sqrt{y}}\right)},$$

where U_k *is the* k*-th Chebyshev polynomials of the second kind.*

Proof Extending the proof of Lemma 2 to the case of up-smooth words over the alphabet [k], (2.9) gives that the generating function $W_k(y,0)$ satisfies

$$W_k(y,0) = \frac{y}{\frac{1}{W_{k-1}(y,0)+1} - y}, \quad W_0(y,0) = 0.$$

Now the proof proceeds by induction on k. Since $U_{-1}(t) = 0$ and $U_0(t) = 1$ then the lemma holds for $k = 0$. Assume that the lemma holds for $k-1$ and let us prove it for k. Let $t = \frac{1}{2\sqrt{y}}$, by the above recurrence and the induction hypothesis we have

$$W_k(y,0) = \frac{y}{\frac{1}{\frac{U_{k-2}(t)}{U_k(t)}+1} - y} = \frac{y(U_k(t) + U_{k-2}(t))}{(1-y)U_k(t) - yU_{k-2}(t)}.$$

Using the fact that the Chebyshev polynomials of the second kind satisfy the recurrence relation

$$U_k(t) = 2tU_{k-1}(t) - U_{k-2}(t), \tag{3.1}$$

we obtain

$$W_k(y,0) = \frac{\sqrt{y}U_{k-1}(t)}{U_k(t) - \sqrt{y}U_{k-1}(t)} = \frac{U_{k-1}(t)}{U_{k+1}(t)},$$

which completes the proof. □

Similar arguments as in the proof of Lemma 2 for the case up-smooth words lead to

$$W_k(y,z) = y\frac{1-z^k}{1-z}(1 + W_k(y,0)) + y\sum_{j=2}^{k} \frac{z^{j-2} - z^k}{1-z} W_k(y;j),$$

which is equivalent to

$$W_k(y,z) = y\frac{1-z^k}{1-z} - \frac{y}{z}W_k(y,0) + \frac{y}{z(1-z)}W_k(y,z) - \frac{z^k y}{1-z}W_k(y,1).$$

In order to solve this function equation we use the kernel method. In other words, if we substitute $y = z(1-z)$ then we obtain that

$$W_k(y,1) = \frac{1}{z^{k+1}}(z(1-z^k) - (1-z)W_k(z(1-z),0)).$$

Hence, by Lemma 4 we have

$$W_k(z(1-z),1)$$
$$= -1 + \frac{1}{z^{k+1}} \left(\frac{zU_{k+1}\left(\frac{1}{2\sqrt{z(1-z)}}\right) - (1-z)U_{k-1}\left(\frac{1}{2\sqrt{z(1-z)}}\right)}{U_{k+1}\left(\frac{1}{2\sqrt{z(1-z)}}\right)} \right).$$

Using the fact that

$$zU_{k+1}\left(\frac{1}{2\sqrt{z(1-z)}}\right) - (1-z)U_{k-1}\left(\frac{1}{2\sqrt{z(1-z)}}\right) = \frac{z^{k+1}}{\sqrt{z(1-z)}^{k+1}},$$

which can be proven by induction on k by using (3.1), we obtain the following result.

Theorem 5 *The generating function for the number of up-smooth words of length n over the alphabet $[k]$ is given by*

$$W_k(y,1) + 1 = \frac{1}{\sqrt{y}^{k+1} U_{k+1}\left(\frac{1}{2\sqrt{y}}\right)},$$

where U_k is the k-th Chebyshev polynomials of the second kind.

The above theorem and Lemma A.1 in [6] give that

$$W_k(y,1)+1$$
$$= \frac{2}{(k+2)\sqrt{y}^k} \sum_{j=1}^{k+1} \frac{(-1)^j \sin^2(j\pi/(k+2))}{1-2\sqrt{y}\cos(j\pi/(k+2))}$$
$$= \sum_{s\geq 0} \frac{2^{s+1}}{k+2} \sum_{j=1}^{k+1} (-1)^j \sin^2(j\pi/(k+2)) \cos^s(j\pi/(k+2)) \sqrt{y}^{s-k},$$

which, by finding the coefficient of y^n, implies the following result.

Theorem 6 *The number of up-smooth words of length n over the alphabet $[k]$ is given by*

$$\frac{2^{2n+1+k}}{k+2} \sum_{j=1}^{k+1} (-1)^j \sin^2(j\pi/(k+2)) \cos^{2n+k}(j\pi/(k+2)).$$

3.2 Up-smooth compositions

A composition $\pi = \pi_1\pi_2\cdots\pi_m$ of n is a word over alphabet $\mathbb{N}$ such that $\pi_1 + \pi_2 + \cdots + \pi_m = n$. Let $F_i(x, z)$ be the generating function for the number of up-smooth compositions of n with m parts and first part equal to i, that is,

$$F_i(x,z) = \sum_{n,m\geq 0} \sum_{\pi = i\pi_2\cdots\pi_m} x^n z^m,$$

where the internal sum is over all up-smooth compositions $i\pi_2\cdots\pi_m$ of n with m parts. Replacing pq^{i-1} by x^i in Theorem 1 amd Lemma 2, we obtain the following results.

Theorem 7 *The generating function for the number of up-smooth compositions of n with m parts and first part equal to i is given by*

$$F_i(x,z) = \frac{\sum_{j\geq 1} \frac{z^j(-1)^{j-1}x^{j^2-j+ij}}{(x;x)_{j-1}}}{1 - \sum_{j\geq 1} \frac{z^j(-1)^{j-1}x^{j^2}}{(x;x)_j}}. \tag{3.2}$$

Theorem 8 *The generating function for the number of up-smooth compositions of n with m parts and first part equal to 1 is given by*

$$F_1(x,z) = \cfrac{xz}{-xz + \cfrac{1}{1 + \cfrac{x^2z}{-x^2z + \cfrac{1}{1 + \cfrac{x^3z}{-x^3z + \cfrac{1}{\ddots}}}}}}.$$

4 Smooth geometric words

Let $c(m)$ be the probability that a geometric word π with m letters is smooth (i.e. $|\pi_{i+1} - \pi_i| \leq 1$, for all $i = 1, 2, \ldots, m-1$), and let $C(z)$ denote the generating function of the numbers $c(m)$,

$$C(z) = \sum_{m\geq 0} c(m) z^m.$$

For example,

$$c(3) = \sum_{i=1}^{\infty} pq^{i-1} \sum_{j=\max(1,i-1)}^{i+1} pq^{j-1} \sum_{k=\max(1,j-1)}^{j+1} pq^{k-1} = p^2(1+2q).$$

In this section we obtain a formula for $C(z)$. The adding-a-new-slice technique does not work in this case, so we use a different approach.

Let π be a geometric sample and denote by $parts_i(\pi)$ the number of occurrences of the integer i as a part of π. Then each sample π with m parts and $parts_1(\pi) = d$ can be represented as

$$\pi^{(0)}1\pi^{(1)}1\cdots\pi^{(d-1)}1\pi^{(d)} \text{ with } d \geq 0,$$

where $\pi^{(j)}$ is a geometric word over the alphabet $\{2, 3, \ldots\}$. We refer to this representation as the *d-minimal part decomposition*. The contribution to the generating function $C(z)$ of the 0-minimal part decomposition is $C(qz)$, and the contribution of a d-minimal part decomposition, $d \geq 1$, gives

$$(pz)^d E(qz)B(qz)(EB(qz))^{d-1},$$

where $E(z)$ (respectively $B(z)$, $EB(z)$) is the generating function for the number of geometric samples π with m parts such that $\pi 0$ (respectively, 0π, $(0\pi 0)$ is a smooth sample. Clearly, by the reversal operation $\pi_1 \cdots \pi_m \mapsto \pi_m \cdots \pi_1$, we have that $B(z) = E(z)$. Thus $C(z)$ satisfies the relation

$$C(z) = C(qz) + \frac{pz(E(qz))^2}{1 - pzEB(qz)}. \tag{4.1}$$

Next we find a relationship between the generating functions $E(z)$ and $EB(z)$. By rewriting the d-minimal part decomposition of a smooth sample π such that $\pi 0$ is also a smooth sample, π can be represented as

$$\pi^{(0)}1\pi^{(1)}1\cdots\pi^{(d-1)}1\pi^{(d)}0,$$

with $d \geq 0$ and $\pi^{(d)} = \emptyset$. Now we consider the function $E(z)$. The 0-maximal part decomposition contributes just 1 (the empty word), while the d-minimal part decomposition, $d \geq 1$, gives $(pz)^d E(qz)(EB(qz))^{d-1}$. Hence

$$E(z) = 1 + \frac{pzE(qz)}{1 - pzEB(qz)}. \tag{4.2}$$

Also for $EB(z)$, we rewrite the d-minimal part decomposition for a smooth geometric word π such that $0\pi 0$ is also a smooth word, and obtain a representation of π in the form

$$0\pi^{(0)}1\pi^{(1)}1\cdots\pi^{(d-1)}1\pi^{(d)}0,$$

with $d \geq 0$ and $\pi^{(0)} = \pi^{(d)} = \emptyset$. Thus the contribution of the 0-minimal part decomposition is 1, that of the 1-minimal part decomposition is pz (the word

1), and that of a d-minimal part decomposition, $d \geq 2$, is $(pz)^d(EB(qz))^{d-1}$. Thus

$$EB(z) = 1 + pz + \frac{(pz)^2 EB(qz)}{1 - pzEB(qz)} = 1 + \frac{pz}{1 - pzEB(qz)}. \qquad (4.3)$$

On applying the relation (4.3) an infinite number of times, we obtain the following result.

Lemma 9 *The generating function $EB(z)$ is given by*

$$EB(z) = 1 + \cfrac{pz}{1 - pz - \cfrac{p^2qz^2}{1 - pqz - \cfrac{p^2q^3z^2}{1 - pq^2z - \cfrac{p^2q^5z^2}{1 - pq^3z - \ddots}}}}.$$

Using Lemma 9 together with (4.2) we find an explicit formula for $E(z)$.

Lemma 10 *The generating function $E(z)$ is given by*

$$E(z) = 1 + \sum_{j=1}^{\infty} \frac{p^j q^{\binom{j}{2}} z^j}{\prod_{i=1}^{j} a_{i,j}(z)},$$

where

$$a_{i,j}(z) = 1 - pq^{i-1} - \cfrac{pq^{2i-1}z}{1 - pq^iz - \cfrac{p^2q^{2i+1}z^2}{1 - pq^{i+1}z - \cfrac{p^2q^{2i+3}z^2}{1 - pq^{i+2}z - \cfrac{p^2q^{2i+5}z^2}{\ddots}}}}.$$

Then Lemma 10 together with (4.1) and (4.2) gives the following result for the function $C(z)$.

Theorem 11 *The generating function $C(z)$ is given by*

$$C(z) = 1 + \sum_{j=1}^{\infty} E(q^jz)(E(q^{j-1}z) - 1),$$

where $E(z)$ is given in Lemma 10.

References

[1] M. Archibald and A. Knopfmacher, The average position of the dth maximum in a sample of geometric random variables, *Statistics and Probability Letters* 79(2009), 864-872.

[2] P. Flajolet and G. N. Martin, Probabilistic counting algorithms for data base applications, *Journal of Computer and System Sciences* 31(1985), 182–209.

[3] P. Flajolet and H. Prodinger, Level number sequences for trees, *Discrete Mathematics* 65(1997), 149–156.

[4] A. Knopfmacher, T. Mansour and A. Munagi, Smooth compositions and smooth words, *Submitted.*

[5] A Knopfmacher, T. Mansour, A. Munagi and H Prodinger, Staircase words and Chebyshev polynomials, *Applicable Analysis and Discrete Mathematics* 4(2010), 81–95.

[6] T. Mansour, Smooth partitions and Chebyshev polynomials, *Bulletin London Math. Soc.* 41(2009), 961–970.

[7] T. Papadakis, I. Munro, and P. Poblete, Average search and update costs in skip lists, *BIT* 32(1992), 316–332.

[8] H. Prodinger, Combinatorics of geometrically distributed random variables: Left-to-right maxima, *Discrete Mathematics* 153(1996), 253–270.