$\mathbb{Z}$-Cyclic Wh(28)

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Abstract

Whist tournament designs are known to exist for all $v \equiv 0, 1 \pmod 4$. Much less is known about the existence of $\mathbb{Z}$-cyclic whist designs. Previous studies [5, 6] have reported on all $\mathbb{Z}$-cyclic whist designs for $v \in \{4, 5, 8, 9, 12, 13, 16, 17, 20, 21, 24, 25\}$. This paper is a report on all $\mathbb{Z}$-cyclic whist tournament designs on 28 players, including a detailed summary of all known whist specializations related to a 28 player $\mathbb{Z}$-cyclic whist design. Our study shows that there are 7,910,127 $\mathbb{Z}$-cyclic whist designs on 28 players. Of these designs 2,568,510 possess the Three Person Property, 240,948 possess the Triplewhist Property and none possess the Balancedwhist Property. Introduced here is the concept of *the mirror image of a $\mathbb{Z}$-cyclic whist design*. In general, utilization of this concept reduces the computer search for $\mathbb{Z}$-cyclic whist designs by nearly fifty percent.

keywords: Whist Tournaments, $\mathbb{Z}$-Cyclic Designs, Triplewhist Designs, Three Person Whist Designs, Balanced Whist Designs.

1 Introduction

Definition 1.1 *A* **whist tournament** *on v players, denoted Wh(v), is a $(v, 4, 3)$ (near) resolvable BIBD. Each block, (a, b, c, d), of the BIBD is referred to as a whist game (alt. table) in which the partnership $\{a, c\}$ opposes the partnership $\{b, d\}$. The whist conditions require that every player* *is a partner of every other player exactly once and is an opponent of every other player exactly twice. The (near) resolution classes of the BIBD are called the* **rounds** *of the Wh(v).*

In 1897, E. H. Moore [13] introduced whist tournament designs into the mathematical literature. Moore's paper contains references to specific whist designs for $v = 4, 8, 12, 16, 20, 24, 28, 32, 36$ and provides formulae for Wh(4^n) where n is any positive integer and Wh($3p+1$) where p is any prime of the form $p \equiv 1 \pmod 4$. In the 1970s it was established that Wh(v) exist for all $v \equiv 0, 1 \pmod 4$ [1]. If $v = 4n$ then the Wh(v) consists of $4n-1$ rounds and every player plays in exactly one game of each round. If $v = 4n+1$, the Wh(v) consists of $4n+1$ rounds and every player plays in exactly one game in all but one round in which the player *sits out.* In this study we focus on the special case $v = 4n = 28$. In particular, the goal is to produce all whist designs on 28 players that possess the property of being $\mathbb{Z}$-cyclic.

1.1 $\mathbb{Z}$-Cyclic Whist Tournaments

Definition 1.2 *A whist design is said to be* **$\mathbb{Z}$-cyclic** *if the players are elements in $\mathbb{Z}_m \cup C$ where $m = v$, $C = \emptyset$ when $v \equiv 1 \pmod 4$ and $m = v-1$, $C = \{\infty\}$ when $v \equiv 0 \pmod 4$. It is also required that the rounds be cyclic. That is to say, the rounds can be labeled, $R_1, R_2, \ldots$, in such a way that R_{j+1} is obtained by adding $+1 \pmod m$ to every element in R_j. When ∞ is present one defines $\infty + 1 = \infty$.*

Since the collection of rounds of a $\mathbb{Z}$-cyclic Wh(v) forms a cyclic set it follows that the entire design is obtainable from any one of its rounds. This representative round is called the **initial round**. For $v = 4n$ it is conventional to take the round in which ∞ and 0 are partners as the initial round. For $v = 4n+1$ the initial round, conventionally, is that for which 0 sits out. These conventions will be adhered to in this study. Since the early 1990s a considerable amount of information has appeared that relates to the existence of $\mathbb{Z}$-cyclic Wh(v). In spite of this activity there are still many open questions regarding this type of whist design.

Example 1.1 Using the method of symmetric differences [1] one can verify that the following seven games form the initial round of a $\mathbb{Z}$-cyclic Wh(28).
$(\infty, 24, 0, 5)$, $(20, 19, 21, 10)$, $(23, 13, 25, 2)$, $(7, 1, 17, 14)$,
$(6, 18, 9, 11)$, $(8, 26, 12, 4)$, $(16, 3, 22, 15)$.

2 Preliminaries

Definition 2.1 *Let G be an abelian group of order $2k+1$, with k a positive integer. Let e_G denote the identity in G . The set of unordered pairs $S = \{(x_i, y_i) : x_i, y_i \in G, i = 1, 2, \ldots, k\}$ is said to be a* **starter** *in G if and only if the following are true: (1) $\bigcup_1^k \{x_i, y_i\} = G \setminus \{e_G\}$; and (2) $\bigcup_1^k \{\pm(y_i - x_i)\} = G \setminus \{e_G\}$.*

Since the order of appearance of an element within a set is unimportant, a starter is considered to be the same starter if the pairs are rearranged. Two starters S_1, S_2 in the same group G are considered to be different if there is at least one instance wherein $(a, b) \in S_1$, $(a, c) \in S_2$ with $b \neq c$. The next two theorems are well known and their proofs will be omitted.

Theorem 2.1 *Let G be an abelian group such that $|G| = 2k+1$. Then the set $PS = \{(x_i, -x_i) : x_i \neq e_G, i = 1, 2, \ldots, k\}$ with $\bigcup_1^k \{x_i, -x_i\} = G \setminus \{e_G\}$ is a starter in G.*

It is typical, in combinatorics literature, to refer to PS as the **patterned starter** in G.

Theorem 2.2 *Let G be an abelian group such that $|G| = 2k + 1$. Let $S = \{(x_i, y_i) : i = 1, 2, \ldots, k\}$ be a starter in G. Then the set $S^* = \{(-x_i, -y_i) : i = 1, 2, \ldots, k\}$ is also a starter in G.*

Corollary 2.3 *Let G and S be as in Theorem 2.2. If S is not the patterned starter in G then S and S^* are different starters.*

Proof: Since S is not the patterned starter there must be at least one (indeed more than one) pair in S, say (x_i, y_i) such that $y_i \neq -x_i$. Now $-x_i$ must belong to some (unordered) pair in S. Without loss of generality assume that $(x_j, y_j) \in S$ is such that $x_j = -x_i$. Note that $y_j \neq -y_i$ for otherwise $\{\pm(y_j - x_j)\} = \{\pm(y_i - x_i)\}$ which contradicts that S is a starter in G. Thus $(-x_i, -y_i) \in S^*$ and $(-x_i, y_j) \in S$ with $y_j \neq -y_i$. ∎

The starter S^* will be referred to as **the mirror image** of the starter S. Theorem 2.2 and its corollary provide a simple proof of the following theorem.

Theorem 2.4 *Let G be an abelian group such that $|G| = 2k + 1$. The number of different starters in G is odd.*

Remark 2.5 *Let G, S and S^* be as in Theorem 2.2. Take a circle of arbitrary radius and attach the elements of G to its circumference in such a way that x and $-x$ are at opposite ends of a horizontal chord of the* *circle. e_G can be placed at the center or at the "north (alt. south) pole" of the circle. After erasing the circle, consider the elements of G as the vertices of a complete graph $\mathcal{G}$ on $2k+1$ vertices. Any starter in G is then a near perfect matching for $\mathcal{G}$. Using the north-south diameter of the original circle as a mirror it is easy to see that the near perfect matching S^* is the mirror image of the near perfect matching S and vice-versa. Clearly the near perfect matching PS is its own mirror image. Several nice illustrations can be found at [3].*

Theorem 2.6 *Let $G = \mathbb{Z}_{2k+1}$ and let $S = \{(x_i, y_i) : i = 1, 2, \ldots, k\}$ be a starter in $\mathbb{Z}_{2k+1}$. If u is a unit in $\mathbb{Z}_{2k+1}$ then the set $S^u = \{(ux_i, uy_i) : i = 1, 2, \ldots, k\}$ is also a starter in $\mathbb{Z}_{2k+1}$.*

Proof: (1) Suppose that $\bigcup_1^k \{ux_i, uy_i\} \neq \mathbb{Z}_{2k+1} \setminus \{0\}$. Clearly, this latter inequality implies that for some pair i, j at least one of the following must be true: (i) $ux_i = ux_j$; (ii) $ux_i = uy_j$; (iii) $uy_i = ux_j$; (iv) $uy_i = uy_j$. In each case multiplication by u^{-1} leads to a contradiction that S is a starter in $\mathbb{Z}_{2k+1}$. Suppose, now, that $\bigcup_1^k \{\pm(uy_i - ux_i)\} \neq \mathbb{Z}_{2k+1} \setminus \{0\}$. Thus, it must follow that one or more of the following is true: (1) $uy_i - ux_i = 0$ for some i; (2) $uy_i - ux_i = uy_j - ux_j$ for some i, j; or (3) $uy_i - ux_i = -(uy_j - ux_j)$ for some i, j. In any of (1), (2), (3) multiplication by u^{-1} produces a contradiction to the fact that S is a starter in $\mathbb{Z}_{2k+1}$. ∎

Corollary 2.7 *Let $G = \mathbb{Z}_{2k+1}$ and let S be a starter in $\mathbb{Z}_{2k+1}$. If x is a unit in $\mathbb{Z}_{2k+1}$, $x \neq x^{-1}$ such that $(1, x) \in S$, then S and $S^{x^{-1}}$ are different starters in $\mathbb{Z}_{2k+1}$.*

Proof: The starter $S^{x^{-1}}$ contains the pair $(1, x^{-1})$. ∎

Remark 2.8 *Note that the set of partner pairs in a $\mathbb{Z}$-cyclic Wh(v), after removal of the pair $\{\infty, 0\}$ (if present), must be a starter in $\mathbb{Z}_m$ where m is as defined in Definition 1.2. Consequently, the basic approach of our attempt to find all $\mathbb{Z}$-cyclic Wh(v) for a specific v has been to find all possible (different) starters in $\mathbb{Z}_m$ and to determine which, if any, of these starters can be arranged into the initial round of a $\mathbb{Z}$-cyclic Wh(v). Of course this latter arrangement process must be exhaustive so as to allow for the possibility that a given starter could give rise to many different Wh(v). For such counting purposes it is important that one has criteria and/or conventions that enable one to establish that two given whist designs, both on v players, are either the same or different. Two conventions used here are the following: (1) within any round of a Wh(v) a change in the order in which the games are written does not produce a new whist tournament; (2) within any game of a Wh(v) an interchange of the N-S, E-W pairs does not produce a new whist tournament.*

It is also important to indicate that although an interchange of N-S, E-W pairs within one or more games of a Wh(v) does not produce a different whist tournament, such manipulations may very well uncover the fact that a given whist tournament possesses some additional feature (see, for example, the triplewhist materials below). The following theorem and its corollary are proven in [5].

Theorem 2.9 *Let $v \equiv 0, 1 \pmod 4$. Set $n = \lfloor v/4 \rfloor$. Let $\mathcal{IR}$ denote the n games of the initial round of a $\mathbb{Z}$-cyclic Wh(v). If x is a unit in the ring $\mathbb{Z}_m$ then $x\mathcal{IR}$ is the set of n games (xa_i, xb_i, xc_i, xd_i), $i = 1, \ldots, n$ where the (a_i, b_i, c_i, d_i) are the games in $\mathcal{IR}$ and the multiplication is taken modulo m. If, however, ∞ is involved, say $a_i = \infty$ then $xa_i = \infty$. The games $x\mathcal{IR}$ form the games in the initial round of a $\mathbb{Z}$-cyclic Wh(v). Furthermore, this latter $\mathbb{Z}$-cyclic Wh(v) possesses all specializations that the original $\mathbb{Z}$-cyclic Wh(v) possessed.*

Corollary 2.10 *Let $v \equiv 0, 1 \pmod 4$. Set $n = \lfloor v/4 \rfloor$. Let $\mathcal{IR}$ denote the n games of the initial round of a $\mathbb{Z}$-cyclic Wh(v). Let x be the partner of 1 in this initial round. If x is a unit in the ring $\mathbb{Z}_m$ such that $x \neq x^{-1}$ then $x^{-1}\mathcal{IR}$ is the initial round of a $\mathbb{Z}$-cyclic Wh(v) that is different from that of the original $\mathbb{Z}$-cyclic Wh(v).*

Corollary 2.11 *Let $v \equiv 0, 1 \pmod 4$. Set $n = \lfloor v/4 \rfloor$. Let $\mathcal{IR}$ denote the n games of the initial round of a $\mathbb{Z}$-cyclic Wh(v). Define $-\mathcal{IR}$ to be the set of n games $(-a_i, -b_i, -c_i, -d_i)$, $i = 1, \ldots, n$ where the (a_i, b_i, c_i, d_i) are the games in $\mathcal{IR}$ and the additive inverse is taken modulo m. If, however, ∞ is involved, say $a_i = \infty$ then $-a_i = \infty$. The games $-\mathcal{IR}$ form the games in the initial round of a $\mathbb{Z}$-cyclic Wh(v). If the starter associated with the partner pairs of the initial round of the original whist design is not the patterned starter then this latter whist design is different than the original whist design. Furthermore, this latter $\mathbb{Z}$-cyclic Wh(v) possesses all specializations that the original $\mathbb{Z}$-cyclic Wh(v) possessed.*

Proof: Set $x = -1$ in Theorem 2.9. If S denotes the starter associated with the partner pairs for the original initial round then the starter associated with the partner pairs of the new initial round is S^*. Since S is not the patterned starter these two starters are different via Corollary 2.3. Thus there exists a pair $(a, b) \in S$ and a pair $(a, c) \in S^*$ such that $b \neq c$. Consequently the game containing player a in the original initial round is different than the game containing a in the new initial round. ∎

Combining the materials of this section leads to the following theorem.

Theorem 2.12 *Let S be a starter in $\mathbb{Z}_{2k+1}$ such that S is not the patterned starter. If k is an odd integer, $k = 2n - 1$ then $2k + 1 = 4n - 1$, $X =$*

$\mathbb{Z}_{4n-1} \cup \{\infty\}$ *and if* k *is an even integer,* $k = 2n$ *then* $2k+1 = 4n+1$, $X = \mathbb{Z}_{4n+1}$. *(a)* S *can be arranged into the initial round of a* $\mathbb{Z}$*-cyclic whist design on* $v = |X|$ *players if and only if* S^* *can be arranged into the initial round of a* $\mathbb{Z}$*-cyclic whist design on* $v = |X|$ *players. (b) If* u *is a unit in* $\mathbb{Z}_{2k+1}$ *such that* $(1, u) \in S$ *then* S *can be arranged into the initial round of a* $\mathbb{Z}$*-cyclic whist design on* $v = |X|$ *players if and only if* $S^{u^{-1}}$ *can be arranged into the initial round of a* $\mathbb{Z}$*-cyclic whist design on* $v = |X|$ *players.*

Parts (a) and (b) of Theorem 2.12 provide separate strategies for our computer approach. In each case the interpretation is that given one solution a second solution is obtained with minimal effort.

Example 2.1 Let S denote the set of initial round partner pairs in the $\mathbb{Z}$-cyclic Wh(28) of Example 1.1. An application of Part (a) of Theorem 2.12 yields the initial round of a $\mathbb{Z}$-cyclic Wh(28) exhibited in (a) below. Setting $u = 14$ in Part (b) of Theorem 2.12 yields the initial round of a $\mathbb{Z}$-cyclic Wh(28) exhibited in (b). (a)

$(\infty, 3, 0, 22)$, $(7, 8, 6, 17)$, $(4, 14, 2, 25)$, $(20, 26, 10, 13)$,
$(21, 9, 18, 16)$, $(19, 1, 15, 23)$, $(11, 24, 5, 12)$.

(b)

$(\infty, 21, 0, 10)$, $(13, 11, 15, 20)$, $(19, 26, 23, 4)$, $(14, 2, 7, 1)$,
$(12, 9, 18, 22)$, $(16, 25, 24, 8)$, $(5, 6, 17, 3)$.

3 Whist Designs with Special Properties

From time to time over the past 115 years additional requirements have been imposed upon the structure of whist designs. For a complete listing of these specialized structures see [5]. Here we focus only on the structures that make sense when $v \equiv 0 \pmod 4$. Thus we mention whist designs that possess one or more of the following: (1) the triplewhist property; (2) the three person property; (3) the patterned starter property; (4) the balanced whist property. A complete analysis of the $\mathbb{Z}$-cyclic Wh(28)s that possess the patterned starter property (that is to say, the set of initial round partner pairs form the patterned starter) can be found in [8] and will not be discussed here. Our exhaustive computer search led to the conclusion that no $\mathbb{Z}$-cyclic Wh(28) possesses the balanced whist property (for the definition see [10]). Consequently our discussion of the balanced whist property is confined to the following theorem.

Theorem 3.1 *There does not exist a* $\mathbb{Z}$*-cyclic Wh(28) that possesses the balanced whist property.*

Note that Theorem 3.1 supplies an answer to a question posed in [10].

3.1 Triplewhist Designs

In a whist game (a, b, c, d) the opponent pairs $\{a, b\}$, $\{c, d\}$ are called **first kind opponents** and the opponent pairs $\{a, d\}$, $\{b, c\}$ are called **second kind opponents.**

Definition 3.1 *[13] A whist tournament on v players is said to be a* **triplewhist tournament,** *TWh(v), if every player opposes every other player exactly once as an opponent of the first kind (and, hence, exactly once as an opponent of the second kind).*

It is known that TWh(v) do not exist for $v \in \{4, 5, 9, 12, 13\}$ and do exist for all other $v \equiv 0, 1 \pmod 4$ except, possibly, $v = 17$ [2]. The Wh(28) given above in Example 1.1 is a triplewhist tournament (as are those of Example 2.1). We commented earlier that for counting purposes an interchange of partner positions does not produce a different whist design. On the other hand such transformations might show that a whist design possesses a particular property. For example, if, in Example 1.1, the game $(23, 13, 25, 2)$ had been given as $(25, 13, 23, 2)$ then the triplewhist property would not be evident.

3.2 Whist Designs Having The Three Person Property

Definition 3.2 *[7] A whist tournament on v players is said to be a* **three person whist tournament,** *3PWh(v), if the intersection of any two games is at most* 2.

Certainly the 3P Property is intrinsic to the design. Consequently a whist design either has this property or it does not. A difference criterion for a $\mathbb{Z}$-cyclic Wh(v) to have the 3P Property is presented in [9]. The TWh(28) presented in Example 1.1 possesses the Three Person Property.

4 A Summary of the Computer Search

Our computer search process was to find all possible starters for $\mathbb{Z}_{27}$ and then subject each starter to all meaningful permutations (see Remark 2.8) to see which, if any, of these permutations resulted in the initial round of a $\mathbb{Z}$-cyclic Wh(28). Theorem 2.12 enabled us to minimize the computer search. In previous studies [5, 6] Part (b) of Theorem 2.12 was used exclusively, however, the present opinion is that utilization of Part (a) is more efficient. In either case the methodology is to determine an exhaustive approach that fixes a (partner) pair in a starter, determine all starters that contain this fixed pair and determine whether or not the pairs of each such starter can be arranged to form the games of the initial round of a $\mathbb{Z}$-cyclic whist design. For Part (b) the exhaustive set of fixed pairs is given by $\{(1,x) : 2 \leq x \leq 26\}$ and for Part (a) the exhaustive set of fixed pairs is given by $\{(x,y) : \pm(y-x) = \{13,14\}\}$. The results of both approaches are presented in our detailed summary. Notice that for $\mathbb{Z}_{27}$ the approach for Part (a) involves fewer cases. Indeed, Part (a) reduces the number of cases to consider by nearly fifty percent. Once the totality of Wh(28) designs was obtained, they were permuted and analyzed for the presence of the triplewhist property and/or three person property. This latter process terminated as soon as the specialization was discovered or if the program ran to completion without success.

In the table below the starter data appears in the column **St**(28) and indicates that the line item corresponding to x represents the total number of starters in $\mathbb{Z}_{27}$ that contain the pair $(1,x)$. If x is a unit in $\mathbb{Z}_{27}$ and $x \neq x^{-1}$ then the corresponding line item is doubled when summing the column entries (see Corollary 2.10). It is to be noted that the totals reported here agree with those presented in [4] and on the website [3]. Our presentation, however, provides a finer analysis than that found in [4] or at [3]. The number of corresponding whist designs (i.e. the initial round of the Wh(v) contains a game in which 1 and x are partners) appear in the column **Wh**(28) and the numbers that appear in this column are to be treated as in the St(28) column. The column **3P** shows the number of Wh(28) that possess the three person property. These numbers are to be counted as in the other two columns.

x	x^{-1}	St(28)	Wh(28)	3P
2	14	4946702	309439	90701
3	DNE	5128885	315210	103203
4	7	5127697	315461	103716
5	11	5170611	321560	106041
6	DNE	5133987	315892	104402
8	17	5156963	320000	105413
9	DNE	5127691	314825	102663
10	19	5130387	315582	102534
12	DNE	5138363	317230	103945
13	25	5124153	312969	102994
15	DNE	5112367	313297	102327
16	22	5135903	317120	104020
18	DNE	5129851	316278	103325
20	23	5168467	321581	105684
21	DNE	5126265	315197	103353
24	DNE	5138871	314335	103310
26	26	5144579	320439	99776
Totals		128102625	7910127	2568510

In the following table the column **Span Pair** indicates an unordered pair that is fixed in the starter and the column **St**(28) indicates the totality of starters that contain that pair. The column **Mirror Pair** gives the mirror image of the span pair. The columns **Wh**(28) and **3P** have the same interpretation as in the table above. With the exception of the entries for the span pair 7 – 20 all column entries are to be doubled.

Span Pair	Mirror Pair	St(28)	Wh(28)	3P
1 – 14	13 – 26	4946702	309439	90701
2 – 15	12 – 25	5126265	315197	103353
3 – 16	11 – 24	5138363	317230	103945
4 – 17	10 – 23	5170611	321560	106041
5 – 18	9 – 22	5127691	314825	102663
6 – 19	8 – 21	5133987	315892	104402
7 – 20	7 – 20	5144579	320439	99776
1 – 15	12 – 26	5112367	313297	102327
2 – 16	11 – 25	5156963	320000	105413
3 – 17	10 – 24	5138871	314335	103310
4 – 18	9 – 23	5129851	316278	103325
5 – 19	8 – 22	5168467	321581	105684
6 – 20	7 – 21	5128885	315210	103203
Totals		128102625	7910127	2568510

The next two tables report on the number of $\mathbb{Z}$-cyclic Wh(28) that possess the triplewhist property and the number of TWh(28) that possess the three person property. The descriptions are as above.

Span Pair	Mirror Pair	TWh(28)	3P
1 – 14	13 – 26	9228	2824
2 – 15	12 – 25	9498	3064
3 – 16	11 – 24	9546	3161
4 – 17	10 – 23	9855	3302
5 – 18	9 – 22	9620	3074
6 – 19	8 – 21	9785	3262
7 – 20	7 – 20	9162	2852
1 – 15	12 – 26	9636	3148
2 – 16	11 – 25	9737	3153
3 – 17	10 – 24	9780	3236
4 – 18	9 – 23	9775	3290
5 – 19	8 – 22	9907	3280
6 – 20	7 – 21	9526	3209
Totals		240948	78858

x	x^{-1}	TWh(28)	3P
2	14	9228	2824
3	DNE	9526	3209
4	7	9737	3226
5	11	9855	3302
6	DNE	9785	3262
8	17	9737	3153
9	DNE	9620	3074
10	19	9704	3160
12	DNE	9546	3161
13	25	9467	3094
15	DNE	9636	3148
16	22	9675	3242
18	DNE	9775	3290
20	23	9907	3280
21	DNE	9498	3064
24	DNE	9780	3236
26	26	9162	2852
Totals		240948	78858

5 $\mathbb{Z}$-cyclic 3^9 Frames

Definition 5.1 *[11] A frame is a group divisible design, $GDD_\lambda(X, \mathcal{G}, \mathcal{B})$ such that (1) the size of each block is the same, say k, (2) the block set can be partitioned into a family $\mathcal{F}$ of partial resolution classes and (3) each $F_i \in \mathcal{F}$ can be associated with a group $G_j \in \mathcal{G}$ so that F_i contains every point in $X \setminus G_j$ exactly once.*

Frames are powerful tools that can be used to construct resolvable and near resolvable designs [11]. G. Ge and L. Zhu, in their excellent paper [12], provide theorems which demonstrate how frames can be utilized to construct $\mathbb{Z}$-cyclic whist designs. If the block size of a frame is $k = 4$ then the blocks are considered to be whist games. If the collection of blocks has the property that every pair of elements (players) from distinct groups appear together in exactly 3 blocks and within these 3 blocks they appear exactly once as partners then the frame is called a whist frame and is denoted by WhFrame. Note that the notation WhFrame(h^w) refers to a frame whose element set, X, contains hw elements, whose blocks are of size 4 and has w groups each of size h. Each partial resolution class is then called a round of the WhFrame. If the blocks of a WhFrame satisfy any additional conditions such as every pair of players from distinct groups meet exactly once as opponents of the first kind (and, hence, exactly once as opponents of the second kind) then the notation for the frame will reflect this property. Thus one speaks of TWhFrames, BWhFrames, etc. It is also possible to define a Z-cyclic WhFrame [12].

Definition 5.2 *Suppose $X = \mathbb{Z}_m, m = hw$ and $\mathbb{Z}_m$ has a subgroup H of order h. Suppose there is a WhFrame(h^w) that has a special round R_1, called the initial round of the frame, whose elements form a partition of $X \setminus H$ and is such that it, together with all the other rounds can be arranged in a cyclic order, say $R_1, R_2, \ldots$ so that R_{j+1} can be obtained by adding $+1$ modulo m to every element in R_j then the frame is said to be $\mathbb{Z}$-cyclic.*

It is sometimes possible to generate a ($\mathbb{Z}$-cyclic) WhFrame(h^w) directly from a ($\mathbb{Z}$-cyclic) Wh(v).

Example 5.1 Consider the $\mathbb{Z}$-cyclic Wh(28) whose initial round is given by the following seven games.

$$(\infty, 9, 0, 18), \quad (25, 20, 26, 22), \quad (8, 1, 11, 14), \quad (5, 3, 16, 15),$$
$$(19, 24, 23, 7), \quad (12, 4, 17, 10), \quad (6, 21, 13, 2).$$

If, in Definition 5.2, we set $X = \mathbb{Z}_{27}$, $H = \{0, 9, 18\}$, $h = 3$ and $w = 9$ then the initial round of a $\mathbb{Z}$-cyclic WhFrame(3^9) can be constructed from the above Wh(28) by removing the game $(\infty, 9, 0, 18)$.

This latter example clearly indicates that any $\mathbb{Z}$-cyclic Wh(28) whose initial round contains either the game $(\infty, 9, 0, 18)$ or the game $(\infty, 18, 0, 9)$ can be used to construct a $\mathbb{Z}$-cylic WhFrame(3^9). Any such $\mathbb{Z}$-cyclic Wh(28) will be referred to as a **frame-producing Wh**(28). On the other hand, adjoining the game $(\infty, 9, 0, 18)$ to the initial round of a $\mathbb{Z}$-cyclic WhFrame(3^9) produces the initial round of a $\mathbb{Z}$-cyclic (frame-producing) Wh(28). Trivially, then, we have the following theorem.

Theorem 5.1 *There exists a $\mathbb{Z}$-cyclic frame-producing Wh(28) if and only if there exists a $\mathbb{Z}$-cyclic WhFrame(3^9).*

It is worthwhile noting that although $\mathbb{Z}_{27}$ has a subgroup of order 9, namely $H = \{0, 3, 6, 9, 12, 15, 18, 21, 24\}$, it is impossible to construct a $\mathbb{Z}$-cyclic WhFrame(9^3). Theorem 5.1 together with the fact that we have constructed all $\mathbb{Z}$-cyclic Wh(28) designs enables us to determine all $\mathbb{Z}$-cyclic WhFrames(3^9) via a simple search procedure. The results are as follows.

x	x^{-1}	WhFr(3^9)
2	14	361
3	DNE	310
4	7	282
5	11	361
6	DNE	306
8	17	404
9	DNE	0
10	19	0
12	DNE	310
13	25	312
15	DNE	306
16	22	330
18	DNE	0
20	23	361
21	DNE	310
24	DNE	306
26	26	404
Totals		7404

Span Pair	Mirror Pair	WhFr(3^9)
1 – 14	13 – 26	361
2 – 15	12 – 25	310
3 – 16	11 – 24	310
4 – 17	10 – 23	361
5 – 18	9 – 22	0
6 – 19	8 – 21	306
7 – 20	7 – 20	404
1 – 15	12 – 26	306
2 – 16	11 – 25	404
3 – 17	10 – 24	306
4 – 18	9 – 23	0
5 – 19	8 – 22	361
6 – 20	7 – 21	310
Totals		7404

6 Orbits

It is well known that in the ring $\mathbb{Z}_n$, n a positive integer, the set of units is a multiplicative group with 1 as identity. For $\mathbb{Z}_{27}$ the group of units is a cyclic group, of order 18, with generators $2, 5, 11, 14, 20, 23$. We denote this group by $\mathcal{U}(27)$. Each $u \in \mathcal{U}(27)$ induces a mapping, Φ_u of $\mathbb{Z}_{27}$ onto itself defined by $\Phi_u(x) = y$ where $ux \equiv y \pmod{27}$. The set of all 7,910,127 $\mathbb{Z}$-cyclic Wh(28) will be denoted by $\mathcal{W}h(28)$. The action of $\mathcal{U}(27)$ on $\mathcal{W}h(28)$, via the maps Φ_u, induces a partition of $\mathcal{W}h(28)$. The cells of this partition will be called **orbits**. There are 439,453 such orbits. All but 3 of these orbits are of order 18. The remaining 3 orbits are of order 9. Illustrations of $\mathbb{Z}$-cyclic Wh(28) that have these orbit orders are given below.

Example 6.1 An orbit of order 9 is generated by a $\mathbb{Z}$-cyclic Wh(28) that is invariant (in accordance with Remark 2.8) under the action of the subgroup of order 2, $\{1, 26\}$. The initial round of a $\mathbb{Z}$-cyclic Wh(28) that generates an orbit of order 9 is given by the following 7 tables.

$(\infty, 6, 0, 21)$, $(1, 13, 26, 14)$, $(10, 12, 17, 15)$, $(4, 5, 23, 22)$,
$(2, 9, 25, 18)$, $(8, 11, 19, 16)$, $(3, 7, 24, 20)$.

Note that this Wh(28) is a $\mathbb{Z}$CPS-Wh(28).

Example 6.2 Any orbit of order 18 is generated by a $\mathbb{Z}$-cyclic Wh(28) that is not invariant under the action of any subgroup of $\mathcal{U}(28)$. The initial round of such a design is given by the following 7 tables.
$(\infty, 2, 0, 13)$, $(1, 22, 14, 23)$, $(5, 8, 7, 17)$, $(6, 20, 16, 26)$,
$(4, 9, 11, 12)$, $(10, 21, 18, 25)$, $(3, 19, 15, 24)$.

Of course, the triplewhist property and the three person property are invariant under the maps Φ_u. It is also the case that the sub-group $\{0, 9, 18\}$ is invariant under these maps thus the orbit generated by any frame producing Wh(28) consists entirely of frame producing Wh(28). It so happens that every Triplewhist orbit is of order 18 and every orbit containing frame producing Wh(28)s is of order 18. Consequently there are 13,386 Triplewhist orbits and 393 orbits containing frame producing Wh(28) designs.

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