Efficient Computation of the Modular Chromatic Numbers of Trees

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Abstract

A modular k-coloring, $k \geq 2$, of a graph G without isolated vertices is a coloring of the vertices of G with the elements in $\mathbb{Z}_k$ (where adjacent vertices may be colored the same) having the property that for every two adjacent vertices in G the sums of the colors of their neighbors are different in $\mathbb{Z}_k$. The minimum k for which G has a modular k-coloring is the modular chromatic number $\text{mc}(G)$ of G. It is known that $2 \leq \text{mc}(T) \leq 3$ for every nontrivial tree T. We present an efficient algorithm that computes the modular chromatic number of a given tree.

Keywords: modular coloring, modular chromatic number.
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1 Introduction

The field of graph coloring is one of the most popular research areas in graph theory. Among the most studied vertex colorings are, of course, proper colorings. In a proper coloring of a graph G, a color is assigned to each vertex of G so that adjacent vertices are assigned distinct colors. Hence a proper coloring distinguishes the two vertices in every pair of adjacent vertices and the minimum number of colors required of a proper coloring of G is the well-known chromatic number $\chi(G)$. A coloring c of the vertices of a

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graph G, which may or may not be proper, is called *neighbor-distinguishing* if every two adjacent vertices of G are distinguished from each other in some manner by c. Therefore, proper colorings are neighbor-distinguishing.

In [2] new neighbor-distinguishing colorings of graphs that need not be proper were introduced. For a vertex v of a graph G, let $N(v)$ denote the neighborhood of v (the set of vertices adjacent to v). For a graph G without isolated vertices, let $c : V(G) \to \mathbb{Z}_k$ $(k \geq 2)$ be a vertex coloring of G where adjacent vertices may be colored the same. The *color sum* $\sigma(v)$ of a vertex v of G is defined as the sum in $\mathbb{Z}_k$ of the colors of the vertices in $N(v)$, that is,

$$\sigma(v) = \sum_{u \in N(v)} c(u).$$

The coloring c is called a *modular k-coloring* of G if $\sigma(u) \neq \sigma(v)$ in $\mathbb{Z}_k$ for every pair u, v of adjacent vertices of G. A coloring c is a *modular coloring* of G if c is a modular k-coloring of G for some integer $k \geq 2$ and the *modular chromatic number* $\mathrm{mc}(G)$ of G is the minimum k for which G has a modular k-coloring.

If c is a modular k-coloring of a graph G, then $\sigma(u) \neq \sigma(v)$ in $\mathbb{Z}_k$ for every pair u, v of adjacent vertices of G. Thus the coloring c^* of the vertices of G defined by $c^*(v) = \sigma(v)$, $v \in V(G)$, is a proper vertex coloring of G with at most k colors. This implies that the chromatic number of G is a lower bound for $\mathrm{mc}(G)$.

Observation 1.1 [2] *For every graph G without isolated vertices,* $\mathrm{mc}(G) \geq \chi(G)$.

By Observation 1.1, it follows that $\mathrm{mc}(T) \geq 2$ for every nontrivial tree T. Furthermore, a sharp upper bound of $\mathrm{mc}(T)$ was presented in [2].

Theorem 1.2 [2] *For every nontrivial tree T,* $2 \leq \mathrm{mc}(T) \leq 3$.

A *caterpillar* is a nontrivial tree the removal of whose leaves results in a path. In [1] modular colorings of trees were further studied and a complete characterization of caterpillars having modular chromatic number 2 was presented.

To illustrate these concepts, let us look at the two trees T and T^* in Figure 1. Both are caterpillars of order 10 having diameter 7 and a coloring from $V(T)$ to $\mathbb{Z}_2$ as well as a coloring from $V(T^*)$ to $\mathbb{Z}_3$ are given, where each vertex is labeled by its color followed by the corresponding color sum. Since the second coordinates of the labels of every two adjacent vertices in

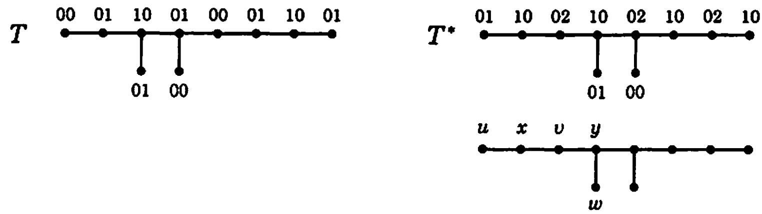

Figure 1: Modular colorings of trees

T are different, this is a modular 2-coloring of T. Similarly, the coloring of T^* given here is a modular 3-coloring of T^*. Then by Observation 1.1 the modular chromatic number of T equals 2. To see that $\text{mc}(T^*) = 3$, assume, to the contrary, that there exists a modular 2-coloring c of T^*. By the symmetry of the tree, we may assume that $\sigma(u) = \sigma(v) = \sigma(w) = 1$. If $c(x) = c(y)$, then $\sigma(v) = 2c(x) \neq 1$, which is impossible. Otherwise, $\sigma(u) = c(x) \neq c(y) = \sigma(w)$, another contradiction. Therefore, such a coloring c does not exist and we conclude that $\text{mc}(T^*) = 3$. In fact, it was shown in [1] that T^* is the tree of smallest order having modular chromatic number 3.

In this work we present an efficient recursive algorithm that determines the modular chromatic number of any given tree. We refer to the book [3] for graph theory notation and terminology not described in this paper.

2 Preliminary Results

Although modular colorings and modular chromatic numbers were originally defined only for graphs without isolated vertices in [2], we define $\text{mc}(K_1) = 1$. Furthermore, for every integer $k \geq 2$ assigning any integer to that isolated vertex v in K_1 results in a modular k-coloring of the graph with $\sigma(v) = 0$.

2.1 Root Adjusted Colorings and Signatures of Trees

For a rooted tree T, let us denote its root by v_T. Let $c : V(T) \to \mathbb{Z}_2$ be a 2-coloring of a rooted tree T. For an integer $\alpha \in \mathbb{Z}_2$, the *α-root adjusted color sum* $\sigma_\alpha(v)$ of a vertex v is defined by

$$\sigma_\alpha(v) = \begin{cases} \sigma(v) + \alpha & \text{if } v = v_T \\ \sigma(v) & \text{otherwise.} \end{cases}$$

The coloring c is called an *α-root adjusted modular 2-coloring* (or simply *α-adjusted coloring*) of T if $\sigma_\alpha(u) \neq \sigma_\alpha(v)$ in $\mathbb{Z}_2$ for every pair u, v of adjacent vertices in T. Hence,

$$c \text{ is a modular 2-coloring of } T \text{ if and only if } \alpha = 0 \tag{1}$$

unless T is trivial.

The *signature* $\text{sig}(T)$ of a rooted tree T is the set of ordered triples of integers in $\mathbb{Z}_2$ such that $(\alpha, \beta, \gamma) \in \text{sig}(T)$ if and only if there exists an α-adjusted coloring of T with $c(v_T) = \beta$ and $\sigma_\alpha(v_T) = \gamma$. When it is clear, we may express each element (α, β, γ) in $\text{sig}(T)$ simply as $\alpha\ \beta\ \gamma$. Note that for every rooted tree T

$$\text{sig}(T) \subseteq \{000, 001, 010, 011, 100, 101, 110, 111\} = \mathbb{Z}_2^3.$$

For example, let us determine the signatures of rooted trees K_1 and K_2. Recall that any coloring c of K_1 is a modular 2-coloring regardless of the color assigned to v_{K_1} and $\sigma(v_{K_1}) = 0$. Therefore, $\sigma_\alpha(v_{K_1}) = \alpha$ for each $\alpha \in \mathbb{Z}_2$. That is, $(\alpha, \beta, \gamma) \in \text{sig}(K_1)$ if and only if $\alpha = \gamma$ and so

$$\text{sig}(K_1) = \{000, 010, 101, 111\}. \tag{2}$$

For K_2, let $V(K_2) = \{v_{K_2}, v\}$ and suppose that c is a 2-coloring of K_2. Then for each $\alpha \in \mathbb{Z}_2$ observe that $\sigma_\alpha(v) = \sigma(v) = c(v_{K_2})$, implying that c is an α-adjusted coloring of K_2 if and only if $\sigma_\alpha(v_{K_2}) \neq c(v_{K_2})$. Hence,

$$\text{sig}(K_2) = \{001, 010, 101, 110\}. \tag{3}$$

By (1) it follows that a rooted tree T has a modular 2-coloring if and only if $(0, \beta, \gamma) \in \text{sig}(T)$ for some $\beta, \gamma \in \mathbb{Z}_2$. That is, to determine whether a given nontrivial rooted tree has modular chromatic number 2 or not it suffices to compute its signature. In the following subsection we discuss a way to find signatures of larger rooted trees.

2.2 Two Graph Operations on Trees

Before turning our attention to rooted trees of order at least 3 and their signatures, we introduce some additional graph operations and notation.

For a rooted tree T, let $x + T$ be the tree obtained from T by adding a new root x and joining x to the root v_T of T. Also, for two rooted trees S and T, let $S + T$ be the rooted tree obtained from S and T by identifying the roots of S and T and the new root v_{S+T} is that vertex belonging to both S and T. (Thus, $S + T = T + S$ for every two rooted trees S and T. Also, $K_1 + T = T$ for every rooted tree T.) For example, both $x + K_2$ and $K_2 + K_2$ are isomorphic to P_3, where $x + K_2$ is rooted at one of the two end-vertices while $K_2 + K_2$ is rooted at the vertex of degree 2.

The following two lemmas allow us to determine $\operatorname{sig}(x+T)$ and $\operatorname{sig}(S+T)$ once we know $\operatorname{sig}(S)$ and $\operatorname{sig}(T)$ of rooted trees S and T.

Lemma 2.1 *Let T be a rooted tree. Then $(\alpha, \beta, \gamma) \in \operatorname{sig}(T)$ if and only if $(\beta + \gamma + 1, \alpha, \gamma + 1) \in \operatorname{sig}(x + T)$.*

Proof. Suppose first that c is an α-adjusted coloring of T with $c(v_T) = \beta$ and $\sigma_\alpha(v_T) = \gamma$. Define the coloring $c' : V(x + T) \to \mathbb{Z}_2$ by $c'(x) = \alpha$ and $c'(v) = c(v)$ for every $v \in V(T)$ and consider the corresponding color sum $\sigma'(v)$ of each vertex $v \in V(x + T)$. Observe that

$$\sigma'(v) = \begin{cases} c'(v_T) = c(v_T) = \beta & \text{if } v = x \\ \sigma_\alpha(v) & \text{otherwise.} \end{cases}$$

Therefore, c' is a $(\beta + \gamma + 1)$-adjusted coloring of $x + T$ since $\sigma'_{\beta+\gamma+1}(x) = \beta + (\beta + \gamma + 1) = \gamma + 1 \neq \sigma'_{\beta+\gamma+1}(v_T)$.

Conversely, if c is a $(\beta + \gamma + 1)$-adjusted coloring of $x + T$ with $c(x) = \alpha$ and $\sigma_{\beta+\gamma+1}(x) = \gamma + 1$, then $c(v_T) = \sigma_{\beta+\gamma+1}(x) - (\beta + \gamma + 1) = \beta$ and $\sigma_{\beta+\gamma+1}(v_T) = \gamma$. It is then straightforward to verify that c restricted to $V(T)$ is an α-adjusted coloring of T possessing the desired property. ∎

Lemma 2.2 *Let T_1 and T_2 be rooted trees.*

(a) *If $(\alpha_i, \beta, \gamma) \in \operatorname{sig}(T_i)$ for $i = 1, 2$, then $(\alpha_1 + \alpha_2 + \gamma, \beta, \gamma) \in \operatorname{sig}(T_1 + T_2)$.*

(b) *If $(\alpha, \beta, \gamma) \in \operatorname{sig}(T_1 + T_2)$, then there exist two integers $\alpha_1, \alpha_2 \in \mathbb{Z}_2$ such that $\alpha = \alpha_1 + \alpha_2 + \gamma$ and $(\alpha_i, \beta, \gamma) \in \operatorname{sig}(T_i)$ for $i = 1, 2$.*

Proof. For $i = 1, 2$ suppose that c_i is an α_i-adjusted coloring of T_i with $c_i(v_{T_i}) = \beta$ and $\sigma_{\alpha_i}(v_{T_i}) = \gamma$. Construct $T = T_1 + T_2$ and define $c : V(T) \to \mathbb{Z}_2$ such that c restricted to $V(T_i)$ equals c_i for $i = 1, 2$. Then the corresponding color sum $\sigma(v)$ of each vertex $v \in V(T)$ is

$$\sigma(v) = \begin{cases} (\gamma - \alpha_1) + (\gamma - \alpha_2) = \alpha_1 + \alpha_2 & \text{if } v = v_T \\ \sigma_{\alpha_i}(v) & \text{if } v \in V(T_i) - \{v_T\},\ i = 1, 2. \end{cases}$$

Therefore, c is an $(\alpha_1 + \alpha_2 + \gamma)$-adjusted coloring of T with $c(v_T) = \beta$ and $\sigma_{\alpha_1+\alpha_2+\gamma}(v_T) = \gamma$.

Next suppose that c is an α-adjusted coloring of $T = T_1 + T_2$ with $c(v_T) = \beta$ and $\sigma_\alpha(v_T) = \gamma$. For $i = 1, 2$ let c_i be the restriction of c to $V(T_i)$ and $\alpha_i = N_i + \gamma$, where $N_i = \sum_{v \in N(v_{T_i})} c_i(v) = \sum_{v \in N(v_{T_i})} c(v)$. Observe that

$$\alpha_1 + \alpha_2 + \gamma = (N_1 + N_2 + \alpha) + \alpha + \gamma = \sigma_\alpha(v_T) + \alpha + \sigma_\alpha(v_T) = \alpha.$$

Furthermore, $\sigma_{\alpha_i}(v_{T_i}) = N_i + \alpha_i = \gamma$ and so one can verify that c_i is an α_i-adjusted coloring of T_i with the desired property for $i = 1, 2$. ■

For example, since $K_2 = x + K_1$, we can derive (3) from (2) and Lemma 2.1 (or derive (2) from (3) and Lemma 2.1).

Figure 2 shows eleven rooted trees S_i $(1 \le i \le 11)$, where the roots are represented by hollow vertices. Observe that for each S_i with $2 \le i \le 11$

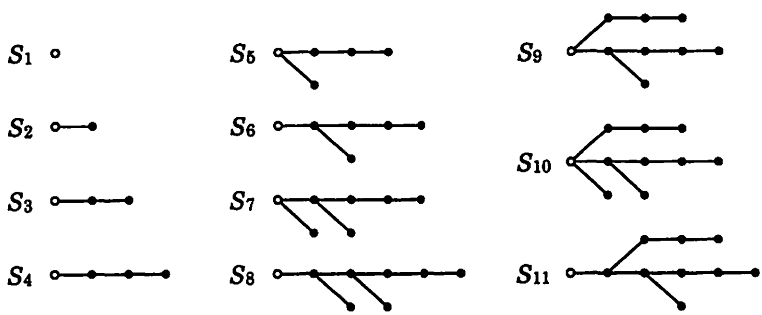

Figure 2: Rooted trees S_i $(1 \le i \le 11)$

either (i) there exists S_j, $1 \le j \le i-1$, such that $S_i = x + S_j$ or (ii) there exist S_{j_1} and S_{j_2}, $1 \le j_1, j_2 \le i-1$, such that $S_i = S_{j_1} + S_{j_2}$. Therefore, the signatures of these rooted trees can be recursively determined using Lemmas 2.1 and 2.2 with (2), which we summarize in (4).

$$\begin{array}{ll} \mathrm{sig}(S_1) = \{000, 010, 101, 111\} & \mathrm{sig}(S_7) = \{010, 110\} \\ \mathrm{sig}(S_2) = \{001, 010, 101, 110\} & \mathrm{sig}(S_8) = \{001, 011\} \\ \mathrm{sig}(S_3) = \{000, 001, 010, 011\} & \mathrm{sig}(S_9) = \{000, 100\} \\ \mathrm{sig}(S_4) = \{000, 001, 100, 101\} & \mathrm{sig}(S_{10}) = \{\} \\ \mathrm{sig}(S_5) = \{001, 101\} & \mathrm{sig}(S_{11}) = \{101, 111\} \\ \mathrm{sig}(S_6) = \{000, 010\} & \end{array} \quad (4)$$

We saw earlier that a rooted tree T has a modular 2-coloring if and only if $(0, \beta, \gamma) \in \mathrm{sig}(T)$ for some integers $\beta, \gamma \in \mathbb{Z}_2$. Therefore, $\mathrm{mc}(S_i) = 2$ for $2 \le i \le 9$ while neither S_{10} nor S_{11} has modular chromatic number 2.

Of course, this is not surprising since both S_{10} and S_{11} are isomorphic to the tree T^* in Figure 1, which is the unique smallest tree having modular chromatic number 3.

2.3 Signature-closed Sets of Trees

Let $\mathcal{S}$ be a nonempty set of rooted trees. We say that $\mathcal{S}$ is *signature closed* if (i) for every $T \in \mathcal{S}$ there exists $S \in \mathcal{S}$ such that $\text{sig}(S) = \text{sig}(x + T)$ and (ii) for every pair $T_1, T_2 \in \mathcal{S}$ there exists $S \in \mathcal{S}$ such that $\text{sig}(S) = \text{sig}(T_1 + T_2)$. It can be verified by Lemmas 2.1 and 2.2 with (4) that the set $\mathcal{S}^* = \{S_1, S_2, \ldots, S_{11}\}$ of the eleven rooted trees shown in Figure 2 is signature closed. Table 1 shows that each member in $\mathcal{S}^*$ satisfies (i); while Table 2 shows that every pair of members in $\mathcal{S}^*$ satisfies (ii). Note that only

$+$	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}
x	S_2	S_3	S_4	S_1	S_6	S_5	S_8	S_9	S_{11}	S_{10}	S_7

Table 1: $S_i, S_j \in \mathcal{S}^*$ such that $\text{sig}(x + S_i) = \text{sig}(S_j)$

$+$	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}
S_1	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}
S_2		S_2	S_2	S_5	S_5	S_7	S_7	S_5	S_{10}	S_{10}	S_5
S_3			S_1	S_4	S_5	S_6	S_7	S_{11}	S_9	S_{10}	S_8
S_4				S_4	S_5	S_9	S_{10}	S_5	S_9	S_{10}	S_5
S_5					S_5	S_{10}	S_{10}	S_5	S_{10}	S_{10}	S_5
S_6						S_6	S_7	S_{10}	S_9	S_{10}	S_{10}
S_7							S_7	S_{10}	S_{10}	S_{10}	S_{10}
S_8								S_{11}	S_{10}	S_{10}	S_8
S_9									S_9	S_{10}	S_{10}
S_{10}										S_{10}	S_{10}
S_{11}											S_{11}

Table 2: $S_i, S_j, S_k \in \mathcal{S}^*$ such that $\text{sig}(S_i + S_j) = \text{sig}(S_k)$

the entries in the upper right half of Table 2 are shown since $S + T = T + S$ for every two rooted trees S and T in general. Note also that $\mathcal{S}^*$ is a minimal signature-closed set containing the trivial tree.

We make another observation here. Consider an unrooted tree $T \cong P_5$. Then there are three rooted trees T_1, T_2, and T_3 that are isomorphic to T, namely $T_1 = x + S_4$, $T_2 = S_5$, and $T_3 = S_3 + S_3$. By Tables 1 and 2 we see that $\text{sig}(T_1) = \text{sig}(T_3) = \text{sig}(S_1) \neq \text{sig}(S_5) = \text{sig}(T_2)$, that is, it is possible that two rooted trees have different signatures and are isomorphic to each other when unrooted. In general, if T is an unrooted tree of order $n \geq 2$ with $V(T) = \{v_1, v_2, \ldots, v_n\}$, then there are n rooted trees $T_1, T_2, \ldots, T_n$ each of which is isomorphic to T and $v_{T_i} = v_i$ for $1 \leq i \leq n$. Then observe that T is modular 2-colorable if and only if T_i is modular 2-colorable for each i, $1 \leq i \leq n$. Therefore, to determine whether a given unrooted tree T is modular 2-colorable or not, it suffices to root T at any vertex and compute the signature of the resulting rooted tree.

3 The Main Result and Algorithm

As mentioned before, our goal is to be able to obtain the signature of a given rooted tree. In order to do this, we state another useful lemma, which provides us with the means to modify a rooted tree without altering its signature. The proof is omitted since it is an immediate consequence of Lemmas 2.1 and 2.2.

Lemma 3.1 *Suppose that* T_1 *and* T_2 *are rooted trees with* $\text{sig}(T_1) = \text{sig}(T_2)$. *Then* (a) $\text{sig}(x + T_1) = \text{sig}(x + T_2)$ *and* (b) $\text{sig}(S + T_1) = \text{sig}(S + T_2)$ *for every rooted tree* S.

Theorem 3.2 *For a given rooted tree* T, *there exists a rooted tree* $S \in \mathcal{S}^*$ *such that* $\text{sig}(S) = \text{sig}(T)$.

Proof. We proceed by induction on the order n of T. If $n = 1$, then $T = S_1 \in \mathcal{S}^*$ and the result is immediate. For an integer $n \geq 1$, suppose that for every rooted tree T' of order at most n there exists $S' \in \mathcal{S}^*$ such that $\text{sig}(S') = \text{sig}(T')$. Now let T be a tree of order $n + 1$ rooted at the vertex v_T. If v_T is an end-vertex, then there exists a rooted tree T' of order n such that $T = x + T'$. Let $S' \in \mathcal{S}^*$ such that $\text{sig}(S') = \text{sig}(T')$. Since $\mathcal{S}^*$ is signature closed, there exists $S \in \mathcal{S}^*$ with $\text{sig}(S) = \text{sig}(x + S')$. Therefore, Lemma 3.1(a) implies that $\text{sig}(S) = \text{sig}(x + S') = \text{sig}(x + T') = \text{sig}(T)$. Otherwise, there exist two rooted trees T' and T'' such that $T = T' + T''$ and observe that the order of each of T' and T'' is at most n. Let $S', S'' \in \mathcal{S}^*$ such that $\text{sig}(S') = \text{sig}(T')$ and $\text{sig}(S'') = \text{sig}(T'')$. Since $\mathcal{S}^*$ is signature closed, $\text{sig}(S) = \text{sig}(S' + S'')$ for some $S \in \mathcal{S}^*$. Hence, $\text{sig}(S) = \text{sig}(S' + S'') = \text{sig}(T' + T'') = \text{sig}(T)$ by Lemma 3.1(b), completing the proof. ∎

The proof of the preceding theorem is constructive and readily yields the following recursive algorithm which, when given a rooted tree T, returns a tree $S \in \mathcal{S}^*$ with $\text{sig}(S) = \text{sig}(T)$.
Case 1. If $T \in \mathcal{S}^*$, then return $S = T$.
Case 2. If $T \notin \mathcal{S}^*$, then T is nontrivial and $N(v_T)$ is nonempty.
Subcase 2.1. If v_T has only one child v, let T' be the subtree of T rooted at v. Recursively find $S' \in \mathcal{S}^*$ with $\text{sig}(S') = \text{sig}(T')$. Return $S \in \mathcal{S}^*$ with $\text{sig}(S) = \text{sig}(x + S')$.
Subcase 2.2. If v_T has n children $v_1, v_2, \ldots, v_n$ where $n \geq 2$, then let T' be the tree obtained from T by deleting the subtree rooted at v_n and let T'' be the tree obtained from T by deleting the subtrees rooted at $v_1, v_2, \ldots, v_{n-1}$. Recursively find $S', S'' \in \mathcal{S}^*$ such that $\text{sig}(S') = \text{sig}(T')$ and $\text{sig}(S'') = \text{sig}(T'')$. Return $S \in \mathcal{S}^*$ with $\text{sig}(S) = \text{sig}(S' + S'')$.

Theorem 3.3 *Given a rooted tree T the algorithm returns a tree $S \in \mathcal{S}^*$ with* $\text{sig}(S) = \text{sig}(T)$. *The running time of the algorithm is linear in the number of vertices of the tree T.*

Proof. The proof of Theorem 3.2 guarantees that the algorithm terminates correctly. The number of recursive calls is at most the number of vertices in the tree. During each call either T is identified as a member of $\mathcal{S}^*$ or a table look-up is used to produce the tree in $\mathcal{S}^*$. So each call can be performed in constant time and so the running time is linear in the number of vertices of the tree. ∎

4 Existence of 3-Modular Colorings of Trees

We note in closing that Theorem 3.2 can be modified to show that for every tree T there exists a modular 3-coloring of T and so $\text{mc}(T) \leq 3$, which gives an alternative proof of Theorem 1.2.

Let $c : V(T) \to \mathbb{Z}_k$ be a k-coloring of a rooted tree T. For an integer $\alpha \in \mathbb{Z}_k$, the *α-root adjusted color sum* $\sigma_\alpha(v)$ of a vertex v is defined by

$$\sigma_\alpha(v) = \begin{cases} \sigma(v) + \alpha & \text{if } v = v_T \\ \sigma(v) & \text{otherwise.} \end{cases}$$

The coloring c is called an *α-root adjusted modular k-coloring* (or simply an *(α, k)-adjusted coloring*) of T if $\sigma_\alpha(u) \neq \sigma_\alpha(v)$ in $\mathbb{Z}_k$ for every pair u, v of adjacent vertices in T. Hence, c is a modular k-coloring of T if $\alpha = 0$ but the converse does not hold in general. (Recall that the converse does hold for $k = 2$, which is stated in (1).) The *k-signature* $\text{sig}_k(T)$ of T is the set of ordered triples of integers in $\mathbb{Z}_k$ such that $(\alpha, \beta, \gamma) \in \text{sig}_k(T)$ if and only if there exists an (α, k)-adjusted coloring of T with $c(v_T) = \beta$ and $\sigma_\alpha(v_T) = \gamma$. If $\mathcal{S}$ is a nonempty set of rooted trees, then $\mathcal{S}$ is *k-signature closed* if (i) for every $T \in \mathcal{S}$ there exists $S \in \mathcal{S}$ such that $\text{sig}_k(S) = \text{sig}_k(x + T)$ and (ii) for every pair $T_1, T_2 \in \mathcal{S}$ there exists $S \in \mathcal{S}$ such that $\text{sig}_k(S) = \text{sig}_k(T_1 + T_2)$.

Note that a rooted tree T has a modular k-coloring if and only if $(0, \beta, \gamma) \in \text{sig}_k(T)$ for some integers $\beta, \gamma \in \mathbb{Z}_k$. In particular, T has a modular 3-coloring if and only if $(0, \beta, \gamma) \in \text{sig}_3(T)$ for some integers $\beta, \gamma \in \mathbb{Z}_3$.

It turns out that the set $\mathcal{S}_3 = \{S_1, S_2, S_3\}$ is 3-signature closed, where S_1, S_2, and S_3 are the first three rooted trees in Figure 2. Also, it can be verified that

$$010 \in \text{sig}_3(S_i) \quad \text{for } 1 \leq i \leq 3. \tag{5}$$

By an argument similar to the one used in the proof of Theorem 3.2, we can show that for an arbitrary rooted tree T there exists a tree $S \in \mathcal{S}_3$ such that $\text{sig}_3(S) = \text{sig}_3(T)$. Then by (5) it follows that S is modular 3-colorable and so is T, that is, $\text{mc}(T) \leq 3$.

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