A STAIRCASE ILLUMINATOR FOR SIMPLY CONNECTED ORTHOGONAL POLYGONS

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ABSTRACT. Let S be an orthogonal polygon in the plane bounded by a simple closed curve. Assume that every two boundary points of S have a common staircase illuminator whose edges are north and east. Then S contains a staircase path μ_0 whose edges are north and east such that μ_0 illumines every point of S. Without the requirement that the illuminators share a common direction, the result fails.

1. INTRODUCTION.

We begin with some definitions from [5] and [6]. A set B in the plane is called a *box* if and only if B is a convex polygon whose edges are parallel to the coordinate axes. A set S in the plane is an *orthogonal polygon* if and only if S is a connected union of finitely many boxes. Point q of S is a *point of local nonconvexity (lnc point)* of S if and only if, for every neighborhood N of $q, N \cap S$ fails to be convex. An edge e of S is a *dent edge* if and only if both endpoints of e are lnc points of $S \cap H$, for H an appropriate closed halfplane determined by the line of e.

Let λ be a simple polygonal path in $\mathbb{R}^2$ whose edges $[v_{i-1}, v_i], 1 \leq i \leq n$, are parallel to the coordinate axes. For points x, y in S, the path λ is called an *orthogonal $x-y$ path in S* or simply an *$x-y$ path in S* if and only if λ lies in S and contains the points x and y. In this case $\lambda(x,y)$ will represent the subpath of λ from x to y (ordered from x to y). The path λ is an *$x-y$ geodesic* if and only if λ is an $x-y$ path of minimal Euclidean length in S. (Clearly an $x-y$ geodesic need not be unique.) The path λ is called a *staircase path* (or simply a *staircase*) if and only if the associated vectors alternate in direction. That is, for an appropriate labeling, for i odd the vectors $\overrightarrow{v_{i-1}v_i}$ have the same horizontal direction, and for i even the vectors $\overrightarrow{v_{i-1}v_i}$ have the same vertical direction. The edge $[v_{i-1}, v_i]$ will be called north, south, east, or west according to the direction of vector $\overrightarrow{v_{i-1}v_i}$.

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Similarly, we use the terms north, south, east, west, northeast, northwest, southeast, southwest to describe the relative position of points.

For points x and y in set S, we say x *sees* y (x is *visible* from y) *via staircase paths* if and only if there is a staircase path in S that contains both x and y. Point x *clearly sees* y (y is *clearly visible* from x) *via staircase paths* if and only if, for some neighborhood N of y, x sees each point of $N \cap S$ via staircase paths. A set S is *staircase convex (orthogonally convex)* if and only if, for every pair x, y in S, x sees y via staircase paths. Similarly, a set S is *starshaped via staircase paths (orthogonally starshaped)* if and only if, for some point p in S, p sees each point of S via staircase paths, and the set of all such points p is the *staircase kernel* of S, $Ker\, S$. For λ a staircase path with $S \cap \lambda$ connected and for x a point in S, λ is called a *staircase illuminator for* x if and only if x sees via staircase paths in S at least one point of λ. When this occurs, we say that λ *illumines* point x. Path λ is a *staircase illuminator for set* S if and only if λ illumines every point of S.

We will use a few standard terms from graph theory. For $F = \{B_1, \dots, B_n\}$ a finite collection of distinct sets, the *intersection graph* G of F has vertex set $b_1, \dots, b_n$. Furthermore, for $1 \le i < j \le n$, the points b_i, b_j determine an edge in G if and only if the corresponding sets B_i, B_j in F have a nonempty intersection. A graph G is a *tree* if and only if G is connected and acyclic. A sequence $v_1, \dots, v_k$ of vertices in G is a *walk* if and only if each consecutive pair v_i, v_{i+1} determines an edge of $G, 1 \le i \le k-1$. A walk $v_1, \dots, v_k$ is a *path* if and only if its points are distinct. Finally, for $B_1, \dots, B_n$ a collection of distinct boxes in $\mathbb{R}^2$, we say that their union is a *chain* of boxes (relative to our ordering) if and only if the intersection graph of $\{B_1, \dots, B_n\}$ is the path $b_1, \dots, b_n$ (where b_i represents the set B_i in the intersection graph, $1 \le i \le n$). That is, relative to our labeling, for $1 \le i < j \le k, B_i \cap B_j \neq \emptyset$ if and only if $j = i + 1$. As Victor Chepoi [8] has observed, every simply connected orthogonal polygon S may be represented as a union of boxes whose intersection graph is a tree. An appropriate decomposition into boxes occurs if we use a horizontal cut at each point of local nonconvexity of S.

Many results in convexity that involve the usual idea of visibility via straight line segments have analogues that use the notion of visibility via staircase paths. (See [3]-[7].) For example, the familiar Krasnosel'skii theorem [12] in the plane states that, for S nonempty and compact in $\mathbb{R}^2$, S is starshaped via segments if and only if every three points of S are visible (via segments in S) from a common point. In the staircase analogue [4], for S a simply connected orthogonal polygon in the plane, S is starshaped via staircase paths if and only if every two points of S are visible (via staircase paths in S) from a common point. Notice that, in the staircase version, the Helly number three is reduced to two. Moreover, in an interesting study concerning rectilinear spaces, Chepoi [8] has generalized the planar result to any finite union of boxes in $\mathbb{R}^d$ whose corresponding intersection graph is a tree.

A similar situation occurs in results involving clear visibility and lnc points. By [2, Theorem 1], a nonempty compact connected set S in $\mathbb{R}^2$ is starshaped via segments if and only if every three of its lnc points are clearly visible (via segments in S) from a common point. In a staircase analogue in [6], for S a simply connected orthogonal polygon in $\mathbb{R}^2$, S is starshaped via staircase paths if and only if every two points belonging to its dent edges are clearly visible (via staircase paths in S) from a common point. Again in [7, Lemma 3] (stated in Result B of this paper), the number two plays an important role, this time to guarantee that a subset of a simply connected orthogonal polygon S lie in a subset of S that is starshaped via staircase paths.

In this paper, we consider a variation of the starshaped set problem. Instead of showing that an orthogonal polygon S is starshaped, the idea is to show that S has a staircase illuminator. Some related results using segment visibility appear in a paper by Bezdek, Bezdek, and Bisztriczky [1]. Among their results is the following theorem: For S a smooth domain in $\mathbb{R}^2$, if every three points of S are illumined (via segments) by some translate in S of segment T, then S contains an illuminator that is a translate of T. A related result for staircase visibility appears in [3]: Let S be a simply connected orthogonal polygon in the plane, and let T be a horizontal (or vertical) segment such that $T' \cap S$ is connected for every translate T' of T. If every two points of S are illumined (via staircase paths) by a translate of T, then some translate of T illumines every point of S. Here we seek a corresponding result for staircase illuminators in S, independent of the translation requirement.

Throughout the paper, *cl* S and *bdry* S will denote the closure and the boundary, respectively, for set S. Readers may refer to Valentine [14], to Lay [13], to Danzer, Grünbaum, Klee [9], and to Eckhoff [10] for discussions concerning Helly-type theorems, visibility via segments, and starshaped sets. Readers may refer to Harary [11] for information on intersection graphs, trees, and other graph theoretic concepts.

2. The Results.

We will use the following theorems. The first is from [6, Theorem 4], the second from [7, Lemma 3].

Result A. Let S be a simply connected orthogonal polygon in the plane. If every (point of each) dent edge of S is clearly visible via staircase paths from point p in S, then p sees via staircase paths every point of S. That is, $p \in Ker\, S$.

Result B. Let S be a simply connected orthogonal polygon in the plane, and let $A \subseteq S$. If, for every pair a, b in A, a and b see a common point of S via staircase paths, then A lies in an orthogonal polygon in S that is starshaped via staircase paths.

To begin, we establish a preliminary lemma. The following definition will be useful.

Definition 1. Let S be an orthogonal polygon in the plane bounded by a simple closed curve (and hence simply connected). For each dent edge D of S, let E_D, E'_D denote the two distinct edges in $bdry\, S$ adjacent to D. Select a corresponding neighborhood N_D of D such that N_D is closed and convex, with $N_D \cap bdry\, S \subseteq D \cup E_D \cup E'_D$. Let $A = \cup\{N_D \cap bdry S : D$ a dent edge of $S\}$.

Lemma 1. Let S, A denote the sets in Definition 1. If point p in S sees each point of A via staircase paths in S, then $p \in KerS$.

Proof. We will show that every (point of each) dent edge of S is clearly visible via staircase paths in S from point p. Let D be any dent edge of S, with N_D the corresponding neighborhood in Definition 1, with $L \equiv L_D$ the associated line, and with L_1, L_2 the corresponding open halfplanes determined by L. For an appropriate labeling of L_1 and L_2, $E_D \cup E'_D \subseteq cl L_2$. Since p sees each point of A via staircase paths in $S, p \notin L_2$, so $p \in L \cup L_1$. If $p \in L$, then certainly each point of D is clearly visible from p via staircase paths in S. It remains to consider the case for $p \in L_1$.

Without loss of generality, assume that line L is vertical, with L_1 (and hence p) east of L. (See Figure 1.) Let D have endpoints d_0, d'_0, where d_0 is strictly north of d'_0. If p is strictly east of any point of D, the result is clear, so (again without loss of generality) assume that p is strictly northeast of d_0. If there exists a $p - d_0$ staircase meeting L only at d_0, again the result is clear. We assert that this must occur: Otherwise, every $p - d_0$ staircase $\lambda(p, d_0)$ meets L at some first point c_λ strictly north of d_0. Select staircase $\lambda_0(p, d_0)$ so that the associated $c_0 \equiv c_{\lambda_0}$ is as close to d_0 as possible. Observe that we may choose points c, d strictly east of c_0, d_0, respectively, with d south of c, so that c is on the last segment of $\lambda_0(p, c_0)$ and d is in $N_D \cap S$. For any such pair $c, d, [c, d] \nsubseteq S$. In turn, this implies the existence of a dent edge $D_2 \subseteq [c_0, d_0]$, where the two edges adjacent to D_2 lie in $L \cup L_1$. However, for the south vertex c'_0 of D_2, there exist points of A near c'_0 that are not visible from p via staircase paths in S. Of course, this contradicts our hypothesis and establishes our assertion. That is, set S contains a $p - d_0$ staircase path that meets L only at d_0.

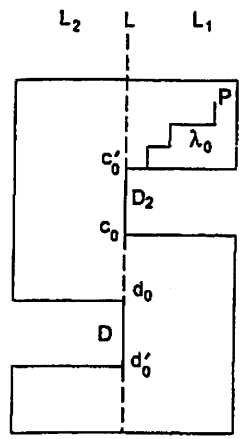

Figure 1.

Therefore, point p clearly sees each point of D via staircase paths in S. Since this is true for every dent edge D, we may use Result A to conclude that p sees via staircase paths in S every point of S. That is, $p \in KerS$, finishing the proof.

Corollary 1. Let S, A denote the sets in Definition 1. If every two points of A see a common point of S via staircase paths in S, then S is starshaped via staircase paths.

Proof. By Result B, A lies in an orthogonal polygon in S that is starshaped via staircase paths. For p in the associated kernel, p sees each point of A via staircase paths in S. Hence by Lemma 1 above, $p \in KerS$, and S is starshaped via staircase paths.

We will use this result in the following form:

Corollary 1$'$. Let S, A denote the sets in Definition 1. If S is not starshaped via staircase paths, then there exists a pair of points a, a' in A such that a and a' see no common point of S via staircase paths in S. Certainly a and a' are associated with two distinct dent edges of S.

We are ready to establish our main result.

Theorem 1. Let S, A denote the sets in Definition 1, and let d denote one of the two permissible directions for staircase paths, either north and east (south and west) or north and west (south and east). If every two points of A have a common staircase illuminator in the direction d, then S

contains a staircase path μ_0 in the direction d such that μ_0 illumines every point of S.

Proof. For convenience, assume that the direction d is north and east (south and west). To begin, orient the boundary of S in a clockwise direction. In an obvious way, this orientation assigns to each edge of *bdry* S a unique direction north, south, east, or west.

Assume that S is not staircase starshaped, for otherwise the result is trivial. Hence by Corollary $1', A$ contains pairs of points a, a' that see no common point of S via staircase paths in S. Further, every such a, a' are associated with two distinct dent edges in S. For every such pair a, a', select a corresponding northeast illuminator λ for a and a'. (That is, each edge of λ is north or east.) We assume that λ is minimal in the sense that no proper subset of λ is a staircase illuminator for a and a'. Moreover, since S is an orthogonal polygon, we may select λ so that, from all such illuminators, λ has minimal length, say $m(a, a')$. In this way, for each such pair a, a', we associate a positive number $m(a, a')$. The set of all the selected numbers $m(a, a')$ is finite and contains a largest member, say m_0. For m_0, select a corresponding a_0, b_0 in A and an associated shortest northeast illuminator λ_0 for a_0, b_0 whose length is m_0.

Relative to our orientation along *bdry* S, the two dent edges associated with a_0 and b_0 either have the same direction, opposing directions, or consecutive directions (one east or west, one north or south). Because the arguments for these various possibilities are quite similar, we restrict our attention to the case in which the dent edges have opposing directions. For convenience, assume that one dent edge, say n, is north while the other, say s, is south. Assume also that the line of n is east of the line of s and that λ_0, oriented from southwest to northeast, begins at a point southwest of s (and of a_0) and ends at a point southeast of n (and of b_0). Then every minimal illuminator λ for a_0, b_0 also will begin southwest of s (and of a_0) and will end southeast of n (and of b_0). (See Figure 2.) Let T denote the union of all such minimal illuminators for a_0, b_0.

We make some preliminary observations about T. It is easy to see that T is a staircase convex orthogonal polygon. Further, T is bounded on the west by an edge of T. The south endpoint v of this edge is a point of T furthest west as well as a point of T furthest south. Similarly, T is bounded on the east by an edge of T. The north endpoint w of this edge is a point of T furthest east as well as a point of T furthest north. Finally, these observations are independent of the particular orientation of the dent edges associated with a_0 and b_0. They follow from the minimality of the paths λ used to define T.

Next, using horizontal cuts (or using vertical cuts) at every point of local nonconvexity of S, in an obvious way we subdivide S into boxes whose corresponding intersection graph is a tree. Let $\mathcal{B}$ denote this collection of boxes. By comments in [5, Lemma 1], for s, t in S and for $\delta = \delta(s,t)$ any $s-t$ geodesic in S, there is a unique shortest chain of boxes in $\mathcal{B}$ containing δ. Thus any point s of S not in T lies in a (unique) shortest chain C_s of boxes in $\mathcal{B}$ from T to s, beginning in a box that intersects T and ending in a box that contains s. Each point u in such a chain we reach from T via a geodesic δ_u joining u to a closest point of T. The path δ_u leaves T along an edge of δ_u that is north, south, east, or west. Moreover, a minimal chain of boxes C_u containing δ_u will be a subchain of C_s. For future reference, let δ_{a_0} denote a geodesic from a_0 to a closest point of T, ordered from T to a_0, and let C_{a_0} represent the corresponding shortest chain of boxes in $\mathcal{B}$ containing $\delta(a_0)$. Similarly, define $\delta(b_0)$ and C_{b_0}.

We join to T certain members of $\mathcal{B}$ to obtain a new set S': Recall that T is staircase convex. Join to T all boxes of $\mathcal{B}$ that meet T. Certainly in the resulting union T_1, all points of T_1 see some point of T via staircase paths in T_1. Next, join to T_1 all boxes of $\mathcal{B}$ that meet T_1 such that, in the resulting union T_2, all points of T_2 see some point of T via staircase paths in T_2. By an obvious induction, in finitely many steps we obtain a subset S' of S such that S' is the union of T with at least some of the boxes in our collection $\mathcal{B}$, each point of S' sees some point of T via staircase paths in S', and S' is maximal. By our construction, both a_0 and b_0 are in S'.

We assert that $S' = S$. Suppose on the contrary that some box P of $\mathcal{B}$ intersects S' but cannot be joined to S' by the scheme described above. Then for some point p in P, no orthogonal path in S (in $S' \cup P$) from T to p is a staircase path. That is, any geodesic $\delta_p \equiv \delta$ in $S' \cup P$ from T to p requires two incompatible directions, either both north and south or both east and west. Thus the addition of P to S' introduces a dent edge of S as a dent edge of $S' \cup P$. We will consider various cases below. Without loss of generality we may assume that $p \in A$, that δ connects p to a closest point of T, and that δ is ordered from T to p. Observe that δ cannot pass through C_{b_0} or even a subchain of C_{b_0} for, with such a configuration, points p and a_0 would require a corresponding staircase illuminator of length greater than the length of λ_0. By a parallel argument, δ cannot pass through C_{a_0} or a subchain of C_{a_0}.

Case 1. Suppose that the first edge of δ is north. (See Figure 2.) There are four possibilities for δ: Edges of δ may alternate north and east, then end in a south edge at p. Edges of δ may alternate north and west, then end in a south edge at p. Edges may alternate north and west, then end in an east edge at p. Or edges may alternate north and east, then end in a west edge at p. However, any of these four configurations would make it impossible for p and b_0 to share an appropriate northeast illuminator in S. Thus Case 1 cannot occur.

Case 2. If the first edge of δ is south, a similar argument shows that it is impossible for p and a_0 to share an appropriate northeast illuminator in S. Hence Case 2 cannot occur.

Case 3. If the first edge of δ is east, then it is impossible for p and b_0 to share an appropriate illuminator. Again, this case cannot occur.

Case 4. Assume that the first edge of δ is west. (Again see Figure 2.) As above, there are four possibilities for δ. If edges of δ alternate west and north, then end in a south edge at p, the points p and b_0 cannot share an appropriate northeast illuminator in S.

If edges of δ alternate west and south, then end in a north edge at p, any minimal northeast illuminator for p and b_0 has length strictly greater than the length of λ_0. That is, $m(p, b_0) > m(a_0, b_0) \equiv m_0$, contradicting our choice of m_0 as maximal. Hence this configuration cannot occur.

If edges of δ alternate west and north, then end in an east edge at p, or if edges of δ alternate west and south, then end in an east edge at p, again $m(p, b_0) > m(a_0, b_0) \equiv m_0$, a contradiction. We conclude that Case 4 cannot occur.

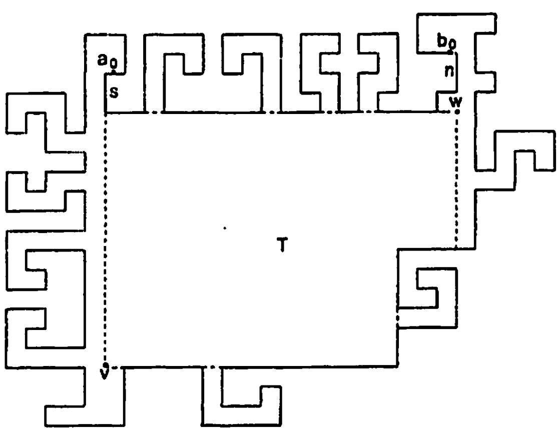

Figure 2.

Therefore, our supposition is false. Every box of $\mathcal{B}$ has been joined to T to produce S', and $S' = S$, the desired result. Thus every point of S sees some point of T via staircase paths in S.

Finally, we select a staircase illuminator μ_0 for $S, \mu_0 \subseteq T$. As noted in our preliminary observations, there is a unique point v of T that lies both furthest west and furthest south. Similarly, there is a point w of T furthest east and furthest north. Let μ_0 be any northeast staircase in T joining v to w. Since each point of S sees some point of T via staircase paths in S, it is easy to show that each point of S sees some point of μ_0 via staircase paths in S, and μ_0 satisfies the theorem, finishing the proof.

In conclusion, it is interesting to observe that we cannot weaken the hypothesis of Theorem 1 to require only that every two points of S have a common staircase illuminator. Consider the following example.

Example 1. Let S denote the orthogonal polygon in Figure 3. Every two points of S have a common staircase illuminator (either northeast or northwest). Notice that any staircase illuminator for points a and c is northeast, while any staircase illuminator for points b and c is northwest. However, points a, b, c have no common staircase illuminator in S.

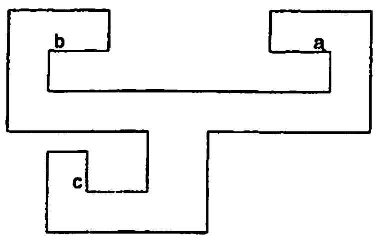

Figure 3

Of course, if we replace the number two by the number four, we have the following variation of Theorem 1.

Theorem 1′. Let S, A denote the sets in Definition 1. If every four points of S have a common staircase illuminator, then S contains a staircase path that illumines every point of S.

Proof. It is easy to see that the hypothesis of Theorem 1 is satisfied.

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