A note on the pancyclism of block intersection graphs for universal friendship hypergraphs

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Abstract

In this note, we consider the i-block intersection graphs (i-BIG) of a universal friendship 3-hypergraph and show that they are pancyclic for $i = 1, 2$. We also show that the 1-BIG of a universal friendship 3-hypergraph is Hamiltonian-connected.

1 Introduction

A *friendship 3-hypergraph* on n elements is a set system $(X, \mathcal{B})$ where $|X| = n$, $\mathcal{B}$ is a set of 3-subsets (or triples) of X such that for every three distinct elements $x, y, z \in X$, there exists a unique element $w \in X$ such that the triples $\{x, y, w\}, \{x, z, w\}, \{y, z, w\} \in \mathcal{B}$.

A friendship 3-hypergraph $(X, \mathcal{B})$ has a *universal friend* u if for every pair of distinct elements $x, y \in X$, the triple $\{u, x, y\} \in \mathcal{B}$. When a friendship 3-hypergraph has a universal friend, we call it a *universal friendship 3-hypergraph.*

Since we will be dealing exclusively with triples, we will use the term hypergraph instead of 3-hypergraph.

In [4], it was shown that the only way to construct a universal friendship hypergraph is to take a Steiner triple system $(V, \mathcal{S})$ of order $v \equiv 1, 3 \pmod 6$ and append all triples of the form $\{u, x, y\}$ where u is an element not in V and $x, y \in V$. The element u is the universal friend. Therefore, universal friendship hypergraphs exists if and only if the number of elements

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is $\equiv 2, 4 \pmod 6$. Example 1 shows a universal friendship hypergraph on eight elements. In [2], it was shown that there exists friendship hypergraphs that do not contain a universal friend. These hypergraphs are much harder to construct and will not be considered in this note.

Example 1 *The following is a universal friendship hypergraph on 8 elements and 28 triples with element* 0 *being the universal friend and the unique* $STS(7)$ *is given as the first row.*

137	*124*	*235*	*346*	*457*	*156*	*267*
013	*012*	*023*	*034*	*045*	*015*	*026*
037	*024*	*035*	*046*	*057*	*056*	*067*
017	*014*	*025*	*036*	*047*	*016*	*027*

□

Given a universal friendship hypergraph $(X, \mathcal{B})$ and a non-negative integer i we define the i-block intersection graph (i-BIG) of $(X, \mathcal{B})$ to be the graph whose vertex set are the members of $\mathcal{B}$ and two vertices B and B^* are adjacent if and only if they have exactly i elements in common. Let $G_{\mathcal{B},i}$ denote the i-BIG of the set system $(X, \mathcal{B})$. Since the vertices of the graph $G_{\mathcal{B},i}$ are the triples of $(X, \mathcal{B})$, the terms vertex and triple will be used interchangeably in regards to $G_{\mathcal{B},i}$. In a universal friendship hypergraph, we will always use 0 to denote the universal friend, and when the number of elements in a universal friendship hypergraph is not explicitly stated, it is assume to have $v + 1$ elements where v is the order of the contained Steiner triple system. We will assume the the point set of a hypergraph is $\{0, 1, 2, ..., v\}$ and $v \geq 7$. We will write xyz to denote the triple $\{x, y, z\}$ whenever it is convenient to do so.

We begin with the observation that $G_{\mathcal{B},0}$ is not Hamiltonian for any universal friendship hypergraph $(X, \mathcal{B})$. To see this, note that the $\binom{v}{2}$ vertices of the form $\{0, x, y\}$ form an independent set of $G_{\mathcal{B},0}$. Therefore, if $G_{\mathcal{B},0}$ contains a Hamilton cycle, then each vertex of the form $\{0, x, y\}$ must be adjacent to exactly two vertices from the Steiner triple system on the cycle. This would imply that there must be at least as many Steiner triple system triples as there are triples of the form $\{0, x, y\}$. Therefore we must have $v(v-1)/6 \geq v(v-1)/2$. But this is clearly not possible. Therefore, $G_{\mathcal{B},0}$ is not Hamiltonian.

We are interested in the 1-BIG and 2-BIG of an universal friendship hypergraph. Our main results are that the 1-BIG and 2-BIG of an universal friendship hypergraph are pancyclic. We also show that the 1-BIG of an universalship hypergraph is Hamiltonian connected. We deal with the 1-BIG in Section 2 and the 2-BIG in Section 3.

2 The 1-BIG

In order to show that $G_{\mathcal{B},1}$ is Hamiltonian, we will require the following result by Chvátal and Erdös [1].

Theorem 2.1 *(Theorem 3 of [1]) Let G be an s-connected graph containing no independent set of s vertices. Then G is Hamiltonian-connected (that is, every pair of vertices is joined by a Hamiltonian path).*

Theorem 2.2 *Let $G_{\mathcal{B},1}$ be the 1-BIG of a universal friendship hypergraph on at least 8 elements. Then $G_{\mathcal{B},1}$ is Hamiltonian.*

Proof It is known that for $v \geq 7$, the 1-BIG of a STS(v) is Hamiltonian [3]. Let C be a Hamilton cycle in this graph. Consider the subgraph H of $G_{\mathcal{B},1}$ induced by the vertices that contain the universal friend. Note that C and $V(H)$ partition the vertices of $G_{\mathcal{B},1}$. The graph H is the Kneser graph $KG(v,2)$. Its independence number is $v-1$ and connectivity is $\binom{v-2}{2}$, for $|v| \geq 6$. As $v \geq 8$ by assumption, H satisfies the conditions of Theorem 2.1. Therefore H is Hamiltonian-connected.

Consider two adjacent vertices on C, say u and v. It is clear that there exists distinct vertices $x, y \in H$ such that $u \to x$ and $v \to y$. Construct a Hamilton path P of H with endpoints x and y. Finally remove the edge $u \to v$ from C and attach the P to C using the two edges $u \to x$ and $v \to y$ to obtain a Hamilton cycle of $G_{\mathcal{B},1}$. □

The proof actually gives something stronger than the Hamiltonicity of $G_{\mathcal{B},1}$. It implies that you can always find a Hamilton cycle in $G_{\mathcal{B},1}$ that has a subpath consisting of all the vertices corresponding to the triples of the Steiner triple system of $\mathcal{B}$.

We now show that $G_{\mathcal{B},1}$ is pancyclic. To do this, we will give an explicit constructions for the cycle lengths 3 to $\frac{4v(v-1)}{6} - 1$.

Theorem 2.3 *$G_{\mathcal{B},1}$ is pancyclic.*

Proof There are $\frac{4v(v-1)}{6}$ vertices in the graph $G_{\mathcal{B},1}$ where $\frac{v(v-1)}{6}$ of them correspond to the triples of an STS(v) and the remaining correspond to triples containing the universal friend. Using a technique similar to the proof of Theorem 2.2 and using only a suitable-length subpath of a Hamilton cycle of the Steiner triple system of the friendship hypergraph, it is easy to see that $G_{\mathcal{B},1}$ has cycles of length between $\frac{3v(v-1)}{6} + 1$ to $\frac{4v(v-1)}{6}$.

So we remains to show $G_{\mathcal{B},1}$ has cycles of length between 3 to $\frac{3v(v-1)}{6}$. To do this, we consider only the vertices of the graph that correspond to triples containing the universal friend 0.

We order these triples in the following manner:

$\{0,1,2\}$	$\{0,2,3\}$	$\{0,3,4\}$	...	$\{0,v-2,v-1\}$	$\{0,v-1,v\}$
$\{0,1,3\}$	$\{0,2,4\}$	$\{0,3,5\}$	...	$\{0,v-2,v\}$	
...	...	...	...	...	
...	...	...	...	...	
...	...	...	...	...	
$\{0,1,v-1\}$	$\{0,2,v\}$				
$\{0,1,v\}$					

For example, when $v = 9$, we have

012	023	034	045	056	067	078	089
013	024	035	046	057	068	079	
014	025	036	047	058	069		
015	026	037	048	059			
016	027	038	049				
017	028	039					
018	029						
019							

Notice that for every row except the first, any two triples in consecutive columns are adjacent in $G_{\mathcal{B},1}$. So we will just permute row 1 so it also has this property. We permute row 1 to be ($\{0,1,2\}$,$\{0,3,4\}$,...,$\{0,v-2,v-1\}$, $\{0,2,3\}$,$\{0,4,5\}$,...$\{0,v-1,v\}$). For $v = 9$, we have

012	034	056	078	023	045	067	089
013	024	035	046	057	068	079	
014	025	036	047	058	069		
015	026	037	048	059			
016	027	038	049				
017	028	039					
018	029						
019							

With this ordering of the triples containing the universal friend, construct a path P starting at $\{0,1,2\}$ and continuing to $\{0,v-1,v\}$ by taking triples from each row starting from row 1 and moving from left to right. When we reach the end of a row, we just continue extending the path from the left side of the next row. Notice that the last vertex of P will be $\{0,2,v\}$. The only triple containing the universal friend which is not on the path P is $\{0,1,v\}$.

Notice that if a triple $0xy$ on P has only 0 in common with 012, then we can start at 012 and follow P until we get to $0xy$ and join the two vertices 012 and $0xy$ to get a cycle. Therefore, the only cycle lengths l that we cannot get using the path P and the approach in the previous sentence are those where the vertex $01x$ or $02y$ have distance $l-1$ from 012 along this path.

For example, using the construction stated above, we can get a cycle of length three 012 $\rightarrow$ 034 $\rightarrow$ 056 $\rightarrow$ 012, but cannot get a cycle on length 5 because the subpath $012 \to 034 \to 056 \to 023$ ends with a triple that has two elements in comment with 012 on P, and therefore 012 and 023 are not adjacent in $G_{\mathcal{B},1}$.

However, we can get around this by changing the construction slightly whenever we run into this difficulty. We begin by handling the special cases of cycle lengths of $3v(v-1)/6-5, 3v(v-1)/6-4, 3v(v-1)/6-2, 3v(v-1)/6-1$ and $3v(v-1)/6$.

To get a cycle of length $3v(v-1)/6$, we modify the path P slightly by placing $\{0,1,v-1\}$ between $\{0,2,4\}$ and $\{0,3,5\}$, placing $\{0,2,v\}$ between $\{0,3,5\}$ and $\{0,4,6\}$ and finally placing $\{0,1,v\}$ between $\{0,2,5\}$ and $\{0,3,6\}$. We now have a Hamiltonian path in $G_{\mathcal{B},1}$ with endpoints $\{0,1,2\}$ and $\{0,3,v\}$ by following the same construction method as for P. As these two vertices are adjacent, we have a desired cycle of length $3v(v-1)/6$. You can throw away the vertex $\{0,1,v\}$ and get a cycle of length $3v(v-1)/6-1$ and you can get a cycle of length $3v(v-1)/6-2$ by throwing away the vertices $\{0,1,v-1\}$ and $\{0,2,v\}$. This shows how to construct cycles of length $3v(v-1)/6-2, 3v(v-1)/6-1$ and $3v(v-1)/6$.

Now consider the subpath S of P starting at $\{0,1,2\}$ and ending in $\{0,1,v-2\}$. On S, we can move $\{0,1,v-2\}$ between $\{0,3,v-1\}$ and $\{0,4,v\}$ to get a new path in which the endpoints $\{0,1,2\}$ and $\{0,4,v\}$ are adjacent in $G_{\mathcal{B},1}$. So this shows there is a cycle of length $3v(v-1)/6-5$. Similarly, consider the subpath S of P starting at $\{0,1,2\}$ and ending in $\{0,2,v-1\}$. On S we can move $\{0,1,v-2\}$ and $\{0,2,v-1\}$ between $\{0,3,v-1\}$ and $\{0,4,v\}$ to get a new path in which the endpoints $\{0,1,2\}$ and $\{0,4,v\}$ are adjacent in $G_{\mathcal{B},1}$. So this shows there is a cycle of length $3v(v-1)/6-4$.

We now handle the reminder of the cycle lengths. We need to consider subpaths of P starting at $\{0,1,2\}$ and ending at $\{0,1,x\}$ or $\{0,2,x\}$. These paths will have the correct number of vertices for the cycle lengths we are interested in but the endpoints are not adjacent. We will manipulate these paths slightly to make the endpoints adjacent while maintaining the same number of vertices as in the original path.

Consider the subpath S of P, starting at $\{0,1,2\}$ and ending in a vertex $\{0,1,x\}$. We can assume $3 \leq x \leq v-3$, because of the special cases already handled. Then we can replace $\{0,1,x\}$ with the next occurring vertex (on P) of the form $\{0,3,x+2\}$. Note that the triple $\{0,y,v\}$ immediately preceding $\{0,1,x\}$ on S never contains 3 nor $x+2$, and therefore is adjacent to $\{0,3,x+2\}$ $G_{\mathcal{B},1}$. To see this, note that $x+2 \neq v$, $v \neq 3$ and $y \neq 3$. Also, it is easy to see that $x+2+y = v+4$, due the ordering of the triples and to the way P is constructed. As $v+4$ is odd, then $x+2$ and y have different parity. Therefore so $x+2 \neq y$.

Note that $\{0,3,x+2\}$ is adjacent to $\{0,1,2\}$ in $G_{\mathcal{B},1}$. This gives a new path whose endpoints are adjacent and has the same number of vertices as

S and hence gives a cycle of the desired length.

In the last scenario, consider the subpath S of P starting at $\{0,1,2\}$ and ending in a vertex $\{0,2,x\}$. We may assume $x \leq v-2$ as for $x = v-2, v-1$, as we have already handle these lengths in the special cases. If $x = 3$, then we can replace $\{0,2,3\}$ on S with $\{0,3,7\}$ and we get a new path with same length and whose endpoints are adjacent. On the other hand, if $x \neq 3$, then $\{0,1,2\}$ and $\{0,2,x\}$ are on different rows. The vertex on S preceding $\{0,2,x\}$ must be $\{0,1,x-1\}$. On the path S, replace $\{0,2,x\}$ with $\{0,3,x+1\}$, which is not already on S. It can be seen that $\{0,3,x+1\}$ is adjacent to $\{0,1,x-1\}$ and $\{0,1,2\}$ in $G_{\mathcal{B},1}$. This gives another path with same number of vertices as S and whose endpoints are adjacent. This takes care of all the cycle lengths and therefore the graph $G_{\mathcal{B},1}$ is pancyclic.
□

We now show that the connectivity of $G_{\mathcal{B},1}$ is larger than its independence number. This result along with Theorem 2.1 shows that $G_{\mathcal{B},1}$ is Hamiltonian-connected.

We begin by showing the independence number of $G_{\mathcal{B},1}$ is at most $v-1$, where v is the number of elements in the Steiner triple system. We let $X = \{0,1,...,v\}$ denote the point set of the friendship hypergraph where 0 is the universal friend.

Lemma 2.4 *If $\mathcal{N}$ is an independent set of $G_{\mathcal{B},1}$, then $|\mathcal{N}| \leq v-1$.*

Proof Consider the number of vertices in $\mathcal{N}$ corresponding to triples from the Steiner triple system used to construct $(X, \mathcal{B})$. If this number is zero, then all vertices in $\mathcal{N}$ must be triples that contain the universal friend 0. Removing the universal friend from these triples, we get a 1-intersecting family (of 2-sets) and hence by the Erdős-Ko-Rado Theorem, there are at most $\binom{v-1}{1} = v-1$ triples in $\mathcal{B}$, if they all contain the universal friend.

Otherwise, if there are $l > 0$ triples $\mathcal{L}$ in $\mathcal{N}$ that come from the Steiner triple system, then these triples must be disjoint and there $l \leq \lfloor v/3 \rfloor$. We now look at how many triples containing the universal friend can be in $\mathcal{N}$. Now, either there is a triple $\{0,x,y\}$ containing the universal friend such that the pair $\{x,y\}$ appears in some triple of $\mathcal{L}$ or for every triple $\{0,x,y\}$ containing the universal friend, neither x nor y belongs to any triples of $\mathcal{L}$. We consider both scenarios.

1. Suppose the triple $\{0,x,y\} \in \mathcal{N}$ and a triple $\{x,y,z\} \in \mathcal{L}$. Then it may be that the triples $\{0,x,z\}$ and $\{0,y,z\}$ belong to $\mathcal{N}$. If so, then these are the only triples containing the universal friend that can belong to $\mathcal{N}$. To see this, consider another triple $\{0,a,b\} \in \mathcal{N}$. If $x = a$, then it must be that $y \neq b$. Therefore, as this triple must have either empty intersection or intersect in two elements with the

triple $\{x, y, z\}$, we see that $\{0, a = x, b \neq y\}$ can be in $\mathcal{N}$ only if $b = z$ giving the triple $\{0, x, z\}$. Similarly, if $y = a$, then it must be that $x \neq b$, and therefore $\{0, a = y, b \neq x\}$ can be in $\mathcal{N}$ only if $b = z$ giving the triple $\{0, y, z\}$. So, in this case there are at most three triples containing the universal friend. As $l \leq \lfloor v/3 \rfloor$, $|\mathcal{N}| \leq \lfloor v/3 \rfloor + 3$.

2. Suppose for every triple $\{0, x, y\} \in \mathcal{N}$, neither x nor y appear in any of the triples of $\mathcal{L}$. As there are l triples in $\mathcal{L}$, by the Erdős-Ko-Rado Theorem, the maximum number of triples containing the universal friend can be $\binom{v-3l-1}{1} = v - 3l - 1$. Therefore the maximum number of triples in $\mathcal{N}$ is $v - 3l - 1 + l = v - 2l - 1$ where $l \geq 0$.

So we have shown that the maximum independent set has size at most $max\{\lfloor v/3 \rfloor + 3, v - 1, v - 2l - 1\}$ which is $v - 1$, as $v \geq 7$ and $l \geq 0$.

□

We now show that the vertex connectivity of $G_{\mathcal{B},1}$ is at least v.

Lemma 2.5 *The vertex connectivity of $G_{\mathcal{B},1}$ is at least v.*

Proof It suffices to show that between any two non-adjacent vertices of the 1-block intersection graph $G_{\mathcal{B},1}$ there are at least v internally-disjoint paths between the two vertices. We consider all pairs of vertices of $G_{\mathcal{B},1}$ that are not adjacent in $G_{\mathcal{B},1}$. Here are the pairs of non-adjacent vertices of $G_{\mathcal{B},1}$ that need to be considered (up to isomorphism).

1. $\{0,1,2\}, \{0,1,3\}$: In this case there are $\binom{v-3}{2}$ triples of the form $\{0, x, y\}$ where $x, y \notin \{1,2,3\}$ that are adjacent to both $\{0,1,2\}$ and $\{0,1,3\}$. Also, there is a triple from the Steiner system which is of the form $\{1, x, y\}$ where $x, y \notin \{2,3\}$. The path $012 \to 1xy \to 013$ along with the other paths give $\binom{v-3}{2} + 1$ internally disjoint paths between the two triples (where each path has length 2), which is at least v for $v \geq 7$.

2. $\{0,1,2\}, \{1,2,3\}$: In this case, we can use each of $\{0,3,x\}$ where $x \in \{4,5,...,v\}$ as the internal vertex of internally disjoint paths of length 2 giving $v - 3$ such paths. Now as $v \geq 7$, each of the 3 paths $012 \to 045 \to 026 \to 123$, $012 \to 046 \to 027 \to 123$ and $012 \to 056 \to 017 \to 123$ are internally disjoint with all the other given paths. This gives v internally disjoint paths between the two triples.

3. $\{0,1,2\}, \{3,4,5\}$: In this case, each of the vertices $\{0,3,x\}$, $\{0,4,x\}$ $\{0,5,x\}$, $x \in \{6,7,...,v\}$ can be used to form a path of length two between $\{0,1,2\}$ and $\{3,4,5\}$. Finally the path $012 \to 067 \to 024 \to 345$ is internally disjoint from the $3(v-5)$ paths listed earlier. As

$v \geq 7$, there are at least $3(v-5)+1 \geq v$ internally disjoint paths between $\{0,1,2\}$ and $\{3,4,5\}$.

4. $\{1,2,3\},\{4,5,6\}$: We note that in this case, $v \geq 9$, as the unique Steiner triple system of order 7 does not contain disjoint triples. We can construct nine internally disjoint paths of length two, each using a triple of the form $\{0,x,y\}$ as the sole internal triple of the path, where $x \in \{1,2,3\}$ and $y \in \{4,5,6\}$. Now for each ordered pair $(x,y) \in \{(1,4),(2,5),(3,6)\}$, we can form $\lfloor (v-6)/2 \rfloor$ internally disjoint paths of length 3 using two internal triples of the form $\{0,x,z\}$ and $\{0,y,w\}$ where z,w are distinct elements from $\{7,8,...,v\}$. This gives a total of $9+3\lfloor (v-6)/2 \rfloor \geq v$ internally disjoint paths between $\{1,2,3\},\{4,5,6\}$.

In each case, there exists at least v internally disjoint paths between two non-adjacent vertices. □

Combining Lemmas 2.4, 2.5 and Theorem 2.1, we have the following result.

Theorem 2.6 *The graph $G_{B,1}$ is Hamiltonian-connected.*

It should be pointed out that Lemma 2.4, 2.5 also imply that $G_{B,1}$ is Hamiltonian (see [1]).

3 The 2-BIG

In this section we will show that the 2-BIG of a universal friendship hypergraph is Hamiltonian and pancyclic. We will do this by giving an explicit construction.

Theorem 3.1 *$G_{B,2}$ is Hamiltonian*

Proof We will give an explicit construction of a Hamilton cycle in $\mathcal{G}_{B,2}$. To contruct the cycle, let $\{0,1,v\}$ be the first vertex of the cycle to be constructed. We will keep adding vertices to it until we get a Hamilton cycle. Let $\{1,x,v\}$ be a triple in the STS and make it the 2nd vertex of the partial cycle and $\{0,1,x\}$ the 3rd vertex of the partial cycle. For each remaining STS triple $1ab$ add to the cycle $01a \rightarrow 1ab \rightarrow 01b$. If there are any universal triples of the form $01x$ that has not been placed on the partial cycle after all the STS triples of the form $1ab$ has been placed, we arbitrarily extend the partial cycle using these remaining universal triples. At this point all triples of the form $\{0,1,x\}$ as well as all STS triples beginning with 1 have been placed onto the partial cycle.

Now suppose all the STS triples whose smallest element that is less than i has been placed on the cycle and suppose the last vertex placed on the partial cycle is $\{0, i-1, x\}$. There are two scenarios to consider

1. If the triple $\{i, x, y\}$ is a triple in the STS where $i < x, y$, then extend the cycle by adding the subpath $0ix \rightarrow ixy \rightarrow 0iy$.

2. If the triple $\{i, x, y\}$ is not a triple in the STS for any $y > i$ and $i \neq x$, then extend the partial cycle by adding the vertex $0ix$ to the end of the cycle.

For the remaining STS triples iab beginning with i, extend the partial cycle with $0ia \rightarrow iab \rightarrow 0ib$. Finally, if there are any triples of the form $0ix$ left over, just add them to the end of the cycle after all the STS triples beginning with i have been processed.

Repeat this process until all triples have been placed. Note that the last triple to be placed on the cycle will be $\{0, v-1, v\}$, which is adjacent to $\{0, 1, v\}$, the first vertex we placed on the cycle. Therefore, we have constructed a Hamilton cycle of $G_{\mathcal{B},2}$ □

Example 2 illustrates the construction process in the proof of Theorem 3.1.

Example 2 *Consider the following STS(9) whose triples are partitioned based on the smallest element in the triple.* {123, 147, 159, 168}, {249, 258, 267}, {348, 357, 369}, {456}, {789}.

After processing triples of STS beginning with 1, we have the (partial cycle) 019 → 159 → 015 → 012 → 123 → 013 → 014 → 147 → 017 → 016 → 168 → 018.

After processing triples of STS beginning with 2, we have the (partial cycle) 019 → 159 → 015 → 012 → 123 → 013 → 014 → 147 → 017 → 016 → 168 → 018 → 028 → 258 → 025 → 024 → 249 → 029 → 026 → 267 → 027 → 023.

After processing triples of STS beginning with 3, we have the (partial cycle) 019 → 159 → 015 → 012 → 123 → 013 → 014 → 147 → 017 → 016 → 168 → 018 → 028 → 258 → 025 → 024 → 249 → 029 → 026 → 267 → 027 → 023 → 034 → 348 → 038 → 035 → 357 → 037 → 036 → 369 → 039.

After processing triples of STS beginning with 4, we have the (partial cycle) 019 → 159 → 015 → 012 → 123 → 013 → 014 → 147 → 017 → 016 → 168 → 018 → 028 → 258 → 025 → 024 → 249 → 029 → 026 → 267 → 027 → 023 → 034 → 348 → 038 → 035 → 357 → 037 → 036 → 369 → 039 → 049 → 045 → 456 → 046 → 047 → 048.

After processing triples of STS beginning with 5,6, we have the (partial cycle) 019 → 159 → 015 → 012 → 123 → 013 → 014 → 147 → 017 →

016 → 168 → 018 → 028 → 258 → 025 → 024 → 249 → 029 → 026 → 267 → 027 → 023 → 034 → 348 → 038 → 035 → 357 → 037 → 036 → 369 → 039 → 049 → 045 → 456 → 046 → 047 → 048 → 058 → 057 → 058 → 056 → 069 → 067 → 068.

After processing triples of STS beginning with 7 we have the (partial cycle) 019 → 159 → 015 → 012 → 123 → 013 → 014 → 147 → 017 → 016 → 168 → 018 → 028 → 258 → 025 → 024 → 249 → 029 → 026 → 267 → 027 → 023 → 034 → 348 → 038 → 035 → 357 → 037 → 036 → 369 → 039 → 049 → 045 → 456 → 046 → 047 → 048 → 058 → 057 → 058 → 056 → 069 → 067 → 068 → 078 → 789 → 079.

Finally, the triple 089 *is the last triple to be added to the partial cycle giving* 019 → 159 → 015 → 012 → 123 → 013 → 014 → 147 → 017 → 016 → 168 → 018 → 028 → 258 → 025 → 024 → 249 → 029 → 026 → 267 → 027 → 023 → 034 → 348 → 038 → 035 → 357 → 037 → 036 → 369 → 039 → 049 → 045 → 456 → 046 → 047 → 048 → 058 → 057 → 058 → 056 → 069 → 067 → 068 → 078 → 789 → 079 → 089.

Note that 019 *and* 089 *have two elements in common and therefore are adjacent in the* 2*-BIG.* □

We now use a similar approach to show that $G_{\mathcal{B},2}$ is pancyclic.

Theorem 3.2 *$G_{\mathcal{B},2}$ is pancyclic.*

Proof Using the Hamilton cycle constructed in the proof of Theorem 3.1, it is easy to see that removing any number of the vertices corresponding to triples of the STS will provide cycles of length between $\frac{3v(v-1)}{6}$ to $\frac{4v(v-1)}{6}$. So we only need to consider constructing cycles of length between 3 and $\frac{3v(v-1)}{6} - 1$ in $G_{\mathcal{B},2}$. To do this, we will consider cycles using only of triples containing of the universal friend.

For each $1 \le i \le v-1$, let $f(i)$ denote the number of triples of the form $\{0, i, j\}$ where $i < j$. Clearly $\sum_{i=1}^{v-1} f(i) = \frac{3v(v-1)}{6}$.

Suppose we want to construct a cycle of length k between 3 and $f(1)$. Then any subset of k triples containing 0 and 1 will give a cycle of length k in $G_{\mathcal{B},2}$.

In general, suppose we would like to construct a cycle of length k between $\sum_{j=1}^{i-1} f(j) + 1$ and $\sum_{j=1}^{i} f(j)$ where $1 < i < v$. To do this, form a path starting with $\{0, 1, v\}$ and ending in $\{0, i-1, i\}$ using all the triples that contain the pairs $\{0,1\}$,$\{0,2\}$,...,$\{0, i-1\}$. This can easily be done. One way is $01v \to \{0, 1, v-1\} \to \cdots \to 012 \to 02v \to \cdots \to 023 \to \cdots \to \{0, i-2, i-1\} \to \{0, i-1, v\} \to \cdots \to \{0, i-1, i\}$. Now this path has length $\sum_{j=1}^{i-1} f(j)$. So now, to get a cycle of length k, we just need to extend this path by adding $k - \sum_{j=1}^{i-1} f(j)$ vertices. You can extend the path by using any $k - \sum_{j=1}^{i-1} f(j)$ triples of the form $0ij$ ($j = i+1$ to v) where the last triple added to the path is $0iv$. This will give a cycle of the desired length as $0iv$ and $01v$ are adjacent vertices in $G_{\mathcal{B},2}$. □

In $G_{\mathcal{B},2}$, each STS triple $\{x, y, z\}$ is adjacent to exactly 3 universal friend triples, namely $\{0, x, y\}, \{0, x, z\}$ and $\{0, y, z\}$. A question that comes to mind is whether it is possible to construct a Hamilton cycle in $G_{\mathcal{B},2}$ such that the STS triples lie exactly a distance of 4 from each other on the cycle. Example 3 shows that it is possible for $v = 7, 9$. It would be interesting to see if this is always possible. However, I was not able to generalize this to arbitrary, admissible values of $v > 9$.

Example 3 *Here is a Hamilton cycle of $G_{\mathcal{B},2}$ with $v = 7$ where the STS triples are a distance 4 apart.*

(012, 123, 013, 037, 047, 147, 017, 016,036, 367, 067, 057, 015,156, 056, 026,027, 257, 025, 045, 046, 246, 024, 023, 035, 345, 034, 014).

Here is a Hamilton cycle of $G_{\mathcal{B},2}$ with $v = 9$ where the STS triples are a distance 4 apart.

(789, 079, 049, 019, 129, 012, 028, 024, 246, 026, 069, 036, 367, 067, 047, 057, 257, 027, 029, 023, 238, 038, 048, 068, 168, 016, 046, 056, 569, 059, 035, 034, 349, 039, 037, 017, 147, 014, 018, 013, 135, 015, 025, 045, 458, 058, 089, 078). □

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