On toughness and $[a, b]$-factors*

Qiuju Bian
Science School, Shandong University of Technology,
Zibo, 255049, P. R. China.

Abstract

In this paper, we consider a variation of toughness, and prove stronger results for the existence of $[a, b]$-factors. Furthermore, we show that the results are sharp in some sense.

Keywords: toughness; k-factor; $[a, b]$-factor

1 Introduction

The graphs considered here will be finite undirected graph without multiple edges and loops. Let G be a graph. We use $V(G)$ and $E(G)$ to denote its vertex set and edge set, respectively. For a vertex $x \in V(G)$, we write $N_G(x)$ for the set of vertices of $V(G)$ adjacent to x and use $\delta(G)$ for the minimum degree of G. For a subset S of $V(G)$, We use $G - S$ to denote the subgraph of G induced by $V(G) - S$. A vertex x is often identified with the set {x}.

Let g and f be two integer-valued functions defined on $V(G)$. A spanning subgraph F of G is called a (g, f)-factor if $g(x) \leq d_F(x) \leq f(x)$ holds for all $x \in V(G)$. A (g, f)-factor is called an $[a, b]$-factor or a k-factor if $g(x) = a$ and $f(x) = b$ or $f(x) = g(x) = k$ for all $x \in V(G)$.

Other terminologies and notations not defined here can be found in [1].

The notion of toughness was first introduced by Chvátal in [2]. If G is complete, define $t(G) = \infty$. If G is not complete,

$$t(G) = \min\{\frac{|S|}{\omega(G - S)} | S \subseteq V(G), \omega(G - S) \geq 2\}.$$

The following results are well known to us.

*This work was supported by NNSF of China under Grant No.10471078.

Lemma 1.1[3] *Let G be a graph. If $t(G) \geq k$, $|V(G)| \geq k+1$ and $k|V(G)|$ is even, then G has a k-factor.*

Lemma 1.2[7] *Let G be a graph. If $t(G) \geq a - 1 + \frac{a}{b}$, then G has an $[a, b]$-factor.*

In [4], a variation of toughness was introduced: If G is complete, define $\tau(G) = \infty$. If G is not complete,

$$\tau(G) = \min\{\frac{|S|}{\omega(G-S)-1} | S \subseteq V(G), \omega(G-S) \geq 2\}.$$

Obviously, $\tau(G) \geq t(G)$. The following results are the improvement of Lemma 1.1 with the order of G large enough.

Lemma 1.3[4] *Suppose $\tau(G) \geq 1$ and $|V(G)|$ is even. Then G has a 1-factor.*

Lemma 1.4[4] *Suppose $\tau(G) \geq 2$ and $|V(G)| \geq 3$. Then G has a 2-factor.*

Lemma 1.5[5] *Suppose $\tau(G) \geq k$, $k|V(G)|$ is even, and $|V(G)| \geq k^2 - 1$. Then G has a k-factor.*

We discuss the relationship between $\tau(G)$ and the existence of $[a, b]$-factors, and improve Lemma 1.2 with given assumption.

Theorem 1.6 *Let G be a connected graph with $|V(G)| \geq 3$ and $b > 1$. Then G has a $[1, b]$-factor if $\tau(G) > \frac{1}{b}$.*

Theorem 1.7 *Let G be a graph with $|V(G)| > ab - a$ and $2 \leq a < b$. If $\tau(G) > a - 1 + \frac{a}{b}$, then G has a $[a, b]$-factor.*

The proof of these two theorems will be given in next section and the following lemmas are needed.

Lemma 1.8[8] *A graph G has a (g, f)-factor if and only if for any two nonadjacent subsets $S, T \subseteq V(G)$,*

$$f(S) - g(T) + d_{G-S}(T) - h(U) \geq 0,$$

where $h(U)$ is the number of components C of $U = G - (S \cup T)$ such that $f(x) = g(x)$ for all $x \in V(C)$ and $f(V(C)) + e_G(V(C), T) \equiv 1 (\mod 2)$.

Lemma 1.9[6] *Let G be bipartite or $f(x) < g(x)$. Then G has a (g, f)-factor if and only if for any subset $S \subseteq V(G)$,*

$$f(S) - g(T) + d_{G-S}(T) \geq 0,$$

where $T = \{x | x \in V(G) - S, d_{G-S}(x) \leq g(x) - 1\}$.

The result in Theorem 1.6 is sharp. To see this, consider $G_1 = K_m \vee (mb+1)K_1$ where "$\vee$" means join and m is an arbitrary positive integer. It is easy to find out that $\tau(G) = \frac{m}{mb+1-1} = \frac{1}{b}$ and $b|S| - i(G-S) = bm - (bm+1) = -1 < 0$ if set $S = V(K_m)$. By Lemma 1.9 G_1 has no $[1,b]$-factor but $\tau(G) = \frac{1}{b}$.

To see Theorem 1.7 is sharp in some sense, we construct the following graph G_2. $V(G_2) = A \cup B \cup C$ whereA, B, C are disjoint with $|A| = |B| = (nb+1)(a-1)$ and $|C| = n(a-1)$. Both A and C are cliques in G_2, while B is isomorphic to $(nb+1)K_{a-1}$. Other edges in G_2 are a perfect matching between A and B and all the Pairs between B and C. Let $X = (A-u) \cup v \cup C$ where $u \in A$ and $v \in B$ is matched to u in G_2. Then $|X| = (nb+n+1)(a-1)$ and $\omega(G-X) - 1 = nb+1$. This follows that

$$\tau(G) = \frac{(nb+n+1)(a-1)}{nb+1} < a-1+\frac{a-1}{b}.$$

If we let $S = C$ and $T = B$, $b|S| - a|T| + d_{G-S}(T) = bn(a-1) - a(a-1)(nb+1) + (a-1)^2(nb+1) = -(a-1) < 0$. G_2 has no $[a,b]$-factor. It is easy to see $\tau(G)$ can be made arbitrarily close to $a-1+\frac{a-1}{b}$ when n is large enough.

Unfortunately, we doubt the condition $|V(G)| > ab - a$ is not sharp for $a > 1$.

2 Proof of Theorems

Proof of Theorem 1.6. Suppose that G has no $[1,b]$-factor. There exists a subset S of $V(G)$ such that

$$b|S| - i(G-S) \leq -1,$$

where $i(G-S)$ denoted by isolated vertices in $G-S$. Clearly, $S = \emptyset$. Otherwise, $i(G-\emptyset) = i(G) \geq 1$, contradicts the assumption. Thus $\omega(G-S) \geq i(G-S) \geq b|S|+1 \geq 2$. Therefore

$$\tau(G) \leq \frac{|S|}{\omega(G-S)-1} \leq \frac{|S|}{b|S|+1-1} = \frac{1}{b},$$

a contradiction. □

Proof of Theorem 1.7 Suppose, by the contrary, G has no $[a,b]$-factors. There exists S of $V(G)$ such that

$$\delta(S,T) = b|S| - a|T| + d_{G-S}(T) \leq -1, \quad (1)$$

where $T = \{x | x \in V(G) - S, d_{G-S}(x) \leq a-1\}$. Choose S satisfying (1) and make $|S|$ as large as possible. Let $U = V(G) - (S \cup T)$. Then

(i) $|N_G(y) \cap T| \leq a-1$ for any $y \in T$.

(ii) $|N_G(z)) \cap T| \leq b-1$ for any $z \in U$.

Since $0 \leq b|S \cup z| - a|T| + d_{G-(S\cup z)}(T) \leq \delta(S,T) + b - e_G(z,T)$, we have $e_G(z,T) \leq b-1$, that is $|N_G(z) \cap T| \leq b-1$ for any $z \in U$.

Let G^* be the graph obtained from G by joining each vertex of S to all the other vertices. Then $|V(G^*)| = |V(G)|$, $\tau(G^*) \geq \tau(G)$ and G^* has no $[a,b]$-factor. Obviously,

(iii) $N_{G^*}(x) = V(G^*) - x$ for any $x \in S$.

(iv) $b|S| - a|T| + d_{G^*-S}(T) - \omega(U) \leq -1$.

Set $G_0 = G^*, S_0 = S, T_0 = T$ and $U_0 = U$. We shall construct subgraphs G_i with $\tau(G_i) \geq a - 1 + \frac{a-1}{b}$ inductively. Define

$$\delta_i = b|S_i| - a|T_i| + d_{G^*-S}(T_i) - \omega(U_i),$$

and

$$\beta_i = \min_{y \in T_i} d_{G^*-S}(y).$$

(Later, we shall show that $T_i \neq \emptyset$.) Clearly, $\beta_i \leq a-1$. If $|V(G_i)| \leq a+1$, then stop. otherwise, choose $y_i \in T_i$ such that $d_{G^*-S}(y_i) = \beta_i$, and define $\gamma_i = d_{G_i - S_i}(y_i)$. Obviously, $\beta_i \geq \gamma_i$. If $|S_i| < a - 1 - \gamma_i$, then stop. Otherwise, choose any subset S_i' of S_i with $|S_i'| = a-1-\gamma_i$. Set $T_i' = N_{G^*}(y_i) \cap T_i$, $U_i' = N_{G^*}(y_i) \cap U_i$, $S_{i+1} = S_i - S_i'$, $T_{i+1} = T_i - (T_i' \cup y_i)$, $U_{i+1} = U_i - U_i'$ and $G_{i+1} = G_i - (S_i' \cup T_i' \cup y_i \cup U_i')$. Then $V(G_{i+1}) = S_{i+1} \cup T_{i+1} \cup U_{i+1}$ and $|S_i'| + |T_i'| + |U_i'| = a-1$.

Claim 1 $\tau(G_i) \geq a - 1 + \frac{a-1}{b}$.

If $\tau(G_i) < a - 1 + \frac{a-1}{b}$, then G_i is not complete and there exists a subset X of $V(G_i)$ such that $\omega(G_i - X) \geq 2$ and $|X| < (a-1+\frac{a-1}{b})(\omega(G_i - X) - 1)$. By (iii), $X \supseteq S_i$. Set

$$Y_i = \cup_{i=0}^{i-1}(S_i' \cup T_i' \cup U_i') \cup X.$$

Then $Y \supseteq S$. Thus $y_0, y_1, \cdots, y_{i-1}$ are isolated vertices in $G^* - Y$ and

$$\omega(G^* - Y) \geq i + \omega(G_i - X) \geq 2.$$

On the other hand,

$$\begin{aligned}
|Y| &= i(a-1) + |X| \\
&< i(a-1) + (a-1+\frac{a-1}{b})(\omega(G_i - X) - 1) \\
&\leq (a-1+\frac{a-1}{b})(\omega(G^* - Y) - 1) + i(a-1) - i(a-1+\frac{a-1}{b}) \\
&< (a-1+\frac{a-1}{b})(\omega(G^* - Y) - 1),
\end{aligned}$$

which contradicts to $\tau(G^*) \geq a-1+\frac{a-1}{b}$.

Claim 2 $\delta_{i+1} < \delta_i$.

$$\begin{aligned}
\delta_{i+1} &= b|S_{i+1}| - a|T_{i+1}| + d_{G^*-S}(T_{i+1}) - \omega(U_{i+1}) \\
&\leq b|S_i| - b|S_i'| - a|T_i| + a|T_i' \cup y_i| + d_{G^*-S}(T_i) - d_{G^*-S}(T_i' \cup y_i) \\
&\quad -\omega(U_i) + |U_i'| \\
&\leq \delta_i - b(a-\gamma_i) + (a-\beta_i)(|T_i'|+1)(\gamma_i - |T_i'|) \\
&= \delta_i - (a-\beta_i-1)(b-|T_i'|) - (b+1)(\beta_i-\gamma_i) - (b-a) \\
&< \delta_i,
\end{aligned}$$

since $\omega(U_{i+1}) \geq \omega(U_i) - |U_i'|$. Moreover, if $\beta_i > \gamma_i$, then $\delta_{i+1} < \delta_i - (b+1)$. If $\delta_{i+1} = \delta_i - (b-a)$, then $\beta_i = \gamma_i = a-1$. Note that Claim 2 implies that $\delta_i < \delta_0 \leq -1$ and $\delta_i \leq -2$ for $i \geq 1$.

Claim 3 $T_i \neq \emptyset$.

Suppose $T_i = \emptyset$, then $\omega(U_i) = \omega(G_i - S_i) \geq 2$. Otherwise, $\omega(U_i) \leq 1$. For $i \geq 1$, $-2 \geq \delta_i = b|S_i| - \omega(U_i) \geq -1$, a contradiction. Since $-1 \geq \delta_0$ and $T_0 = \emptyset$, then $S_0 \neq \emptyset$. Thus $-1 \geq \delta_0 = b|S_0| - \omega(U_0) \geq b-1 > 0$, a contradiction too. Therefore $|S_i| \geq (a-1+\frac{a-1}{b})(\omega(U_i)-1)$ by Claim 1. Hence

$$\begin{aligned}
\delta_i &= b|S_i| - \omega(U_i) \\
&\geq b(a-1+\frac{a-1}{b})(\omega(U_i)-1) - \omega(U_i) \\
&= (ba-b+a-2)\omega(U_i) - (ba-b+a-1) \\
&\geq (b+1)(a-1) - 2 > 0,
\end{aligned}$$

which is a contradiction.

For some m, either $|V(G_m)| \leq a+1$ or $|S_m| < a-1-\gamma_m$.

Suppose $|S_m| < a-1-\gamma_m$ and let $X = N_{G_m}(y_m)$. Then y_m is an isolated vertex in $G_m - X$. Since $\tau(G_m) \geq a-1+\frac{a-1}{b}$, $|X| = |S_m| + \gamma_m < a-1$ and $X = V(G_m) - y_m$. This means that $|V(G_m)| = |X|+1 < a$. Then we consider $|V(G_m)| \leq a+1$ in the following.

Claim 4 G_m is complete.

Suppose G_m is not complete. Then there exists a subset $Y \subset V(G_m)$ such that $\omega(G_m - Y) \geq 2$. Hence $|Y| \leq |G_m| - \omega(G_m - Y) \leq a-1$. Therefore $a-1+\frac{a-1}{b} \leq \tau(G_m) \leq \frac{|Y|}{\omega(G_m-Y)-1} \leq a-1$. This is impossible.

Claim 5 $|V(G_m)| \neq a+1$.

If $|V(G_m)| = a+1$, then $m \neq 0$ and G_m is an a-factor itself.

$$\begin{aligned} -2 &\geq \delta_m = b|S_m| - a|T_m| + d_{G^*-S}(T_m) - \omega(U_m) \\ &\geq a|S_m| - a|T_m| + d_{G_m-S_m}(T_m) - h(U_m) + (b-a)|S_m| \\ &\quad + h(U_m) - \omega(U_m) \\ &\geq 0 + h(U_m) - \omega(U_m) \\ &\geq -1, \end{aligned}$$

since $\omega(U_m) \leq 1$. This is a contradiction.

Then $|V(G_m)| \leq a$, let $\beta = \beta_m$.

Case 1 $\beta \leq a-2$.

By the proof of Claim 2,

$$\begin{aligned} \delta_{i+1} &\leq \delta_i - (a-\beta_i-1)(b-|T_i'|) - (b-a) \\ &\leq \delta_i - (a-\beta_i-1)(b-\beta_i) - (b-a) \\ &\leq \delta_i - (a-\beta-1)(b-\beta) - (b-a), \end{aligned}$$

for $0 \leq i \leq m-1$. Hence

$$\delta_m < \delta_0 - m(a-\beta-1)(b-\beta).$$

On the other hand,

$$\begin{aligned} \delta_m &= b|S_m| - a|T_m| + d_{G^*-S}(T_m) - \omega(U_m) \\ &\geq (\beta-a)|T_m| - |U_m| \\ &\geq (\beta-a)(\beta+1-|U_m|) - |U_m| \\ &= (\beta-a)(\beta+1) - (\beta-a+1)|U_m| \\ &\geq (\beta-a)(\beta+1). \end{aligned}$$

Hence

$$m \leq \frac{\delta_0 + (\beta-a)(\beta+1)}{(a-\beta-1)(b-\beta)} < \frac{\beta+1}{a-\beta-1} \leq a-1.$$

Therefore,

$$|V(G^*)| = ma + |V(G_m)| \leq a^2 - a < ab - a.$$

Contradicts the assumption.

Case 2 $\beta = a-1$.

Then $|V(G_m)| \geq \beta+1 = a$. This is possible only if $S_m = \emptyset$ and $|V(G_m)| = a$, and then

$$\delta_m \geq (\beta-a)|T_m| - |U_m| = -(|T_m| + |U_m|) = -a.$$

By Claim 2, $m \le \delta_0 - \delta_m \le -1 + a$. Hence

$$|V(G^*)| = ma + |V(G_m)| \le a^2 \le ab - a.$$

Contradicts the assumption too. This completes the proof of Theorem. □

References

[1] J.A.Bondy and U.S.R.Murty, Graph Theory with Applications. Macmillan Press Ltd, New York, 1976.

[2] V.Chvátal, Tough graphs and hamiltonian circuits, Discrete Math. 1973, 5: 215- 228.

[3] H.Enomoto, B.Jackson, P.Katerinis and A.Satio, Toughness and the existence of k-factors, J. Graph Theory, 1985, 9: 87-95.

[4] H.Enomoto, Toughness and the existence of k-factors. III, Discrete Math., 1998, 189: 227-282.

[5] H.Enomoto, M.Hagita, Toughness and the existence of k-factors. IV, Discrete Math., 2000, 216: 111-120.

[6] K.Heinrich, P. Hell, D.Kirkpatrick and G.Liu, A simple existence criterion for (g, f)-factors, Discrete Math., 1990, 85: 313-317.

[7] P.Katerinis, Toughness of graphs and the existence of factors, Discrete Math., 1990, 80: 81-92.

[8] L.Lovász, Subgraph with prescribed valencies, J Combin Theory, 1970, 8: 391-416.