Hypertournaments-scores, losing scores, total scores and degrees

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Abstract

Hypertournaments are generalizations of tournaments. We discuss the concept of scores, losing scores, total scores and degrees in k-hypertournaments and present characterizations of sequences to be score, losing score, total score and degree sequence of some k-hypertournament. We further discuss stronger upper and lower bounds for scores and losing scores. We extend the concept of scores, losing scores and degrees to bipartite hypertournaments. In the end we list some open problems in hypertournaments.

Keywords: hypertournament, score, losing score, degree, oriented hypergraph.
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1 Introduction

A tournament is a complete oriented graph. In a tournament the score of a vertex is its outdegree and the sequence of scores listed in non-decreasing order is called the score sequence. Landau [17] characterized the score sequences of a tournament.

Theorem 1.1. *A sequence $S = [s_i]_1^n$ of non-negative integers in non-deceasing order is a score sequence of a tournament if and only if for each $1 \leq k < n-1$,*

$$\sum_{i=1}^{k} s_i \geq \binom{k}{2} \tag{1}$$

with equality for $k = n$.

There are now several proofs of this fundamental result in tournament theory ranging from clever arguments involving gymastics with subscripts, arguments involving arc reorientations of properly chosen arcs, arguments by contradiction, arguments involving the idea of majorization to a constructive argument utilizing network flows and another one involving systems of distinct representatives, two proofs of Griggs and Reid [8] one a new direct proof and the second is self contained.

A k-hypergraph is a pair $H = (V, E)$, where V is the set of vertices and E is the set of k-subsets of V, called k-edges. A hypertournament is a generalization of a tournament, and have been studied by a number of authors, Assous [1], Barhut and Bialostocki [3], Frankl [7], Gutin and Yeo [9]. These authors raise the problem of extending the most important results on tournaments to hypertournaments. A k-hypertournament is a complete k-hypergraph with each k-edge endowed with an orientation, that is a linear arrangement of the vertices contained in the edge.

Given two non-negative integers n and k with $n \geq k > 1$, a k-hypertournament on n vertices is a pair (V, A), where v is a set of vertices with $|V| = n$ and A is a set of k-tuples of vertices, called arcs, such that for any k-subset S of V, A contains exactly one of the $k!$ k-tuples whose entries belong to S. Note that if $n < k$, then $A \neq \phi$, and this type is called a null hypertournament. Clearly a 2-hypertournament is an ordinary tournament.

Let $R = [r_1, r_2, \cdots, r_n]$ be an integer sequence. For $1 \leq i < j \leq n$, let $R(r_i^+, r_j^-) = [r_1, r_2, \cdots, r_i + 1, \cdots, r_j - 1, \cdots, r_n]$, $R(r_i^+, r_j^-) = [r'_1, r'_2, \cdots, r'_n]$ will denote a permutation of $R(r_i^+, r_j^-)$ such that $r'_1 \leq r'_2 \leq \cdots, \leq r'_n$. An (x, y)-path in H is a sequence $(x =)v_1 e_1 v_2 e_2 v_3 \cdots v_{t-1} e_{t-1} v_t$ $(= y)$ of distinct vertices $v_1, v_2, v_3, \cdots v_{t-1}, v_t$, $t = 1$, and distinct arcs $e_1, e_2, \cdots, e_{t-1}$ such that v_{i+1} lies on the last entry in e_i, $1 \leq i \leq t-1$. Let $e = (v_1, v_2, v_3, \cdots, v_k)$ be an arc in H and $i < j \leq k$, we denote

$$e(v_i, v_j) = (v_1, v_2, \cdots, v_{i-1}, v_j, v_{i+1}, \cdots, v_{j-1}, v_i, v_{j+1}, \cdots, v_k),$$

that is the new arc obtained from e by exchanging v_i and v_j in e.

Let S be the subset of V, we denote $H(S)$ to be the subhypertournament induced by S. A k-hypertournament H is strong if for any two vertices $x \in V$ and $y \in V$, H contains both an (x, y)-path and a (y, x)-path. A strong component of a k-hypertournament H is a maximal strong subhypertournament of H. For a pair of distinct vertices x and y in H, $A(x, y)$ denotes the set of all arcs of H in which x precedes y.

2 Scores and losing scores in k-hypertournaments

For a given vertex $v \in V$, the score $d_H^+(v)$ (or simply $d^+(v)$) of v is denoted by $d_H^+(v) = \left|\bigcup_{u \in V} A(v,u)\right|$, that is, the number of arcs containing v and in which v is not the last element. Similarly the losing score $d_H^-(v)$ (or simply $d^-(v)$) is the number of arcs containing v and in which v is the last element. The score sequence (losing score sequence) of a k-hypertournament is a non-decreasing sequence of non-negative integers $[s_1, s_2, \cdots, s_n]$ ($[r_1, r_2, \cdots, r_n]$) where s_i (r_i) is a score (losing score) of some vertex in H.

Zhou et al. [31] derived a result analogous to Landau theorem on tournaments, the following are the characterizations of losing score sequences and score sequences of k-hypertournaments.

Theorem 2.1. *Given two non-negative integers n and k with $n \geq k > 1$, a non-decreasing sequence $R = [r_i]_1^j$ of non-negative integers is a losing score sequence of some k-hypertournament if and only if for each j $(k \leq j \leq n)$,*

$$\sum_{i=1}^{j} r_i \geq \binom{j}{k} \tag{2}$$

with equality holding when $j = n$.

Theorem 2.2. *Given two non-negative integers n and k with $n \geq k > 1$, a non-decreasing sequence $S = [s_i]_1^j$ of non-negative integers is a score sequence of some k-hypertournament if and only if for each j $(k \leq j \leq n)$,*

$$\sum_{i=1}^{j} s_i \geq j\binom{n-1}{k-1} + \binom{n-j}{k} - \binom{n}{k} \tag{3}$$

with equality holding when $j = n$.

Recently Pirzada and Zhou [24] gave a new and short proof of Theorem 2 by using the contradiction argument. Further Zhou et. al [31] obtained a necessary and sufficient condition for a score sequence of a strong k-hypertournament. This result generalizes a theorem of Harary and Moser [11] about strong tournaments.

Theorem 2.3. [31] *A non-decreasing sequence $[s_i]_1^n$ $(1 \leq i \leq n)$, of non-negative integers is a score sequence of a strong k-hypertournament with $n > k$ if and only if for $k \leq j \leq n-1$*

$$\sum_{i=1}^{j} s_i > j\binom{n-1}{k-1} + \binom{n-j}{k} - \binom{n}{k} \tag{4}$$

and

$$\sum_{i=1}^{n} s_i = (k-1)\binom{n}{k}. \tag{5}$$

By using a similar argument, we have the following.

Theorem 2.4. *A non-decreasing sequence $[r_i]_1^n$ of non-negative integers with $n > k$ is a losing score sequence of a strong k-hypertournament if and only if for $k \leq j \leq n-1$,*

$$\sum_{i=1}^{j} r_i > \binom{j}{k}. \tag{6}$$

and

$$\sum_{i=1}^{n} r_i = \binom{n}{k}. \tag{7}$$

Koh and Ree [16] gave another proof of Theorem 2 based on finding a system of distinct representatives of a family of sets. This is similar to the proof of Bang and Sharp of Landau's theorem, using Hall's theorem. Koh and Ree [15, 16] defined a k-hypertournament matrix $M(H)$ as the incidence matrix of a k-hypertournament. This M(H) is the $n \times \binom{n}{k}$ matrix whose $(i,j)th$ entry is given by

$$m_{ij} = \begin{cases} 1 & \text{if vertex } v_i \text{ appears in arc } e_j \text{ but not as the last entry,} \\ -1 & \text{if vertex } v_i \text{ appears in arc } e_j \text{ as the last entry,} \\ 0 & \text{otherwise.} \end{cases}$$

Several properties of k-hypertournament matrix are given in Koh and Ree [15, 16].

We note that the concept of a k-hypertournament matrix differs from that of an h-tournament matrix as given by Kirkland [14] and Maybee and Pullman [18]. An h-hypertournament matrix is any square matrix A that satisfies $A + A^T = hh^T - I$, where I is the identity matrix and h is a non-zero column matrix.

Kayibi, Khan and Pirzada [12] have investigated the problem of randomly sampling all k-hypertournaments with a given score (equivalently losing score) sequence by constructing a Markov chain, denoted by $\mathfrak{M}_{mc}$ on the set $\mathcal{M}$ of all k-hypertournament matrices with the same score sequence S based on a switching operation. They proved that $\mathfrak{M}_{mc}$ is ergodic, has a uniform stationary distribution and is rapidly mixing. They then used $\mathfrak{M}_{mc}$ to construct a Markov chain $\mathfrak{H}_{mc}$ on the set $\mathcal{H}$ of all k-hypertournaments with score sequence S and proved ergodicity, uniform stationary distribution and rapid mixing of the new Markov chain.

3 Application of classical inequalities

Khan, Pirzada and Kayibi [13] investigated how classical inequalities can provide information about the behaviour of score and losing score sequences and hence the structure of hypertournaments. In the following results some famous inequalities from mathematical analysis such as Holder, Minkowski and Mahler have been used to obtain results on powers of scores and losing scores.

Theorem 3.1. *Let n and k be two non-negative integers with $n \geq k > 1$. If $R = [r_i]_1^n$, is a losing score sequence of a k-hypertournament, then for $1 < g < \infty$*

$$\sum_{i=1}^{j} r_i^g \geq \frac{j}{k^g}\binom{j-1}{k-1}^g \tag{8}$$

where $1 \leq j < n$. In particular

$$\sum_{i=1}^{n} r_i^g \geq \frac{j}{k^g}\binom{n-1}{k-1}^g \tag{9}$$

with equality if and only if the hypertournament is regular.

Theorem 3.2. *If $[s_i]_1^n$ is non-increasing, $[r_i]_1^n$ non-decreasing, then*

$$\left(\frac{\sum_{i=1}^{j} s_i^g}{j}\right)^{\frac{1}{g}} + \left(\frac{\sum_{i=1}^{j} r_i^g}{j}\right)^{\frac{1}{g}} \geq \binom{n-1}{k-1} \tag{10}$$

where $1 \leq j < n$. In particular

$$\left(\frac{\sum_{i=1}^{j} s_i^g}{n}\right)^{\frac{1}{g}} + \left(\frac{\sum_{i=1}^{j} r_i^g}{n}\right)^{\frac{1}{g}} \geq \binom{n-1}{k-1} \tag{11}$$

with equality if and only if the hypertournament is regular.

Furthermore for any positive integer $1 \leq j < n$, we have the following.

Theorem 3.3.

$$\prod_{i=1}^{j} s_i^{\frac{1}{j}} + \prod_{i=1}^{j} r_i^{\frac{1}{j}} \leq \binom{n-1}{k-1}. \tag{12}$$

In particular for $j = n$,

$$\prod_{i=1}^{n} s_i^{\frac{1}{n}} + \prod_{i=1}^{n} r_i^{\frac{1}{j}} \leq \binom{n-1}{k-1}, \tag{13}$$

with equality if and only if the hypertournament is regular.

Let R^n denote the n-dimensional Euclidean space. The inner product of two vectors $A = (a_1, a_2, \cdots, a_n)$ and $B = (b_1, b_2, \cdots, b_n)$ in R^n is defined as $\langle A, B\rangle = \sum\limits_{i=1}^{n} a_i b_i$. The next result [13] gives an upperbound for the inner product of score and losing score vector in R^n. The bound given in Theorem 9 is best possible in the sense that it is realized by regular hypertournaments.

Theorem 3.4. *If $S = [s_i]_1^n$, $R = [r_i]_1^n$, then*

$$\langle S, R\rangle \leq \frac{k-1}{k}\binom{n}{k}\binom{n-1}{k-1} \tag{14}$$

with equality if and only if the hypertournament is regular.

Koh and Ree [16] have given the following necessary and sufficient conditions for the existence of regular hypertournaments.

Theorem 3.5. *For $n = 3$ and $2 \leq k \leq n-1$, a regular k-hypertournament on n vertices exists if and only if n divides $\binom{n}{k}$.*

Using Theorem 3.1 for $g = 2$ we get a short proof of Theorem 3.4. Theorem 3.4 can also proved by using Theorem 3.3 together with the argument as used in proving Theorem 3.4 from Theorem 3.1.

4 Stronger inequalities for scores and losing scores

Brualdi and Shen [5] obtained stronger bounds for scores in tournaments, which indeed give better necessary and sufficient conditions for score sequences in tournaments. The following results by Pirzada, Khan and Kayibi [27] give bounds for $\sum\limits_{i\in I} r_i$.

Theorem 4.1. *Given non-negative integers n and k with $n \geq k > 1$, a sequence $R = [r_i]_1^n$, of non-negative integers in non-decreasing order is a losing score sequence of some k-hypertournament if and only if for every subset $I \subseteq [n]$,*

$$\sum_{i\in I} r_i \geq \frac{1}{2}\sum_{i\in I}\binom{i-1}{k-1} + \frac{1}{2}\binom{|I|}{k} \tag{15}$$

with equality when $|I| = n$.

Theorem 4.2. *$[r_i]_1^n$ is a losing score sequence if and only if*

$$\sum_{i\in I} r_i \leq \frac{1}{2}\sum_{i\in I}\binom{i-1}{k-1} + \frac{1}{2}\binom{n}{k} - \frac{1}{2}\binom{n-|I|}{k} \tag{16}$$

with equality when $|I| = n$.

Theorem 4.3. *If* $R = [r_i]_1^n$ *is a losing score sequence of a k-hypertournament then for each* $1 \le i \le n$, *we have*

$$\frac{1}{2}\binom{i-1}{k-1} \le r_i \le \frac{1}{2}\binom{i-1}{k-1} + \frac{1}{2}\binom{n-1}{k-1}. \tag{17}$$

Since $s_{n+1-i} + r_i = \binom{n-1}{k-1}$, for $I \subseteq [n] = \{1, 2, \cdots, n\}$, we have

$$\sum_{i \in I} s_{n+1-i} + \sum_{i \in I} r_i = \sum_{i \in I} \binom{n-1}{k-1} \tag{18}$$

Or

$$\sum_{i \in I} s_{n+1-i} = |I| \binom{n-1}{k-1} - \sum_{i \in I} r_i \tag{19}$$

So by using Theorems 4.1 and 4.2, we obtain the following.

Theorem 4.4. $S = [s_i]_1^n$ *in non-decreasing order is a score sequence if and only if for every subset* $I \subseteq [n]$

$$\sum_{i \in I} s_{n+1-i} \le |I| \binom{n-1}{k-1} - \frac{1}{2} \sum_{i \in I} \binom{i-1}{k-1} - \frac{1}{2}\binom{|I|}{k} \tag{20}$$

with equality when $|I| = n$.

Theorem 4.5. $S = [s_i]_1^n$ *is a score sequence if and only if*

$$\sum_{i \in I} s_{n+1-i} \ge |I| \binom{n-1}{k-1} - \frac{1}{2} \sum_{i \in I} \binom{n}{k} + \frac{1}{2}\binom{n-|I|}{k} \tag{21}$$

with equality when $|I| = n$.

The following is a consequence of Theorems 4.3 and 4.4.

Theorem 4.6. *If* $S = [s_i]_1^n$ *is a score sequence of a k-hypertournament, then for each* $1 \le i \le n$

$$\frac{1}{2}\binom{n-1}{k-1} - \frac{1}{2} \le s_{n+1-i} \le \binom{n-1}{k-1} - \frac{1}{2}\binom{i-1}{k-1}. \tag{22}$$

5 Total scores in k-hypertournaments

The total score of a vertex v_i is defined as $t_i = s_i - r_i$. The total score sequence is the sequence of total scores arranged in non-increasing order. Koh and Ree [16] characterized total score sequences in hypertournaments.

Theorem 5.1. *A non-decreasing sequence of integers $[t_i]_1^n$ is a total score sequence of a k-hypertournament of order n if and only if t_i has the same parity as that of $\binom{n-1}{k-1}$ for each $i = 1, 2, \cdots, n$,*

$$\sum_{i=1}^{j} t_i \leq j\binom{n-1}{k-1} - 2\binom{j}{k} \tag{23}$$

with equality when $j = n$.

Using the improved bounds for scores and losing scores presented in Section 4, Pirzada et al. [27] obtained the following stronger upper and lower bounds for total scores.

Theorem 5.2. *A non-decreasing $T = [t_i]_1^n$ is a total score sequence if and only if t_i has the same parity as $\binom{n-1}{k-1}$ for each $i = 1, 2, \cdots, n$ and for every $|I| = [n]$*

$$\begin{aligned} &|I|\binom{n-1}{k-1} - \sum_{i\in I}\binom{i-1}{k-1} - \binom{n}{k} + \binom{n-|I|}{k} \leq \sum_{i\in I} t_i \\ &\leq |I|\binom{n-1}{k-1} - \sum_{i\in I}\binom{i-1}{k-1} - \binom{|I|}{k}. \end{aligned} \tag{24}$$

Theorem 5.3. *If a non-increasing sequence $T = [t_i]_1^n$ is a total score sequence, then*

$$-\binom{i-1}{k-1} \leq t_i \leq \binom{n-1}{k-1} - \binom{i-1}{k-1}. \tag{25}$$

Theorem 5.4. *A non-increasing sequence $T = [t_i]_1^n$ is a total score sequence of a strong k-hypertournament if and only if t_i has the same parity as that of $\binom{n-1}{k-1}$ for each $i = 1, 2, \cdots, n$*

$$\sum_{i=1}^{j} t_i < j\binom{n-1}{k-1} - 2\binom{j}{k} \tag{26}$$

for $1 \leq j \leq n-1$ and $\sum_{i=1}^{n} t_i = (k-2)\binom{n}{k}$.

6 Scores in bipartite hypertournaments

Bipartite hypergraph is a generalization of a bipartite graph. If $U = \{u_1, u_2, \cdots, u_n\}$ and $V = \{v_1, v_2, \cdots, v_n\}$ are vertex sets, then the edge of a bipartite hypergraph is a subset of the vertex sets which contains atleast one vertex from U and atleast one vertex from V. If an edge has exactly h vertices from U and has exactly k vertices from V, it is called an $[h, k]$-edge. An $[h, k]$-bipartite hypergraph is a bipartite hypergraph all of whose edges are $[h, k]$-edges. An $[h, k]$ bipartite hypertournament (or $[h, k]$-BH) is a complete $[h, k]$-bipartite hypergraph with each $[h, k]$-edge endowed with an orientation, that is, a linear arrangement of the vertices contained in the hyperedge.

Equivalently, given non-negative integers m, n, h and k with $m \geq h \geq 1$ and $n \geq k \geq 1$, an $[h, k]$-bipartite hypertournament of order $m + n$ consists of two vertex sets U and V with $|U| = m$ and $|V| = n$, together with an arc set E, a set of $(h+k)$ tuples of vertices, with exactly h vertices from U and exactly k vertices from V, called arcs, such that for any h-subset U_1 of U and k-subset V_1 of V, E contains exactly one of the $(h+k)!$ $(h+k)$-tuples whose h entries belong to U_1 and k entries belong to V_1.

For a given $u_i \in U$, the score $d^+(u_i)$ is the number of $[h, k]$-arcs containing u_i and in which u_i is not the last element. The losing score $d^-(u_i)$ is the number of $[h, k]$-arcs containing u_i and in which u_i is the last element. Similarly we define $d^+(v_j)$ and $d^-(v_j)$ for $v_j \in V$. The losing score lists of an $[h, k]$-bipartite hypertournament is a pair of non-decreasing sequence of non-negative integers $A = [a_i]_1^m$ and $B = [b_j]_1^n$ ($C = [c_i]_1^m$ and $D = [d_j]_1^n$) where a_i (c_i) is the losing score (score) of some vertex $u_i \in U$ and b_j (d_j) is the losing score (score) of some vertex $v_j \in V$.

The following two results br Pirzada et al. [23] provide characterizations of losing score lists and score lists in $[h, k]$-bipartite hypertournaments. These are similar to the characterizations of score lists in bipartite tournaments by Beineke and Moon [4].

Theorem 6.1. *Given non-negative integers m, n, h and k with $m \geq h \geq 1$ and $n \geq k \geq 1$, the non-decreasing sequence $A = [a_i]_1^m$ and $B = [b_j]_1^n$ of non-negative integers are the losing score lists of an $[h, k]$-BH if and only if*

$$\sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j \geq \binom{p}{h}\binom{q}{k} \tag{27}$$

with the equality when $p = m$ and $q = n$.

Theorem 6.2. *$C = [c_i]_1^m$ and $D = [d_j]_1^n$ are the score lists of an $[h, k]$-BH* *if and only if for each p and q ,*

$$\sum_{i=1}^{p} c_i + \sum_{j=1}^{q} d_j \geq p\binom{m-1}{h-1}\binom{n}{k} + q\binom{m}{h}\binom{n-1}{k-1} + \binom{m-p}{h}\binom{n-q}{k} - \binom{m}{h}\binom{n}{k} \tag{28}$$

with the equality when $p = m$ and $q = n$.

Now if an edge has atmost h vertices from U and atmost k vertices from V, it is called (h,k)-edge . An (h,k)-bipartite hypertournament ((h,k)-BH) is a complete (h,k)-bipartite hypergraph. The following characterizations of losing score lists and score lists in (h,k)-BH can be seen in Pirzada and Zhou [19].

Theorem 6.3. *$A = [a_i]_1^m$ and $B = [b_j]_1^n$ are losing score lists of an (h,k)-BH if and only if*

$$\sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j \geq \sum_{i=1}^{h}\sum_{i=1}^{k} \binom{p}{i}\binom{q}{j}. \tag{29}$$

with the equality when $p = m$ and $q = n$.

Theorem 6.4. *$C = [c_i]_1^m$ and $D = [d_j]_1^n$ are the score lists of an (h,k)-BH if and only if*

$$\sum_{i=1}^{p} c_i + \sum_{j=1}^{q} d_j \geq \sum_{i=1}^{h}\sum_{i=1}^{k} p\binom{m-1}{i-1}\binom{n}{j} + q\binom{m}{i}\binom{n-1}{j-1} + \binom{m-p}{i}\binom{n-q}{j} - \binom{m}{i}\binom{n}{j} \tag{30}$$

with the equality when $p = m$ and $q = n$.

7 Degrees in k-hypertournaments

Let $a = (x_1, \cdots, x_k)$ be an arc of a k-hypertournament H. We call x_i the ith entry of a , x_{i+1} the (i+1)th entry of a , x_{i+1} in the successor of x_i and x_i the predecessor of x_{i+1}. Clearly x_k has no successor and x_1 has no predecessor in a. Define a function ρ on a by

$$(x,a) = \begin{cases} k-i & \text{if } x \in a \text{ and } x \text{ is the ith entry of } a \text{ ,} \\ 0 & \text{if } x \notin a. \end{cases}$$

For $v \in V(H)$, we denote $d_H^+ = \sum_{a \in H} \rho(v, a)$ or simply $d^+(v)$ is the degree of v in H. The degree sequence of a k-hypertournament is a non-decreasing sequence of non-negative integers $[d_i]_1^n$, where each d_i is the degree of some vertex in $V(H)$. Zhou and Zhang [32] raised the following conjecture and proved the case $k = 3$. This was settled in affirmative by Wang and Zhou [30].

Theorem 7.1. *Given two positive integers n and k with with $n > k > 1$, a non-decreasing sequence $D = [d_i]_1^n$ of non-negative integers is a degree sequence of some k-hypertournament if and only if*

$$\sum_{i=1}^{j} d_i \geq \binom{j}{2}\binom{n-2}{k-2} \tag{31}$$

for all $1 \leq j \leq n$, with equality for $j = n$.

Pirzada and Zhou [26] extended the concept of degree in a k-hypertournament to k-bipartite hypertournament and obtained the following characterization of degree lists.

Theorem 7.2. *Given non-negative integers m, n and k with $m+n > k > 3$, let $A = [a_i]_1^m$ and $B = [b_j]_1^n$ be non-decreasing sequences of non-negative integers. Then A and B are degree lists of some k-bipartite hypertournament if and only if*

$$\sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j \geq \binom{p+q}{2}\binom{m+n-2}{k-2} - \binom{p}{2}\binom{m-2}{k-2} - \binom{q}{2}\binom{n-2}{k-2} \tag{32}$$

with the equality when $p = m$ and $q = n$.

The extension of scores and losing scores to tripartite hypertournaments can be seen in Pirzada et al. [20] multipartite hypertournaments in Pirzada [25]. The generalization of scores and degrees to oriented k-hypergraphs can be found in [21, 33], while as degree sequences of k-multi hypertournaments are due to Pirzada [22].

8 Open problems

1. The set of distinct scores (losing scores) is called the score (losing score) set of a k-hypertournament. The characterization of score sets in tournaments can be found in [10, 28]. Characterize score sets and losing score sets in hypertournaments.

2. A survey of kings in tournaments can be found in Reid [29]. Characterize kings and serfs in hypertournaments.

3. We can find characterization of self converse score sequences in tournaments is given by Eplett [6] and the characterization of uniquely realizable score sequences in tournaments can be seen in Avery [2]. Characterize self converse and uniquely realizable score sequences in hypertournaments.

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