Fully Automorphic Decompositions of Graphs

Robert A. Beeler
East Tennessee State University
Johnson City, TN 37614-1700 USA
beelerr@etsu.edu

Abstract

A *decomposition* $\mathscr{D}$ of a graph H by a graph G is a partition of the edge set of H such that the subgraph induced by the edges in each part of the partition is isomorphic to G. The *intersection graph* $I(\mathscr{D})$ of the decomposition $\mathscr{D}$ has a vertex for each part of the partition and two parts A and B are adjacent iff they share a common node in H. If $I(\mathscr{D}) \cong H$, then $\mathscr{D}$ is an *automorphic decomposition* of H. If $n(G) = \chi(H)$ as well, then we say that $\mathscr{D}$ is a *fully automorphic decomposition*. In this paper, we examine the question of whether a fully automorphic host will have an even degree of regularity. We also give several examples of fully automorphic decompositions as well as necessary conditions for their existence.

1 Introduction

All graphs in this paper are finite, undirected, simple graphs with no isolated vertices. We will use notation consistent with [3, 9]. In particular, for any graph G, $n(G)$ denotes the the number of vertices in G and $e(G)$ denotes the number of edges in G. As usual, K_n and P_n will denote, respectively, the complete graph and the path on n-nodes. However, the degree of a vertex v in a graph G will be denoted by $\deg_G(v)$. For all other undefined graph theory terminology, refer to West [9].

A *decomposition* $\mathscr{D}$ of a graph H (called the *host*) is a partition of the edge set of H. The subgraphs of H induced by the parts of the partition are the *blocks* of the decomposition [3]. By definition, blocks are edge-disjoint but in general, they may share common nodes. The *intersection graph* $I(\mathscr{D})$ of the decomposition $\mathscr{D}$ has a vertex for each block and two vertices

JCMCC **83** (2012), pp. 87-96

are adjacent iff they share a common node. To help separate the levels of abstraction, the term "vertex" will be used for vertices of the intersection graph and the term "node" will be used for vertices of the host graph and its blocks.

A decomposition $\mathscr{D}$ is *automorphic* iff the host H and $I(\mathscr{D})$ are isomorphic [2]. A decomposition is *cyclic* iff the host H is (isomorphic to) a graph with node set $\mathbb{Z}_n$ (the integers modulo n) in which the translations $x \to x + t$ are automorphisms of H carrying blocks into blocks.

We restrict ourselves to the case where all blocks of the decomposition are isomorphic to a single graph G. This graph is the *prototype* for the decomposition. A decomposition $\mathscr{D}$ with prototype G is said to be a *G-decomposition* of H [3]. In this case, we say that G is a *divisor* of H. If there is an automorphic decomposition of H, then H is an *automorphic host.* Similarly, if there is an automorphic G-decomposition of H, then G is an *automorphic divisor of H* [2].

The paper by Beeler and Jamison [2] gives several results concerning automorphic decompositions. Here $\overline{d}(H)$ denotes the average degree of H.

Theorem 1.1 *[2] If $\mathscr{D}$ is an automorphic G-decomposition of H, then (i) $e(H) = n(H)e(G)$; (ii) The average degree of H is $\overline{d}(H) = 2e(G)$; (iii) $n(G) \leq \chi(H)$, where $\chi(H)$ is the chromatic number of H.*

In this paper, we consider the case of an automorphic G-decomposition of H such that $\chi(H) = n(G)$. Such a decomposition will be referred to *fully automorphic.* Analogous definitions will be used for *fully automorphic divisor* and *fully automorphic host.*

Decompositions and colorings are related concepts (sf. [3]). Hence it is important to clarify that the notion of an automorphic decomposition is completely different from that of an automorphic coloring. Given a graph G, an *automorphic edge (vertex)-coloring* of G is a proper edge (vertex)-coloring such that each automorphism of the graph preserves the coloring [6].

One of the primary questions raised in [2] was whether all automorphic hosts are regular of even degree. This question is the primary motivation behind the introduction of fully automorphic decompositions. This paper will investigate the possible structure of fully automorphic decompositions and provide several examples. We will also prove that certain classes of fully automorphic hosts are regular of even degree.

2 Necessary conditions and regularity results

In this section, we give several necessary conditions for the existence of a fully automorphic decomposition. To facilitate this discussion, we prove four additional necessary conditions for automorphic decompositions in the following lemma.

Lemma 2.1 *In addition to the conditions of Theorem 1.1, the following are necessary for the existence of an automorphic G-decomposition of H:*

(i) $\alpha(H)n(G) \leq n(H)$, *where* $\alpha(H)$ *is the independence number of* H;

(ii) For any node $v \in V(H)$, *at most* $\omega(H)$ *G-blocks meet at* v, *where* $\omega(H)$ *is the clique number of* H;

(iii) The average number of G-blocks that meet at any node of H *is* $n(G)$;

(iv) $n(G) \leq \omega(H)$.

Proof.

(i) Since $\mathscr{D}$ is automorphic, $I(\mathscr{D})$ must have an independent set of size $\alpha(H)$. Thus we must have $\alpha(H)$ independent G-blocks. Each block requires $n(G)$ distinct nodes in H, so $\alpha(H)n(G) \leq n(H)$.

(ii) If more than $\omega(H)$ G-blocks meet at any node of H, then these blocks form a clique of order greater than $\omega(H)$ in $I(\mathscr{D})$, a contradiction.

(iii) Let $\bar{b}$ be the average number of G-blocks that meet at a node of H. Each block has $n(G)$ distinct nodes of H. The total number of blocks is $n(H)$ by Theorem 1.1. Thus $n(H)\bar{b}/n(G) = n(H)$, or equivalently, $\bar{b} = n(G)$.

(iv) Follows immediately from (ii) and (iii). ■

Theorem 2.2 *Suppose that* $\mathscr{D}$ *is a fully automorphic G-decomposition of* H. *In addition to the conditions of Theorem 1.1 and Lemma 2.1, we require:*

(i) $n(G) = \omega(H) = n(H)/\alpha(H) = \chi(H)$;

(ii) In $\mathscr{D}$ *there are* $n(G)$ *G-blocks meeting at any node of* H;

(iii) If G is not the disjoint union of P_2*'s, then* $\delta(H) \geq n(G) + 1$, *where* $\delta(H)$ *is the minimum degree of* H;

(iv) H is not complete.

Proof.

(i) Note that $n(G) \leq \omega(H) \leq \chi(H)$ and $n(G) \leq \frac{n(H)}{\alpha(H)} \leq \chi(H)$ by Theorem 1.1. Further, in a fully automorphic G-decomposition of H we have that $n(G) = \chi(H)$. From this it follows that:

$$n(G) = \omega(H) = n(H)/\alpha(H) = \chi(H).$$

(ii) By Lemma 2.1, we have that at most $\chi(H)$ G-blocks meet at any node of H. On average, $n(G)$ blocks meet at any node of H by Lemma 2.1. We have $n(G) = \omega(H)$ by (i). Thus, there are no nodes of H that have more than $n(G)$ blocks. Thus, at any node of H, we have $n(G)$ G-blocks.

(iii) Assume that G is not the disjoint union of P_2's. Take $u \in V(H)$ such that $deg_H(u) = \delta(H)$. If $\delta(H) = n(G)$, then there is a G-block, $A \in \mathscr{D}$, representing u in $I(\mathscr{D})$ that intersects exactly $n(G)$ other blocks. Let $\mathscr{B}$ be the set of G-blocks that share a common node with A. For each node v of A we must choose $n(G) - 1$ distinct blocks to intersect A at v. Since $|\mathscr{B}| = n(G)$, we have $n(G)$ choices for each node of A, where each choice leaves out a distinct element of $\mathscr{B}$. If we use $n(G)$ different choices, then each pair of elements of $\mathscr{B}$ share a common node with A. This would result in a clique of size $n(G) + 1 > \omega(H)$, a contradiction. Thus, we can make at most $n(G) - 1$ different choices for each node of A. Thus, A shares the same node set with at least $n(G) - 2$ other blocks. Hence, $n(G) - 1$ copies of G share the same node set. It follows that the most edges that this set of shared nodes can have is $n(G)(n(G) - 1)/2$. Thus:

$$(n(G) - 1)e(G) \leq \frac{n(G)(n(G) - 1)}{2} \Rightarrow e(G) \leq \frac{n(G)}{2}.$$

However, this implies that G has an isolated node or G is the disjoint union of P_2's. In either case, we have a contradiction.

(iv) Note that complete graphs of even order do not have automorphic decompositions by [2]. Hence, we need only consider complete graphs of odd order, say $H \cong K_{2n+1}$. Hence $n(G) = 2n + 1$ and $e(G) = e(K_{2n+1})/n(K_{2n+1}) = n$. Since each edge covers at most two distinct nodes, G must contain an isolated node, a contradiction. ∎

In [2], Beeler and Jamison provided regularity results for a certain class of automorphic G-decompositions. In the following theorems, we establish similar results regarding fully automorphic decompositions.

Corollary 2.3 *Suppose that H admits a fully automorphic G-decomposition.*

(i) If G is a d-regular graph, then H must be $n(G)d$-regular;

(ii) If the smallest distinct elements of the degree sequence of G are 1 and a, where $a \geq 2e(G) - n(G) + 1$, then H must be $2e(G)$-regular.

Proof.

(i) Theorem 2.2 implies that $n(G)$ G-blocks meet at every node of H. Since G is d-regular, each of these blocks contribute d edges. Thus, each node of H is incident with $n(G)d$ edges.

(ii) If G is the disjoint union of P_2's, then the result follows from (i). Otherwise, it follows from Theorem 2.2 that $\delta(H) \geq n(G) + 1$. Theorem 2.2 also implies that exactly $n(G)$ G-blocks meet at any node of H. Since the smallest elements of the degree sequence of G are 1 and a, it follows that the smallest admissible degrees for nodes of H are $n(G)$ and $n(G) - 1 + a$. Since $\delta(H) \geq n(G) + 1$, it follows that $\delta(H) \geq n(G) - 1 + a$. However, $a \geq 2e(G) - n(G) + 1$ implies that $\delta(H) \geq 2e(G)$. Since H has no nodes of degree lower than $2e(G)$, it follows from Theorem 1.1 that H is $2e(G)$-regular. ■

Theorem 2.4 *If H admits a fully automorphic P_4-decomposition, then H is 6-regular.*

Proof. Since P_4 is not a disjoint union of P_2's and H admits a fully automorphic P_4-decomposition, it follows from Theorem 2.2 that $\delta(H) \geq 5$. Since $\overline{d}(H) = 6$ by Theorem 1.1, we need only show that H has no nodes of degree five.

Suppose to the contrary that H has a node of degree five. This means that there is a P_4-block A that intersects exactly five others, say B_1, B_2, B_3, B_4, and B_5. At every node of A, exactly three of these blocks must intersect A. For convenience, we denote these blocks by their indices. At most, we can have a clique of order four, and one of these blocks is A. Thus at most three of the B_i mutually intersect. Further, since at most four of these combinations are used, at least two pairs of blocks never appear on a node of A. Without loss of generality, suppose that the pairs 12 and 13 never appear on a node of A. We cannot allow any two of 234, 235, and 245 as this would result in a clique of size $n(G) + 1$. Thus, we have 145 and 345 as well as one of 234, 235, or 245. Hence either B_4 or B_5 intersect A three times. Without loss of generality, assume B_5 intersects A three times. Hence B_5 and A share a common node set. It follows that any other P_4-block may share at most one node with A, otherwise that block will share at least one edge with either A or B_5. However, B_4 shares two nodes with A, a contradiction. ■

3 Constructions using valuations

Let G be a graph with $e(G) = q$. If $x \in \mathbb{Z}_n$, define $|x|_n = \min\{x, n-x\}$. A *$\mathbb{Z}_n$-labeling of G* is an injective map $f : V(G) \to \mathbb{Z}_n$. A *$\mathbb{Z}_n$-valuation* is a $\mathbb{Z}_n$-labeling f of G for which the induced edge labeling $f^*(xy) := |f(x)-f(y)|_n$ is injective and $n/2$ does not appear as an edge label. A $\mathbb{Z}_n$-valuation is *closed* if $f^*(G) = f^*(E)$, where: $f^*(E) = \{f^*(xy) : xy \in E\}$ and $f^*(G) = \{|f(x) - f(y)|_n : x, y \in V, x \neq y\}$ [1].

Rosa [8] introduced several important types of valuations, including graceful labelings. A *graceful labeling* on G is a $\mathbb{Z}_{2q+1}$-valuation in which all of the vertex labels come from $\{0, 1, ..., q\}$. Note that a graceful labeling is a closed $\mathbb{Z}_n$-valuation for all $n \geq 2q+1$. An extensive list of graphs that admit graceful and related labelings can be found in [4].

Let S be a set of numbers in $\mathbb{Z}_n$, all between 1 and $n/2$, inclusive. The *circulant* $C_n(S)$ has vertex set $\mathbb{Z}_n$ and edge set $\{xy : x, y \in V, |x-y|_n \in S\}$.

The following theorem constructs an infinite class of automorphic decompositions using circulants and valuations.

Theorem 3.1 *[2] G is a cyclic automorphic divisor of $C_n(S)$ iff there exists a closed $\mathbb{Z}_n$-valuation f on G such that $S = f^*(E)$.*

Using Theorem 3.1 and the results of [5] and [7] yields examples of fully automorphic decompositions. For completeness, we list the relevant results from these papers.

Theorem 3.2 *[5] Let $G = C_n(a, b)$ be a connected circulant with $|a|_n \neq |b|_n$. Then:*

(i) $\chi(G) = 2$ *if and only if a and b are odd and n is even;*

(ii) $\chi(G) = 4$ *if $3 \nmid n$, $n \neq 5$ and* $b \equiv \pm 2a \pmod{n}$ *or* $a \equiv \pm 2b \pmod{n}$*;*

(iii) $\chi(G) = 4$ *if $n = 13$ and* $b \equiv \pm 5a \pmod{13}$ *or* $a \equiv \pm 5b \pmod{13}$*;*

(iv) $\chi(G) = 5$ *if $n = 5$;*

(v) $\chi(G) = 3$ *in all other cases.*

Theorem 3.3 *[7] Let $G = C_n(a, b, a+b)$ be a connected 6-regular circulant, where $n \geq 7$ and a, b, and $a+b$ are pairwise distinct positive integers. Then:*

(i) $\chi(G) = 7$ *if and only if* $n = 7$*;*

(ii) $\chi(G) = 6$ *if and only if* $G \cong C_{11}(1, 2, 3)$*;*

(iii) $\chi(G) = 5$ *if and only if* $G \cong C_n(1, 2, 3)$ *and* $n \neq 7, 11$ *is not divisible by 4 or G is isomorphic to one of the following circulants:* $C_{13}(1, 3, 4)$, $C_{17}(1, 3, 4)$, $C_{18}(1, 3, 4)$, $C_{19}(1, 7, 8)$, $C_{25}(1, 3, 4)$, $C_{26}(1, 7, 8)$, $C_{33}(1, 6, 7)$, $C_{37}(1, 10, 11)$*;*

(iv) $\chi(G) = 3$ *if and only if* $3|n$ *and* $3 \nmid a, b, a + b$*;*

(v) $\chi(G) = 4$ *in all other cases.*

From these two results and Theorem 3.1, the following corollary is immediate.

Corollary 3.4 *The following graphs H have a fully automorphic G- decomposition: (i)* $H \cong C_{2n}$, $G \cong P_2$*; (ii)* $H \cong C_{3n}(a, 2a)$, $G \cong P_3$, *and* $3 \nmid a, 2a$*; (iii)* $H \cong C_{3n}(a, b, a + b)$, $G \cong K_3$, *and* $3 \nmid a, b, a + b$.

Other than the trivial cases (i.e., a cycle or a complete graph), the chromatic number of circulants has been understudied, except for the results of Heuberger [5] and Meszka et al. [7]. Those results suggest that the order of the host is a multiple of the order of the prototype in a fully automorphic decomposition. This is also suggested by the following result.

Lemma 3.5 *If $C_{mk}(S)$ is a circulant such that for all $a \in S$, $k \nmid a$ and m is sufficiently large, then $\chi(C_{mk}(S)) \leq k$.*

Proof. Let $V(C_{mk}(S)) = \{0, 1, ..., mk-1\}$. We claim that the color classes are the congruence classes modulo k. Since $k \nmid a$ for all $a \in S$, it follows that no two elements in a congruence class share a common edge. Thus we need at most k colors. ■

This implies that if $k|n$ and $k \nmid a$ for all $a \in S$, then at most k colors are required for $C_n(S)$. This observation leads us to study p-modular valuations. A *(closed) p-modular $\mathbb{Z}_{pk}$-valuation* of G is a (closed) $\mathbb{Z}_{pk}$-valuation of G such that for all $uv \in E(G)$, $p \nmid |f(u) - f(v)|_{pk}$.

Corollary 3.6 *Let G be a graph with $n(G) = p$ and take f to be a closed p-modular $\mathbb{Z}_{pk}$-valuation of G with $S = f^*(E(G))$. Then $H = C_{pk}(S)$ admits a fully automorphic G-decomposition.*

Proof. By Theorem 3.1, if f is a closed p-modular $\mathbb{Z}_{pk}$-valuation, it will induce a cyclic automorphic G-decomposition $\mathcal{D}$ of $H = C_{pk}(S)$. Further, $\chi(H) \leq p$ by Lemma 3.5. Since $\mathcal{D}$ is automorphic, we have that $\chi(H) \geq p$ by Theorem 1.1. Thus, it follows that $\chi(H) = p$. ■

Previously we have used the chromatic number to derive regularity results. However, we can use automorphic decompositions to determine the chromatic number. While this does not give the chromatic number of all circulants, it does give the chromatic number of many circulants where the colorability was unknown.

We now give a method for constructing closed p-modular valuations on K_p.

Theorem 3.7 *The complete graph, K_p, has a closed p-modular $\mathbb{Z}_{pk}$-valuation for sufficiently large k.*

Proof. It suffices to construct the required valuation. Let $V(K_p) = \{v_0, ..., v_{p-1}\}$ with $f(v_i) = a_i$. Suppose that $a_i < a_j$ for $i < j$. Further, assume that $a_i \equiv i \pmod{p}$. Note that if $a_{i+1} > 2a_i$ for all i, then the differences will all be distinct, provided the modulus is large enough. By choosing $pk > 2m$ where m is the largest edge label induced by f, we obtain the required order. Since all differences are distinct and each vertex label is from a different congruence class modulo p, we have constructed the required p-modular $\mathbb{Z}_{pk}$-valuation. Since any valuation on K_p is trivially closed, this is a closed p-modular $\mathbb{Z}_{pk}$-valuation. ■

Using Theorem 3.7 and Corollary 3.6, the following proposition is immediate.

Proposition 3.8 *For suitably chosen S and sufficiently large k, $C_{pk}(S)$ has a fully automorphic K_p-decomposition.*

We can use the notion of graceful labelings to give additional examples of fully automorphic divisors.

Theorem 3.9 *Let G be a connected graph of order p, and let f a graceful labeling on G. It follows that f is a closed p-modular valuation on G iff G is a tree.*

Proof. If G is not a tree, then $e(G) \geq n(G)$. Since f is a graceful labeling on G, then the edge labels induced by f are $\{1, 2, ..., e(G)\}$. Since $e(G) \geq n(G)$, it follows that there is an edge labeled $n(G) = p$, contrary to the definition of p-modular valuation.

If G is a tree, then $e(G) = p - 1$. Since f is a graceful labeling, no edge is labeled with an element larger than $p - 1$. It follows that no edge is labeled with a multiple of p. By definition, f is a closed $\mathbb{Z}_{2p-1}$-valuation on G. Thus, f is a closed p-modular $\mathbb{Z}_{2p-1}$-valuation on G. ∎

The following proposition is an immediate consequence of Theorem 3.9 and Corollary 3.6.

Proposition 3.10 *If G is a graceful tree of order p, then $C_{pk}(1, ..., p-1)$ has a fully automorphic G-decomposition for $k \geq 1$.*

4 Open Questions

There are three main questions related to this study that warrant further exploration. First, under what conditions is a graph a fully automorphic divisor? Similarly, what graphs obtain a closed p-modular valuation? Finally, is every fully automorphic host regular of even degree?

Acknowledgements The author would like to thank the anonymous referee for their comments regarding the final revision of this paper.

References

[1] Robert A. Beeler and Robert E. Jamison. Valuations, rulers, and cyclic decompositions of graphs. *Congr. Numer.* 183 (2006), 109–127.

[2] Robert A. Beeler and Robert E. Jamison. Automorphic decompositions of graphs. *Graphs Combin.* 27 (2011), no. 2, 149–160.

[3] Juraj Bosák. *Decompositions of Graphs*, volume 47 of *Mathematics and its Applications (East European Series)*. Kluwer Academic Publishers Group, Dordrecht, 1990. Translated from the Slovak with a preface by Štefan Znám.

[4] Joseph A. Gallian. A dynamic survey of graph labeling. *Electron. J. Combin.* 5 (2010) Dynamic Survey 6.

[5] Clemens Heuberger. On planarity and colorability of circulant graphs. *Discrete Math.* 268 (2003), no. 1-3, 153–169.

[6] Giuseppe Mazzuoccolo. The NP-completeness of automorphic colorings. *Discuss. Math. Graph Theory* 30 (2010), no. 4, 705-710.

[7] Mariusz Meszka, Roman Nedela, and Alexander Rosa. Circulants and the chromatic index of Steiner triple systems. *Math. Slovaca* 56 (2006), no. 4, 371–378.

[8] A. Rosa. On certain valuations of the vertices of a graph. In *Theory of Graphs (Internat. Sympos., Rome, 1966)*, pages 349–355. Gordon and Breach, New York, 1967.

[9] Douglas B. West. *Introduction to Graph Theory.* Prentice Hall Inc., Upper Saddle River, NJ, second edition, 2001.