Domination and absorbance in signed graphs and digraphs: I. Foundations

BELMANNU DEVADAS ACHARYA
Center for Excellence in Interdisciplinary Mathematics
A-9, M.E.R.I.T., Quatab Enclave
Shaheed Jit Singh Marg
U.S.O. Road
New Delhi-110067, India.
e-mail: devadas.acharya@gmail.com

Abstract

A *signed graph* (*digraph*) Σ is an ordered tripe (V, E, σ) (respectively, $(V, \mathcal{A}, \sigma)$), where $|\Sigma| := (V, E)$ ($:= (V, \mathcal{A})$) is a graph (digraph), called the *underlying graph* (*underlying digraph*) of Σ, and σ is a function that assigns to each edge (arc) of $|\Sigma|$ a *weight* $+1$ or -1. Any edge (arc) e of Σ is said to be *positive* or *negative* according to whether $\sigma(e)$=+1 or $\sigma(e) = -1$. A subset $D \subseteq V$ of vertices of Σ is an *absorbent* (respectively, *a dominating set*) of Σ if there exists a *marking* $\mu : V \to \{+1, -1\}$ of Σ such that every vertex u of Σ is either in D or

$$O(u) \cap D \neq \emptyset \text{ and } \sigma(u, v) = \mu(u)\mu(v) \;\; \forall \;\; v \in O(u) \cap D,$$

(respectively,

$$I(u) \cap D \neq \emptyset \text{ and } \sigma(u, v) = \mu(u)\mu(v) \;\; \forall \;\; v \in I(u) \cap D),$$

where $O(u)$ ($I(u)$) denotes the set of vertices v of Σ that are joined by the *outgoing arcs* (u, v) from u (*incoming arcs* (v, u) at u). Further, an absorbent (dominating set) of Σ that is independent is called a *kernel* (*solution*) of Σ. The main aim of this paper is to initiate a study of absorbents and dominating sets in a signed graph (signed digraph), extending the existing studies on these special sets of vertices in a graph (digraph).

Keywords: signed digraph, subsigraph, absorbent, domination, solution.

2010 Mathematics Subject Classification Number: 05C10.

1 Introduction

The notion of *domination* is as old as survival of life on this planet. The theory of evolution of terrestrial life species holds the tenet, "survival of the fittest". Prey-predator mathematical models address various intricate real-world problems involved in the understanding of this complex phenomenon most effectively. A discrete structure called a *signed digraph*, denoted $\Sigma_0 = (V, \mathcal{A}_0, \sigma_0)$, for describing some of the basic components of this important problem in Ecology of the Planet Earth may be constructed as follows: Let V be the set of various life species striving to exist in an ecosystem, where the fact that a species s_i preys on the species s_j is represented by the *arc* $(s_i, s_j) \in \mathcal{A}_0$ and the fact that the population of s_i at a given time t_0 of our observation is directly or inversely proportional to the population of s_j at time t_0 is represented by the value of $\sigma_0(s_i, s_j)$ being $+1$ or -1 respectively; the arc (s_i, s_j) is then said to be *positive* or *negative* accordingly. The signed digraph Σ_0 so defined would then provide us the dynamic structure of interdependent survival of the species in the ecosystem as observed at time t_0. One of the principal questions in ecology is concerned with conservation of ecosystems for the general well being of the life on the planet. It is hence tantamount to consider the problem of maintaining *stability* of the *life space* Σ_0 in the ecosystem. Here arises the notion of *dominance*: a nonempty subset D of $V(\Sigma_0)$, the set of species existing in the ecosystem under observation at time t_0, is a *dominating set* of Σ_0 if every species s is either in D or there exists a *compatibility function* $\mu : V(\Sigma_0) \to \{-1, +1\}$, called a *marking* of Σ_0, such that

$$I(s) \cap D \neq \emptyset \text{ and } \sigma_0(s, s') = \mu(s)\mu(s') \;\; \forall \;\; s' \in I(s) \cap D, \qquad (1)$$

where $I(u)$ denotes the set of vertices v of Σ_0 that are joined by the *incoming arcs* (v, u) at u, or of *in-neighbors* of u for short. Here, the value of $\mu(u)$ is interpreted as the increase or decrease in the population of the given species u according to whether it is $+1$ or -1. In particular, if $\sigma_0(u, v) = +1$ for all arcs $(u, v) \in \mathcal{A}_0$ then the above definition reduces to the standard definition of a dominating set of a digraph as in [7]; it is important to be noted here that the study of even this particular case is scanty as seen from the existing literature and hence needs to be studied carefully.

Hence, the main aim of this article is to identify principal directions of study of the notion of domination in signed graphs and signed digraphs. In signed digraphs, there is another notion, called *absorbent*, akin to that of domination and these two concepts coincide in the case of signed graphs.

2 Domination in Signed Graphs

For all terminology and notation in the theory of digraphs and the theory of graphs, not specifically given here, we refer the reader to the good old standard text-books by Harary et al. [5] and Harary [6], respectively. However, unless mentioned otherwise, graphs considered here could be infinite.

As treated by Chartrand [4] (respectively, Harary et al. [5]), *signed graph* (*signed digraph*) Σ is an ordered tripe (V, E, σ) (respectively, $(V, \mathcal{A}, \sigma)$), where $|\Sigma| := (V, E)$ $(:= (V, \mathcal{A}))$ is a graph (digraph), called the *underlying graph* (*underlying digraph*) of Σ, and σ is a function that assigns to each edge (arc) of $|\Sigma|$ a *sign* '+' or '−', which are some times treated as *weights* $+1$ and -1, respectively. Any edge (arc) e of Σ is said to be *positive* or *negative* according to whether $\sigma(e) = +$ or $\sigma(e) = -$; hence, the set of positive edges (arcs) of Σ is denoted $E^+(\Sigma)$ (respectively, $\mathcal{A}^+(\Sigma)$) and $E^-(\Sigma) := E \setminus E^+(\Sigma)$ (respectively, $\mathcal{A}^-(\Sigma) := \mathcal{A} \setminus \mathcal{A}^+(\Sigma)$). Further, Σ is said to be *homogeneous* (*heterogeneous*) if all (not all) edges, or arcs if Σ is a signed digraph, are of the same sign; in particular, Σ is said to be *all-positive* (*all-negative*) if the set of its negative (positive) edges or arcs is empty. Additional notation and terminology used above are drawn from the still developing, yet standardized, literature as may be easily tracked from the dynamic survey being continually updated by Zaslavsky [15].

Definition 2.1. Let $\Sigma = (V, E, \sigma)$ be any signed graph. A subset $D \subseteq V$ of vertices of Σ is a *dominating set* of Σ if there exists a *marking* $\mu : V \to \{+1, -1\}$ of Σ such that every vertex u of Σ is either in D or

$$N(u) \cap D \neq \emptyset \text{ and } \sigma(uv) = \mu(u)\mu(v) \ \forall \ v \in N(u) \cap D. \tag{2}$$

Note that if Σ is an all-positive signed graph, which is then essentially a graph in the standard sense, then Definition 1, restricted to finite graphs, reduces to that of a dominating set as treated in Haynes et al. [7] because, with the *all-positive marking* μ^+, which assigns $+1$ to each vertex of Σ satisfies (2). Thus, we have the following very first elementary observation:

Proposition 2.2. *The set $\mathcal{D}_\Sigma$ of all dominating sets of a signed graph Σ is contained in the set $\mathcal{D}_{|\Sigma|}$ of all dominating sets of its underlying graph $|\Sigma|$.*

Proposition 2.2 raises the following natural open question:

Problem 2.3. *Determine the signed graphs Σ such that*

$$\mathcal{D}_\Sigma = \mathcal{D}_{|\Sigma|}. \tag{3}$$

Note that there do exist signed graphs Σ that do not satisfy (3). For example, take a 4-cycle (or, the so-called 'quadrilateral') in which three edges are negative; in this signed graph Σ, no independent dominating set (or, the so-called *kernel*) of its underlying quadrilateral $C_4 =: |\Sigma|$ is a dominating set of Σ.

Hence, one can develop the theory of domination in signed graphs, starting from scratch as in [11] and develop it on similar lines as in [7]; of course, one can expect quite different notions and results in such a theory that would not perhaps be available in the usual theory of domination in graphs as indicated by the example of the signed quadrilateral mentioned above. Towards this end, apart from the usual questions in the theory of domination in graphs, a number of additional questions might arise. For example, notice that in a procedure to mark the vertices of $V \setminus D$ so as to satisfy the condition of Definition 1 some vertices in D can be marked arbitrarily. This observation raises the following open question.

Problem 2.4. *Determine the signed graphs in which there exists a minimal dominating set that allows no freely markable vertex.*

Next, the example of the signed quadrilateral given above signals the following interesting open problem:

Problem 2.5. *Determine the signed graphs in which there exists no kernel.*

Note that Problem 2.5 is 'interesting' because every graph has a kernel, viz., any maximal independent set (cf.: [3])! Thus, Problem 3 is a parallel to the well known, yet unsolved, problem of determining the digraphs having no kernel (cf.: Berge [3]); I wonder, if there is a relation between the two! A signed graph (signed digraph) in which there exists no kernel will be referred to as *kernel-free* signed graph (signed digraph).

By a *subsigraph* of a signed graph Σ we shall mean a signed graph Σ' such that

$$V(\Sigma') \subseteq V(\Sigma),\ E(\Sigma') \subseteq E(\Sigma) \text{ and } \sigma' := \sigma|_{E(\Sigma')},$$

where $\sigma|_{E(\Sigma')}$ denotes the restriction of the 'signature' σ of Σ to the edges of Σ'. Further, given a set S of vertices in Σ, the subsigraph *induced* by S, denoted $\langle S \rangle$, is the subsigraph with S as its vertex set and having all the edges of Σ that join any two vertices in S. We shall say that a given signed graph Σ_1 is *embedded* (or, *embeddable*) in a signed graph Σ_2, written $\Sigma_1 \preceq \Sigma_2$, if there exists an induced subsigraph of Σ_2 that is isomorphic to Σ_1. We have the following some what negative result.

Theorem 2.6. *Every signed graph can be embedded as an induced subsigraph of a signed graph possessing a kernel.*

Proof. Let Σ be any signed graph and let Σ^+ denote the signed graph obtained by augmenting one new positive pendent edge at each vertex of Σ. Then, the set of new vertices so augmented form a kernel of Σ^+ due to the all-positive marking μ^+ of Σ^+. □

Theorem 2.2 is a 'negative result' in the sense that it implies there is no "forbidden subgraph characterization" of a signed graph possessing a kernel. Therefore, it becomes important to explore specific classes of signed graphs that possess kernels.

We have the following four main results toward this end, besides additional open problems arising thereof.

Theorem 2.7. *If Σ is a signed graph such that $|\Sigma|$ has a kernel D having the property that the set E_u^+ of positive edges containing the vertex u is empty for every $u \in D$ then $D \in \mathcal{D}_\Sigma$ (i.e., D is a kernel of Σ).*

Proof. Let D satisfy the hypothesis of the theorem. Then, let μ be the marking of Σ which marks every vertex of D as positive and every vertex of $V(\Sigma)\setminus D$ as negative. By a well known theorem of Berge [3] (Proposition 2, p.309), D is a minimal dominating set of $|\Sigma|$ (and hence $V(\Sigma) \setminus D$ is a dominating set of $|\Sigma|$, by virtue of a theorem of Ore [9]), whence condition (2) holds for every $v \in V(\Sigma)\setminus D$ and the result follows by Definition 2.1. □

A signed graph Σ is *balanced* if its vertex set $V(\Sigma)$ can be divided into two disjoint subsets V_1 and V_2, one of them possibly empty, such that every negative edge of Σ has one of its ends in V_1 and the other in V_2 but no positive edge of Σ has this property (see [4, 5, 14]); such a decomposition of $V(\Sigma)$ is often called in literature a *Harary bipartition* of Σ. The proof of Theorem 2.6 suggests the validity of the following result.

Theorem 2.8. *If Σ is a balanced signed graph such that one of the parts V_1 and V_2, say V_1, in its Harary bipartition is a dominating set in $|\Sigma|$ then it is a dominating set in Σ (i.e., $V_1 \in \mathcal{D}_\Sigma$). Further, if V_1 is a minimal dominating set of $|\Sigma|$ then $V_2 \in \mathcal{D}_\Sigma$ too.*

Proof. The first part of the theorem follows on similar line of arguments as in the proof of Theorem 2.7. The second part follows from the definition of balance of Σ and a theorem of Berge [3]. □

Corollary 2.9. *Every homogeneously signed tree has two disjoint kernels.*

Proof. Let T be any homogeneously signed tree. If T is an all-positive tree then each of the two parts V_1 and V_2 of its usual bipartition (due to bipartiteness of T) is a kernel. Hence, suppose that T is an all-negative signed tree. Then, $\pi = \{V_1, V_2\}$ is its Harary bipartition, whence the result follows from Theorem 2.8. □

It is well known that every finite tree has two disjoint dominating sets (cf.: [7]). That this is true in general for any finite signed tree is established in the following result.

Theorem 2.10. *Every finite signed tree has two disjoint dominating sets.*

Proof. Let Σ be any signed tree and let $\pi = \{V_1, V_2\}$ be the standard bipartition of $|\Sigma|$. As mentioned above, both V_1 and V_2 are kernels of $|\Sigma|$. Label the vertices in V_1 as $u_1, u_2, \ldots$ and label the vertices in V_2 as $v_1, v_2, \ldots$. Starting from v_1, assign $+1$ or -1 arbitrarily to v_1. Hence, assign weights $+1$ or -1 to the 'neighbors' u_i of v_1 (they are all in V_1 as V_2 is independent) such that u_i receives the weight of v_1 or its additive inverse according to whether $v_1 u_i$ is a positive edge or a negative edge in Σ. Next, take $v_2 \in V_2$. If none of the neighbors of v_1 is a neighbor of v_2 then assign $+1$ or -1 to v_2 arbitrarily and then assign $+1$ or -1 to the neighbors of v_2 (again, they are all in V_1 as V_2 is independent) in the same manner as done for the neighbors of v_1. If, on the other hand, the set $N(v_2)$ of neighbors of v_2 contains a neighbor of v_1 then, there must be only one such vertex $u_j \in V_1$, for otherwise, there would be a cycle (in fact a quadrilateral) in Σ contradicting the hypothesis that Σ is a signed tree; hence, in this case, assign to v_2 the weight of u_j or its additive inverse according to whether the sign of the edge $u_j v_2$ is positive or negative. Further, continuing in this manner, if at any stage, the vertex $v_r \in V_2$ has a neighbor that is already marked then it must be unique, for otherwise Σ would contain a cycle. Hence, we may continue this procedure till all the vertices of Σ are exhausted, whence we would have a 'dominating marking' μ of Σ due to the hypothesis that Σ is a finite signed tree as also due to the fact that each of the sets V_1 and V_2 is a dominating set of $|\Sigma|$. Thus, the proof is seen to be complete. □

In the case when Σ is a finite all-negative signed tree as considered in Corollary 3.1, the usual bipartition $\pi = \{V_1, V_2\}$ of $|\Sigma|$ is also its Harary bipartition, which may not necessarily be so for a general balanced signed graph (even for a signed tree) as such. In fact, it may be easily seen that for any finite signed tree Σ the usual bipartition $\pi = \{V_1, V_2\}$ of $|\Sigma|$ is also its Harary bipartition if and only if Σ is a finite all-negative signed tree. However, when Σ is not necessarily a signed tree, the following is easy to prove.

Theorem 2.11. *In a signed graph Σ, having bipartite underlying graph $|\Sigma|$ with bipartition $\pi = \{V_1, V_2\}$, π is the Harary bipartition of Σ if and only if Σ is balanced and all-negative; further, such an isolate-free signed graph has two disjoint kernels.*

Ore [9] has shown that if a graph G does not have isolates, then G has a dominating set D such that $V(G) \setminus D$ is a dominating set; its proof involves the assumption that G is connected and has a spanning tree T that has a pendent vertex, but this may not hold when G is a connected infinite graph in general. For example, a two-way infinite path does not have pendent vertices. However, the proof goes through even if we choose an arbitrary vertex x of T as its *root* and partition the vertices of G into two subsets V_o^x and V_e^x such that V_o^x consists of the vertices at odd distances from x and $V_e^x = V(G) \setminus V_o^x$. In fact, it has been shown further that the complement $\overline{D}$ of every minimal dominating set D in G is a dominating set of G. However, the existence of a minimal dominating set is known only when G is *locally finite*, in the sense that the *edge-degree* $e_G(u) := |E_u|$ is finite for every $u \in V(G)$ (cf.: [9], Theorem 13.1.1, p.206). By this theorem, it is easily seen that for every finite graph G, the set $\mathcal{D}_G^m$ of minimal dominating sets of G is nonempty (as, of course, it is true when G is locally finite). The following open problem does not seem to have been attempted even for graphs.

Problem 2.12. *Characterize isolate-free infinite signed graphs Σ for which $\mathcal{D}_\Sigma^m = \emptyset$.*

Theorem 2.13. *For any balanced signed graph Σ, (3) holds; that is, $\mathcal{D}_\Sigma = \mathcal{D}_{|\Sigma|}$.*

Proof. By Proposition 2.2, $\mathcal{D}_\Sigma \subseteq \mathcal{D}_{|\Sigma|}$. We complete the proof by showing the other way inclusion: $\mathcal{D}_{|\Sigma|} \subseteq \mathcal{D}_\Sigma$. Toward this end, we recall a well known result of Sampathkumar and Bhave [10] that a signed graph is balanced if and only if there exists a marking μ of its vertices such that for every edge uv in the signed graph its sign is the the product $\mu(u)\mu(v)$, and notice that holds irrespective of Σ being finite or infinite (see [1]). Now, since Σ is balanced, by the Sampathkumar-Bhave theorem just mentioned, there exists a *balancing marking* μ of Σ. Hence, let $D \in \mathcal{D}_{|\Sigma|}$ and consider the set D in Σ. Then, it is easy to see that μ satisfies (2), whence we get $D \in \mathcal{D}_\Sigma$ by definition. Since the choice of D was arbitrary in $\mathcal{D}_{|\Sigma|}$, the result follows. □

The reader is encouraged to find a counter-example to note that the converse of Theorem 6 does not hold. Thus, it may be observed in passing here that Problem 1 essentially remains to be settled for unbalanced signed graphs.

An obvious but important point that needs to be noted here, in view of the example of the signed quadrilateral mentioned much above, is the following fact.

Proposition 2.14. *If, for a signed graph Σ, $\mathcal{D}_\Sigma = \mathcal{D}_{|\Sigma|}$ then Σ does have a kernel and, moreover, every kernel of $|\Sigma|$ is a kernel of Σ.*

3 The Domination Number of a Signed Graph

Clearly, for any signed graph Σ, its vertex set $V(\Sigma)$ is trivially a dominating set of Σ. Further, since $\mathcal{D}_\Sigma \subseteq \mathcal{D}_{|\Sigma|}$ we must have in general when Σ is finite

$$\gamma(|\Sigma|) \leq \gamma(\Sigma), \tag{4}$$

where $\gamma(\Sigma)$ denotes the minimum cardinality of a dominating set in Σ, called its *domination number.* We note here that the inequality in (4) could be strict as may be verified by taking the signed graph Σ on the *hexagon*, or the 6-cycle C_6, consisting of three mutually disjoint negative edges. For this signed graph, one has $\gamma(|\Sigma|) = 3$ and $\gamma(\Sigma) = 4$. In fact, one can easily see the validity of the following more general statement.

Corollary 3.1. *For any finite balanced signed graph* Σ, $\gamma(\Sigma) = \gamma(|\Sigma|)$.

Converse of Corollary 3.1 is not true. Take, for example, the unbalanced signed graph Σ on the hexagon C_6 in which the set $E^-(\Sigma)$ of negative edges forms an all-negative subsigraph consisting of one path of length one and one path of length two (then the set $E^+(\Sigma)$ also is seen to form a similar all-positive subsigraph of Σ); it is easily verified that $\gamma(\Sigma) = \gamma(|\Sigma|) = 3$.

It would thus be interesting to solve the following problems.

Problem 3.2. *Study the special properties of minimal (minimum) dominating sets of finite unbalanced signed graphs.*

Problem 3.3. *Determine or estimate the domination numbers of all unbalanced signed cycles on the n-gon* C_n, $n \geq 3$, *the complete graph* K_n, $n \geq 4$, *the complete bipartite graph* $K_{m,n}$, *the n-dimensional hypercube* Q_n, *the 2-dimensional complete square lattice grid* $P_m \times P_n$, *and on the generalized Petersen graph.*

4 The role of switching

Given any signed graph (digraph) $\Sigma = (V, E, \sigma)$ and a marking μ, *switching* Σ with respect to μ (or, the 'μ-switching' of Σ_μ) is to obtain a signed graph (digraph) $S_\mu(\Sigma)$ from Σ by changing the sign of every edge xy for which $\mu(x) \neq \mu(y)$. A signed graph (digraph) Σ_1 is said to *switch* to a signed graph (digraph) Σ_2, written $\Sigma_1 \sim \Sigma_2$, whenever there exists a marking μ of Σ_1 such that $S_\mu(\Sigma_1) \cong \Sigma_2$. It is well known that the binary relation '$\sim$' is an equivalence relation in the class of all signed graphs (digraphs) of a given order [14]. The following result is due to the referee.

Theorem 4.1. *For any signed graph (signed digraph)* Σ, *the set* $\mathcal{D}_\Sigma$ *is invariant under switching.*

Proof. We first consider a signed graph $\Sigma = (V, E, \sigma)$. Let $D \subseteq V$ and $N(D)$ the set of all vertices in $V \setminus D$ that are adjacent to the vertices in D. Clearly,

$$D \in \mathcal{D}_\Sigma \Leftrightarrow [N(D) = V \setminus D] \text{ and } [\partial^\mu_\Sigma(D)] \text{ is balanced}], \tag{5}$$

where $\partial^\mu_\Sigma(D)$ is the subsigraph spanned by the *μ-boundary edges*, which are edges $xy \in E(\Sigma)$ with $x \in D$, $y \in V \setminus D$, is an alternate definition of a dominating set. Neither of the defining properties in (5) is altered by μ-switching of the *μ-marked signed graph* Σ_μ. Therefore, if we switch Σ with respect to μ to another signed graph Σ', we find that $D \in \mathcal{D}_\Sigma$ satisfies the requirements to belong to $\mathcal{D}_{\Sigma'}$. That is, we have $\mathcal{D}_\Sigma \subseteq \mathcal{D}_{\Sigma'}$. Similarly, one can show $\mathcal{D}_{\Sigma'} \subseteq \mathcal{D}_\Sigma$ by interchanging the roles of Σ and Σ' in the above argument as $\sim$ is a symmetric binary relation. The conclusion follows.

The same conclusion follows when Σ is a signed digraph due to the fact that the notion of μ-switching does not involve the directions of the arcs in Σ. □

The following result is well known [14].

Theorem 4.2. *A signed graph (signed digraph) Σ is balanced if and only if $\Sigma \sim |\Sigma|$.*

Thus, invoking Theorem 4.1 and Theorem 4.2, Theorem 2.13 can be seen also as a natural corollary of Theorem 4.2.

5 Relations with clustering and colorings

In graph theory, there is a fundamental result that for any finite graph G there is a *minimum coloring* (i.e., a proper coloring of the vertices using minimum number of colors) φ that contains a maximal monochromatic set of vertices that is a kernel (cf.: [12]) of G. How does this result get extended in the case of minimum colorings of a finite signed graph (cf.: [13])? A proper extension of the result would possibly yield several new directions of research in the theory of signed graphs.

In [2], the authors defined a notion of *coloring* of a signed graph $\Sigma = (V, E, \sigma)$ as an assignment c of colors $c(u)$ to the vertices $u \in V$ so that

$$uv \in E^- \Rightarrow c(u) \neq c(v). \tag{6}$$

Further, they called the corresponding *color partition* $P^c = \{V_1^c, V_2^c, \dots\}$ of V into monochromatic subsets V_i^c a *semiclustering* of Σ.

A semiclustering of a finite signed graph Σ with the least possible number $\chi(\Sigma)$ of monochromatic subsets is called a *minimum semiclustering* of Σ. Since not every signed graph contains a kernel, the following problem assumes significance.

Problem 5.1. *Characterize finite signed graph $\Sigma = (V, E, \sigma)$ which admits a minimum semiclustering that contains a kernel.*

Let us call a signed graph Σ with the property stated in Problem 5.1 a *kernel chromatic semiclustering.* Clearly, if a signed graph does not possess a kernel then it cannot admit a kernel chromatic semiclustering.

Given a semiclustering P^c of a signed graph Σ, a positive edge having its two ends in two different monochromatic subsets of P^c is called a *positive inconsistency.* As well known, a signed graph is said to be *clusterable* if it has a semiclustering having no positive inconsistency, and such a signed graph has the following characterization.

Theorem 5.2. (J. Davis, 1967) *A signed graph Σ is clusterable if and only if it contains no cycle with exactly one negative edge.*

The following result gives a partial solution to Problem 5.1.

Theorem 5.3. *If $\Sigma = (V, E, \sigma)$ is a clusterable signed graph then Σ admits a kernel chromatic semiclustering.*

Proof. Let $P = \{V_1, V_2, \ldots,\}$ be any minimum clustering of Σ and suppose it does not contain any kernel. Our assumption implies that none of the sets in P is a kernel in Σ. Now, since P is a clustering of Σ, each V_i contains no negative edge of Σ (or, *negative inconsistency*) and hence by the result of [12] quoted in the first paragraph of this section, we may assume, without loss of generality, that one of the sets in P, say V_1 is a kernel in $|\Sigma|$. This implies, V_1 is an independent set in $|\Sigma|$ such that $N(V_1) = V \setminus V_1$. Next, since none of the sets V_i is a kernel in Σ, in particular V_1 is not a kernel in Σ. Then, by the alternate definition (5) of domination in Σ and our penultimate argument imply existence of a negative cycle Z in the subsigraph spanned by the edges in $\partial_\Sigma(V_1)$. Without loss of generality, we may assume that Z has the least possible length. Since Σ is clusterable, by Theorem 5.2, two cases arise, viz., Z has no positive edge, or Z has at least two positive edges.

Case 1: Z is all-negative.

Then, Z has an odd length, say $n = 2k + 1$. Since every negative edge joins vertices of two different subsets in P and since V_1 is independent, minimality of n implies that one of the edges of Z does not belong to $\partial_\Sigma(V_1)$, a contradiction.

Case 2: Z contains at least two positive edges.

Since the number of negative edges in Z is odd and all of them are in $\partial_\Sigma(V_1)$, at least one of the positive edges of Z must have an end in V_1 and the other in $V(\Sigma) \setminus V_1$. But, this contradicts the fact that P is a clustering of Σ.

Thus, the proof follows by contraposition. □

The converse of Theorem 5.3 fails to hold, as the following nonclusterable signed graph illustrates.

Example 5.4. Consider the pentagon $C_5 = (v_1, v_2, v_3, v_4, v_5, v_1)$ and let Z be the negative cycle on C_5 in which v_1v_2 is the only negative edge. Consider the minimum semiclustering $P = \{V_1 = \{v_2, v_4\}, V_2 = \{v_1, v_3, v_5\}\}$. Define $\mu : V(C_5) \to \{-1, +1\}$ by letting $\mu(v_1) = +1, \mu(v_2) = \mu(v_3) = \mu(v_4) = \mu(v_5) = -1$. It may then be easily verified that μ is a domination marking for each of the sets V_1 and V_2. We observe that V_1 is a kernel of Z.

Thus, Problem 5.1 remains open for nonclusterable signed graphs.

6 Absorbents and kernels in Signed Digraphs

In this section, the term 'digraph' (as treated in [5]) will be equivalent to the term '1-graph' as treated in [3]; this assumption essentially means that between any two distinct vertices u and v there is at most one arc in each direction, from u to v or from v to u.

The notion of a dominating set in a digraph was first considered by Berge [3] (Ch.14), under the name *absorbent*, motivated by the problem of managing surveillance networks. Subsequently, it has been studied extensively as may be found in [7, 8]. An extension of this notion to signed digraphs is defined below.

Definition 6.1. Let $\Sigma = (V, \mathcal{A}, \sigma)$ be any signed digraph. A subset $D \subseteq V$ of vertices of Σ is an *absorbent* of Σ if there exists a *marking* $\mu : V \to \{+1, -1\}$ of Σ such that every vertex u of Σ is either in D or

$$O(u) \cap D \neq \emptyset \text{ and } \sigma(u, v) = \mu(u)\mu(v) \;\; \forall \;\; v \in O(u) \cap D, \tag{7}$$

where $O(u)$ denotes the set of vertices v of Σ that are joined by the *outgoing arcs* (u, v) from u, or of *out-neighbors* of u for short.

It is easily seen that Definition 6.1 coincides only with a part of the definition of an absorbent given by Berge for digraphs (cf.: [3], p.303), viz., when every digraph is treated as an all-positive signed digraph and the set of *in-neighbors* is ignored from his definition. While generalizing results from the theory of digraphs is one purpose of studying signed digraphs, in this particular case the study of absorbents in signed digraphs seems to serve some purpose of application too: A set D of soldiers on the war front is to be chosen in a surveillance network Σ, in which there are some communication channels that are 'good' or reliable (positive) and some that are 'bad' or less reliable or noisy (negative), so that there is a perfect understanding by every commander $u \in V(\Sigma) \setminus D$ with every soldier v under his command in D about the quality $\sigma(u, v)$ of the communication channel (u, v) so that there is no 'confusion' in either the message sent by u to v or in its receiving and decoding by v for the purpose of taking further necessary action at the latter's end.

Again, clearly since $V(\Sigma) \in \mathcal{D}_\Sigma$ for any signed digraph Σ Proposition 2.2 has the following extension.

Proposition 6.2. *The set $\mathcal{D}_\Sigma$ of all absorbents of a signed digraph Σ is contained in the set $\mathcal{D}_{|\Sigma|}$ of all dominating sets of its underlying digraph $|\Sigma|$.*

Proposition 6.2 also raises the following natural open problem, similar to Problem 2.3.

Problem 6.3. *Determine the signed digraphs Σ such that*

$$\mathcal{D}_\Sigma = \mathcal{D}_{|\Sigma|}. \tag{8}$$

The following observation is rather obvious.

Proposition 6.4. *For any signed symmetric digraph $\Sigma^{\leftrightarrow}$, viz., the signed digraph in which every adjacent pair of vertices are joined to each other by an arc of the same sign in either direction,*

$$\mathcal{D}_{\Sigma^{\leftrightarrow}} = \mathcal{D}_\Sigma,$$

where Σ is its 'underlying signed graph' obtained by replacing each symmetric arc of $\Sigma^{\leftrightarrow}$ by an undirected edge of the same sign as that of the symmetric arc in $\Sigma^{\leftrightarrow}$.

Proposition 6.4 implies that the theory of domination in signed symmetric digraphs $\Sigma^{\leftrightarrow}$ coincides with the theory of domination in their underlying signed digraphs Σ.

A vertex u in a digraph is said to be *reachable* from a vertex v if there is a directed path from v to u. A digraph is then said to be *strongly connected* if every two of its vertices are mutually reachable from each other. A digraph is *weakly connected* if its underlying undirected graph is connected. All such basic terminology for structural aspects of a directed graph as given in Harary et al. [5] also hold for any signed digraph. Hence, in particular, notice that every weakly connected sign symmetric digraph $\Sigma^{\leftrightarrow}$ is strongly connected. Proposition 6.4 prompts the following open question.

Problem 6.5. *Can we extend Proposition 6.4 to strongly connected signed digraphs?*

Next, the relation between the signature σ of a signed digraph Σ and a dominating marking μ of Σ, whenever it exists as per Definition 6.1, suggests the following observation.

Proposition 6.6. *For any signed digraph Σ, $\mathcal{D}_\Sigma \subseteq \mathcal{D}_{|\Sigma|}$, where $|\Sigma|$ denotes the underlying digraph of Σ.*

The converse of Proposition 6.6 is not true. For example, consider the signed digraph Σ of order 6 constructed as follows: Take a signed directed cycle $Z_4 = (a_1, a_2, a_3, a_4, a_1)$ with just one positive arc (a_1, a_2) and adjoin two other vertices x and y by introducing the positive arc (x, a_4) and the three negative arcs $(x, a_2), (y, a_2)$ and (y, a_4). Then, it is not difficult to verify that $\{a_2, a_4\}$ is a *kernel* in $|\Sigma|$, viz., an absorbent that is also independent, but is not even an absorbent in Σ. Thus, the following open problem arises.

Problem 6.7. *Characterize signed digraphs Σ for which $\mathcal{D}_\Sigma = \mathcal{D}_{|\Sigma|}$.*

Signed digraphs for which $\mathcal{D}_\Sigma = \mathcal{D}_{|\Sigma|}$ will be referred to as *absorbance invariant signed digraphs.*

As noted already, however, a digraph H may not have a kernel and hence any signed digraph on H would not have a kernel; but a signed digraph Σ on H may have an absorbent, which is not a kernel. The converse may be amusing due to the 6-vertex signed digraph constructed above, which has no kernel but whose underlying digraph has exactly one kernel, viz., $\{a_2, a_4\}$.

Problem 6.8. *Characterize signed digraphs Σ having no kernel but their underlying digraphs $|\Sigma|$ having one.*

Further, we shall call a kernel-free signed digraph a *strictly kernel-free signed digraph* if its underlying digraph is kernel-free too.

Let $A(\Sigma)$ denote the set of *sources* in Σ, i.e., vertices at each of which there are only outgoing arcs and no incoming arcs. Then, clearly, $A(\Sigma)$ cannot be an absorbent of Σ. However, the following result shows that its complement is an absorbent in Σ.

Proposition 6.9. *Given any signed digraph Σ, $V(\Sigma) \setminus A(\Sigma) \in \mathcal{D}_\Sigma$.*

Proof. Clearly, $V(\Sigma) \setminus A(\Sigma) \in \mathcal{D}_{|\Sigma|}$. Hence, let μ be a marking that assigns $+1$ to each vertex $u \in A(\Sigma)$, $+1$ or -1 to every vertex $v \in O(u) \cap (V(\Sigma) \setminus A(\Sigma))$ according to whether the arc (u, v) is positive or negative in Σ and assigns arbitrarily $+1$ or -1 to any vertex x in $V(\Sigma) \setminus A(\Sigma)$ that has no predecessor in $A(\Sigma)$. It is easy to see that μ so defined satisfies the conditions of Definition 6.1. □

Next, as mentioned above already, by a theorem of Ore, any isolate-free graph G has a dominating set D such that $V(G) \setminus D$ is a dominating set. Is there an analogue of this theorem for isolate-free signed digraphs? Instantly, it may be noticed that in any signed digraph Σ on an *out-star*, i.e., the star $K_{1,n}$, $n \geq 1$ in which every edge is oriented outward from its center c (viz., the vertex of full out-degree), the set of its *sinks* (i.e., vertices from which no arc is outgoing) is the only nontrivial absorbent in Σ; moreover, notice that it is in fact a kernel. This raises the following open problems.

Problem 6.10. *Characterize signed digraphs having a unique minimum absorbent.*

Problem 6.11. *Characterize signed digraphs having a unique minimum kernel.*

However, one can prove the following result, in the same manner in which its graph theoretical analogue is established by Ore [9].

Theorem 6.12. *Let Σ be any signed digraph whose underlying digraph is strongly connected having a finite bound on the out-degrees and in-degrees of its vertices. Then, to any minimal absorbent in Σ there exists another disjoint from it.*

The following three lemmas will be required to establish Theorem 6.12.

Lemma 6.13. *Let Σ be any weakly connected digraph having a finite bound on the in-degrees of its vertices. Then, any absorbent in Σ contains a minimal one.*

Proof. We apply the principle of minimality. Let $\{D_i\}$ be any family of inclusion ordered absorbents with the intersection D_0. Suppose, for some $d \notin D_0$ there is no arc from d to some vertex in D_0. Then, there would be some set D_i such that there are no arcs from D_i to d, contrary to the definition of D_i. □

Lemma 6.14. *Any strongly connected digraph has an absorbent D such that its complement $\overline{D}$ is also an absorbent.*

Proof. Let Σ be any strongly connected digraph. Then, it has a maximal acyclic subdigraph T having a root c such that every other vertex v in Σ is reachable from c via a directed path in T from c to v. Then, the vertices in T and Σ fall into two disjoint sets D and $\overline{D}$ consisting respectively of the vertices with an even and odd directed distance from c in T. Evidently then D and $\overline{D}$ are absorbents in Σ. □

Note that Lemma 6.14 does not hold if the requirement of Σ being strongly connected is dropped from its hypothesis.

Lemma 6.15. *In any strongly connected digraph, the complement $\overline{D}$ of a minimal absorbent D is an absorbent.*

Proof. This follows from the fact that the property of a set of vertices in a digraph is *superhereditary*, in the sense that the superset of any absorbent is again an absorbent. □

Proof of Theorem 6.12. This follows now from Lemmas 6.13, 6.14 and Definition 6.1. □

Problem 6.16. *Study the properties of absorbents in signed digraphs whose underlying digraphs are not necessarily strongly connected.*

And so on .... As the saying goes in the scientific circles, "The best way to predict the future is to invent it!". Here lies a counter-example to the often chanted philosophy that 'simplicity is ultimate sophistication'!

Ackowledgement

I am deeply indebted to the referees for their incisive and thought-provoking comments that helped me to control the unwieldy flow of thoughts I was pouring from all sides to come out with this concise version of the original manuscript.

References

[1] B.D. Acharya and Mukti Acharya, New algebraic models of social systems, *Indian J. Pure Appl. Math.*, **17**(2) (1986), 150–168.

[2] B.D. Acharya and S. Joshi, *Semibalance in signed graphs*, Proc. First Internat. Instruc. Workshop (held in IASST, Guwahati: Junuary, 2004) on Cybernetics, Man and Systems (Ed.: D. Majumder), IASST, Guwahati, 2006.

[3] C. Berge, *Graphs and Hypergraphs*, Elsevier, North-Holland, Amsterdam, 1973.

[4] G. Chartrand, *Graphs as Mathematical Models*, Prindl, Weber and Schmidt, New York, 1977.

[5] F. Harary, R.Z. Norman and D.W. Cartwright, *Structural Models: An Introduction to the Theory of Directed Graphs*, John Wiley, New York, 1965.

[6] F. Harary, *Graph Theory*, Addison-Wesley Publ. Comp., Reading, Massachusettes, 1969.

[7] T. Haynes, S.T. Hedetneimi and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, 1998.

[8] T. Haynes, S.T. Hedetneimi and P.J. Slater, *Domination in Graphs: Advanced Topics*, Marcel Dekker, 1998.

[9] O. Ore, Theory of Graphs, *Amer. Math. Soc. Colloq.*, Publ. No.**38**, American Mathematical Society, Rhode Islands, 1962.

[10] E. Samapthkumar and V.N. Bhave, Group-valued graphs, *Karnatak University, J. Sci.*, **18** (1973), 325–328.

[11] H.B. Walikar, B.D. Acharya and E. Sampathkumar, *Recent Developments in the Theory of Domination in Graphs, MRI* Lecture Notes in Mathematics, No.1. The Mehta Research Institute of Mathematics and Mathematical Physics, Allahabad (India), 1979.

[12] H.B. Walikar, B.D. Acharya, K. Narayankar and H.G. Shekharappa, *Embedding index of nonindominable graphs*, In: Graphs, Combinatorics, Algorithms and Applications (Eds.: S. Arumugam, B.D. Acharya and S.B. Rao), Narosa Publ. House, Ltd., (2005), 173–179..

[13] T. Zaslavsky, Signed graph coloring, *Discrete Math.*, **39** (1982), 215–228.

[14] T. Zaslavsky, Signed graphs, *Discrete Appl. Math.*, 4(1) (1982), 47–74.

[15] T. Zaslavsky, A mathematical bibliography of signed and gain graphs and allied areas, *Electron. J. Combin.*, 8(1) (1998), 1–124.