A toughness condition for the existence of f-factors in graphs *

Jiansheng Cai †
School of Mathematics and information Sciences, Weifang University, Weifang 261061, P. R. China

Abstract

Let G be a graph and let f be a positive integer-valued function defined on $V(G)$ such that $1 \leq a \leq f(x) \leq b \leq 2a$ for every $x \in V(G)$. If $t(G) \geq \frac{b^2}{a}$, $|V(G)| \geq \frac{b^2}{a} + 1$ and $f(V(G))$ is even, then G has a f-factor.

Key words: Toughness condition; f-factor; k-factor
AMS(2000) subject classification: 05C70

1 Introduction

The graphs considered in this paper will be simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Denote by $d_G(x)$ the degree of a vertex x in G. Let g and f be two integer-valued functions defined on $V(G)$ such that $0 \leq g(x) \leq f(x)$ for all $x \in V(G)$. Then a (g, f)-factor of G is a spanning subgraph F of G satisfying $g(x) \leq d_F(x) \leq f(x)$ for all $x \in V(G)$. If $g(x) = f(x)$ for all $x \in V(G)$, then a (g, f)-factor is called an f-factor. Let a and b be two integers such that $0 \leq a \leq b$. If $g(x) = a$ and $f(x) = b$ for all $x \in V(G)$, then a (g, f)-factor is called an $[a, b]$-factor. If $a = b = k$, then an $[a, b]$-factor is called a k-factor. Denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degree of a vertex in G, respectively. For $A \subseteq V(G)$, denote by $N_G(A)$ the set of neighbors in G of vertices in A. If A and B are disjoint subsets of $V(G)$, then $e_G(A, B)$ denotes the number of edges that join a vertex in A and a vertex in B.

*This work is supported by natural science foundation of shandong province(No.ZR2009AM009), NSFC(No.11001055)

†The corresponding author: Jiansheng Cai, mailaddress: School of Mathematics and information science, Weifang University, weifang, 261061, P.R.China. E-mail: health-cai@163.com

If $A = \{x\}$, then $e_G(x, B)$ denotes the number of edges that join x and a vertex in B. The number of connected components of G is denoted by $\omega(G)$. Let $S, T \subseteq V(G)$ and $S \bigcap T = \emptyset$. If C is a component of $G - (S \bigcup T)$ such that $\sum_{x \in V(C)} f(x) + e_G(C, T) \equiv 1 \pmod 2$, then we say that C is an odd component of $G - (S \bigcup T)$ and we denote by $h(S, T)$ the number of odd components of $G - (S \bigcup T)$. For a subset S of $V(G)$, we denote by $G - S$ the subgraph obtained from G by deleting the vertices in S together with edges incident with vertices in S. In the following we write $f(W) = \sum_{x \in W} f(x)$ and $f(\emptyset) = 0$ for any $W \subseteq V(G)$. In particular, we set $d_{G-S}(T) = \sum_{x \in T} d_{G-S}(x)$ for $S, T \subseteq V(G)$ and $S \bigcap T = \emptyset$. For any $x \in V(G)$, we set $N_G(x)$ denote the neighborhood of x, and denoted by $N_G(A)$ the neighbors union of $x \in A$. We also set $N_G[A] = N_G(A) \bigcup A$.

The notion of *toughness* was introduced by Chvátal[2]:

$$\begin{aligned} t(G) &= max\{t \mid |S| \geq t \cdot \omega(G - S) \ \ if \ \ \omega(G - S) \geq 2\} \\ &= min\{\frac{|S|}{\omega(G - S)} \mid \omega(G - S) \geq 2\} \end{aligned}$$

if G is complete, and $t(G) = \infty$ if G is complete.

Notations and definitions not given here can be found in [1].

Many authors have investigated *(g, f)-factors* and *f-factors* [4,5,6]. There is a well-known necessary and sufficient condition for a graph G to have an *f-factor* which was given by Tutte.

Theorem A. [7] *(1) A graph G has an f-factor if and only if*

$$\delta(S, T) = f(S) + d_{G-S}(T) - f(T) - h(S, T) \geq 0$$

for any disjoint subsets S and T of $V(G)$, where $h(S, T)$ denotes the number of odd components C of $G - (S \bigcup T)$.

(2) $\delta(S, T) \equiv f(V(G)) \pmod 2$.

Hikoe Enomoto et.al investigated the relationship between *toughness* and existence of factors[3,4], gave the following well known result.

Theorem B. [3] *Suppose $|V(G)| \geq k + 1$, $k \mid V(G) \mid$ even, and $t(G) \geq k$. Then G has a k-factor.*

In this paper, we generalized the above Theorem and obtain the following result.

Theorem 1 *Let G be a graph and let f be a positive integer-valued function defined on $V(G)$ such that $1 \leq a \leq f(x) \leq b \leq 2a$ for every $x \in V(G)$. If $t(G) \geq \frac{b^2}{a}$, $|V(G)| \geq \frac{b^2}{a} + 1$ and $f(V(G))$ is even, then G has a f-factor.*

2 proof of Theorem 1

Let G be a graph satisfying the hypothesis of Theorem 1, we prove the theorem by contradiction. Suppose that G has no f-factors. Then $\delta(S,T) < 0$ for some disjoint subsets S and T of $V(G)$ by Theorem A. We take S and T such that

$$\delta(S,T) = f(S) + d_{G-S}(T) - g(T) - h(S,T) < 0 \tag{1}$$

and subject to this, $\mid T \mid$ is as small as possible. This means that $T = \emptyset$; or $\delta(S,T) \geq 0$ for any proper subset T' of T. At first, we prove the following lemma.

Lemma 1. *Choose S and T as the above. If $T \neq \emptyset$, then $\Delta(G[T]) \leq b-2$.*

Proof. We prove that $e_G(x, T-x) \leq b-2$ for any $x \in T$. Put $T' = T-x$, since (S,T) is a pair as above, $\delta(S,T') \geq 0$. Hence by Theorem A,

$$\begin{aligned} 2 &\geq \delta(S,T) - \delta(S,T') \\ &= d_{G-S}(x) - f(x) + h(S,T') - h(S,T) \\ &\geq d_{G-S}(x) - b + h(S,T') - h(S,T). \end{aligned}$$

Since $h(S,T') \geq h(S,T) - e_G(x, G-(S\bigcup T))$

$$e_G(x,T') = d_{G-S}(x) - e_G(x, G-(S\bigcup T)) \leq b-2.$$

We now continue to prove Theorem 1.

When $a = b = 1$, by Theorem B we know that G has a 1-factor. It is obvious that a complete graph K_n has a f-factor if and only if $n \geq \frac{b^2}{a} \geq b+1$ and $f(V(G))$ is even. A graph is connected if and only if its toughness is positive. Hence we may assume that $b \geq 2$, so G is connected. Suppose G has no f-factor. Then there exists a pair (S,T) satisfying the above assumption. Let $U = V(G)-(S\bigcup T)$, $\{C_1, \cdots, C_\omega\}$ be the set of components of U, where $\omega = \omega(U)$. We may assume that $\Gamma(x) = V(G) - \{x\}$ for all $x \in S$, and that every C_i is complete. Furthermore, let $N = e_G(T,U), M = e_G(T,T)$, $s =\mid S \mid$ and $t =\mid T \mid$. Since $a \leq f(x) \leq b$ for every $x \in V(G)$, then

$$\begin{aligned} 0 > \delta(S,T) &= f(S) + N + 2M - f(T) - h(S,T) \\ &\geq as + N + 2M - bt - \omega. \end{aligned} \tag{2}$$

Let S_1 be a maximal independent subset of T and $T_1 = T - S_1$. For $2 \leq i \leq b-1$ let S_i be a maximal independent subset of T_{i-1} and $T_i =$

$T_{i-1} - S_i$. Since $\Delta(G[T]) \leq b-2$ by Lemma 1. $\Delta(G[T_i]) \leq b-i-2$. Hence T_{b-2} is independent, $S_{b-1} = T_{b-2}$, and $T_{b-1} = \emptyset$

First we show the following claims.

Claim 1. $T \neq \emptyset$

Otherwise $T = \emptyset$. If $S = T = \emptyset$, then $0 > \delta(S,T) = -h(S,T)$. Since G is connected, $h(S,T) = 1$ and $\delta(S,T) = -1$. This is a contradiction by the assumption that $f(V(G))$ is even and Theorem A(2). So $S \neq \emptyset$. when $\omega \leq 1$, then

$$\delta(S,T) \geq as - h(S,T) \geq as - \omega \geq 0.$$

This contradicts the assumption. When $\omega \geq 2$, since G is b^2/a-tough, $| S | \geq \frac{b^2}{a}\omega$. Then $a | S | \geq b^2\omega \geq \omega$. Namely, $\delta(S,T) \geq as - \omega \geq 0$. We also get the desired contradiction. Therefore, $T \neq \emptyset$.

Claim 2. *$d_{G-S}(x) \leq b$ for all $x \in T$.*

If $d_{G-S}(x) \geq b+1$ for some $x \in T$, since $f(x) \leq b$ for every $x \in V(G)$ and $h(S,T) \leq h(S, T-\{x\}) + 1$, then

$$\begin{aligned}
\delta(S, T-\{x\}) &= f(S) + d_{G-S}(T-\{x\}) - f(T-\{x\}) - h(S, T-\{x\}) \\
&\leq f(S) + d_{G-S}(T) - b - 1 - f(T) + b - (h(S,T) - 1) \\
&= \delta(S,T) < 0.
\end{aligned}$$

This is a contradiction to the choice of S and T. So $d_{G-S}(x) \leq b$ for all $x \in T$.

Claim 3. $| S | \geq b$.

Since G is $\frac{b^2}{a}$-tough, the minimum degree of G $\delta(G) \geq 2b$. By Claim 2, it follows that

$$| S | + b \geq | S | + d_{G-S}(x) \geq \delta(G) \geq 2b$$

for all $x \in T$, implying $| S | \geq b$.

Claim 4. $M \geq \sum_{j=1}^{b-2} | T_j |$.

According to the definition of M, we know

$$M = \sum_{1 \leq i < j \leq b-1} e_G(S_i, S_j) = \sum_{i=1}^{b-2} e_G(S_i, \bigcup_{j=i+1}^{b-1} S_j) = \sum_{j=1}^{b-2} (S_j, T_j).$$

Since S_j is a maximal independent subset of T_{j-1}, $e_G(S_j, x) > 0$ for any $x \in T_j$. Hence $e_G(S_j, T_j) \geq | T_j |$ and $M \geq \sum_{j=1}^{b-2} | T_j |$. So $M \geq \sum_{j=1}^{b-2} | T_j |$.

Claim 5. $s + | T_1 | + e_G(U, S_1) - \omega \geq \frac{b^2}{a} | S_1 |$

Let $U_1 = \{u \in U \mid e_G(u, S_1) > 0\}$, $L_1 = \{C_i \mid e_G(C_i, S_1) > 0\}$, and $L_2 = \{C_i \mid e_G(u_i, S_1) = 1 \ \ for \ \ some \ \ u_i \in V(C_i)\}$. We may assume

$L_1 = \{C_1, \cdots, C_{\omega_1}\}$ and $L_2 = \{C_1, \cdots, C_{\omega_2}\}(\omega_2 \le \omega_1 \le \omega)$. For each $C_i \in L_2$ choose $u_i \in C_i$ such that $e_G(u_i, S_1) = 1$. Let $U_2 = \{u_i \mid 1 \le i \le \omega_2\}$ and $U_3 = U_1 - U_2$. Then $|U_1| \le e_G(U_1, S_1) - (\omega_1 - \omega_2) = e_G(U, S_1) - (\omega_1 - \omega_2)$ and $|U_2| = \omega_2$. Hence $|U_3| \le e_G(U, S_1) - \omega_1$. Let $G' = G - (S \bigcup T_1 \bigcup U_3)$. Since S_1 is an independent set , vertices in S_1 belong to different component of G'. Hence $\omega(G') \ge |S_1| + \omega - \omega_1$. First consider the case $\omega(G') \ge 2$. Then $|S \bigcup T_1 \bigcup U_3| \ge \frac{b^2}{a}\omega(G') \ge \frac{b^2}{a}(|S_1| + \omega - \omega_1)$ because we have assumed that G is $\frac{b^2}{a}$-tough. On the other hand, $|S \bigcup T_1 \bigcup U_3| \le s + |T_1| + e_G(U, S_1) - \omega_1$. Therefore,

$$s + |T_1| + e_G(U, S_1) - \omega_1 \ge \frac{b^2}{a}(|S_1| + \omega - \omega_1) \ge \frac{b^2}{a}|S_1| + \omega - \omega_1$$

and the claim follows.

Next we consider the case $\omega(G') = 1$. Then $|S_1| = 0$ or $|S_1| = 1$, because $|S_1| + \omega - \omega_1 \le \omega(G')$. Assume that the claim does not hold. Then $s + |T_1| + e_G(U, S_1) - \omega < \frac{b^2}{a}|S_1|$. If $|S_1| = 0$, then $T = \emptyset$. So $T_1 = \emptyset$ and $s - \omega < 0$. If $\omega(G - S) \ge 2$, then we have $|S| = s \ge \frac{b^2}{a}\omega$, which is impossible. This implies $\omega \le 1$ and $|S| = 0$, which contradicts Claim 3. Hence we may assume $|S_1| = 1$ for any choice of S_1. This implies that $G[T]$ is complete. Furthermore, $\omega(G') = 1$ means that $U \subset \Gamma_G(S_1)$. In particular, $e_G(U, S_1) \ge \omega$ and then $s + |T_1| < \frac{b^2}{a}|S_1| = \frac{b^2}{a}$. Since $\omega \ge 2$, $|S \bigcup T| \ge \frac{b^2}{a}\omega(U) \ge \frac{2b^2}{a}$. However, $|S \bigcup T| = s + |T_1| + |S_1| < \frac{b^2}{a} + 1$, a contradiction.

Claim 6. *For* $2 \le i \le b-1$, $s + e_G(T - T_{i-1}, S_i) + |T_i| + e_G(U, S_i) \ge \frac{b^2}{a}|S_i|$

If $S_i = \emptyset$, then $T_i = \emptyset$. and the claim holds. Suppose $S_i \ne \emptyset$, and let $X_i = \Gamma_G(S_i) \bigcap (T - T_{i-1}), Y_i = \Gamma_G(S_i) \bigcap U$ and $G_i = G - \Gamma_G(S_i)$. Then S_i is a set of isolated vertices in G_i. First consider the case $\omega(G_i) \ge 2$. Then $|\Gamma_G(S_i)| = |S \bigcup X_i \bigcup Y_i \bigcup T_i| \ge \frac{b^2}{a}\omega(G_i) \ge \frac{b^2}{a}|S_i|$. On the other hand, $|S \bigcup X_i \bigcup Y_i \bigcup T_i| \le s + e_G(T - T_{i-1}, S_i) + e_G(U, S_i) + |T_i|$. Hence the claim follows in this case. Next we consider the case $\omega(G_i) = 1$, which implies that $|S_i| = 1$ and $V(G) = \Gamma_G(S_i) \bigcup S_i$. Suppose that the claim does not hold. Then

$$|\Gamma_G(S_i)| \le s + e_G(T - T_{i-1}, S_i) + |T_i| + e_G(U, S_i) < \frac{b^2}{a}.$$

This contradicts the assumption that $|V(G)| \ge \frac{b^2}{a} + 1$. So the claim follows.

By Claim 5 and Claim 6,

$$\frac{b^2}{a}t \le (b-1)s + \sum_{i=1}^{b-1} |T_i| + \sum_{i=2}^{b-1} e_G(T - T_{i-1}, S_i) + \sum_{i=1}^{b-1} e_G(U, S_i) - \omega$$

$$\leq \quad (b-1)s + \sum_{i=1}^{b-2} \mid T_i \mid + M + N - \omega, \tag{3}$$

since $\sum_{i=1}^{b-1} \mid S_i \mid = \mid T \mid = t$, $T_{i-2} = \emptyset$, $\sum_{i=2}^{b-1} (T - T_{i-1}, S_i) = M$ and $\sum_{i=1}^{b-1} e_G(U, S_i) = N$. By (2) and (3), we get

$$\frac{b}{a}(as + N + 2M - \omega) < \frac{b^2}{a} t \leq (b-1)s + \sum_{i=1}^{b-2} \mid T_i \mid + M + N - \omega.$$

Then

$$as + (b-a)N + 2(b-a)M < (b-a)\omega$$

If $a = b$, then we get a contradiction by Claim 3. If $a \neq b$, we may have

$$N + 2M + \frac{as}{b-a} < \omega. \tag{4}$$

By (4) and Claim 5, we get

$$N + 2M + \frac{as}{b-a} < s + \mid T_1 \mid + e_G(U, S_1) - \frac{b^2}{a} \mid S_1 \mid.$$

By Claim 4, $M \geq \mid T_1 \mid$. Since $N \geq e_G(U, S_1)$ and $b \leq 2a$, we get

$$M < -\frac{b^2}{a} \mid S_1 \mid.$$

This is a contradiction. So the Theorem is proved.

When $a = b = k$, we can get Theorem B from Theorem 1.

corollary 1. *Let G be a graph. If G is k-tough, $\mid V(G) \mid \geq k+1$ and $k \mid V(G) \mid$ is even, then G has a k-factor.*

Remark. when $a = b = k$, the condition that G is k-tough is sharp. Since there exists a $(k - \epsilon)$-tough(ϵ is an any real positive number) graph G with $k \mid V(G) \mid$ even and $\mid V(G) \mid \geq k+1$ which has no k-factor. But we do not know whether the condition that G is $\frac{b^2}{a}$-tough can be improved.

Acknowledgement. We would like to thank the referees for providing some very helpful suggestions for revising this paper.

References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, Macmillan, London (1976).

[2] V. Chvátal, Tough graphs and hamilitonian circuits, *Discrete Math. 5(1973)215-228.*

[3] H.Enomoto, B.Jackson, P.Katerinis, A.Saito, Toughness and the existence K-fators, *J. Graph Theory 9(1985)87-95.*

[4] H.Enomoto, Toughness and the existenc of k-factors. III, Discrete Math.189(1998)277-282.

[5] G. Liu, Q.Yu, K-factors and extendabilitywith prescribed components, preprint.

[6] G.Liu, (g,f)-factors and factorizations in graphs, *Acta Math. Sinica* 37(1994), 230-237..

[7] W. T. Tutte, The factor of graphs, *Can. T. Math* 4(1952), 314-328.