P_4-Factorization of Cartesian product of complete graphs

R. SAMPATHKUMAR AND P. KANDAN
Department of Mathematics
Annamalai University
Annamalainagar 608 002, Tamil Nadu, India.
e-mail: sampathmath@gmail.com, kandan2k@gmail.com

Abstract

In 1996, Muthusamy and Paulraja have conjectured that for $k \geq 3$, the Cartesian product $K_m \square K_n$ has a P_k-factorization if and only if $mn \equiv 0 \bmod k$ and $2(k-1)|k(m+n-2)$. Recently, Chitra and Muthusamy have partially settled this conjecture for $k = 3$. In this paper, it is shown that for $k = 4$ above conjecture is true if $(m \bmod 12, n \bmod 12) \in \{(0,2),(2,0),(0,8),(8,0),(2,6),(6,2),(6,8),(8,6),(4,4)\}$. The left over cases for $k = 4$ are $(m \bmod 12, n \bmod 12) \in \{(0,5),(5,0),(0,11),(11,0),(1,4),(4,1),(3,8),(8,3),(4,7),(7,4),(4,10),(10,4),(8,9),(9,8),(10,10)\}$.

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1 Introduction

For terminology and notations not defined here we refer Balakrishnan and Ranganathan [1] and we consider finite undirected simple graphs only. We use usual notations: P_n for the path on n vertices, K_n for the complete graph on n vertices, and $K_m(n)$ for the complete m-partite graph with n vertices in each part. Unless otherwise mentioned, $V(K_n) = \{1, 2, \ldots, n\}$.

A *decomposition* of a graph G is a collection $\mathcal{D} = \{H_1, H_2, \ldots, H_k\}$ of nonempty subgraphs of G such that the edge sets $E(H_1), E(H_2), \ldots, E(H_k)$ form a partition of the edge set $E(G)$; we denote this by $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$. A decomposition $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ of G is a *factorization* of G if each F_i is a spanning subgraph of G; in addition if each $F_i \cong F$, then we say that F *factorizes* G and denote this by $F \| G$.

The *Cartesian product* $G_1 \Box G_2$ of two graphs G_1 and G_2 is the simple graph with $V(G_1) \times V(G_2)$ as its vertex set and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G_1 \Box G_2$ if and only if either $u_1 = u_2$ and v_1 is adjacent to v_2 in G_2, or u_1 is adjacent to u_2 in G_1 and $v_1 = v_2$.

In [3], Muthusamy and Paulraja posed the following conjecture.

Conjecture 1.1. [3] *For* $k \geq 3$, $P_k \| (K_m \Box K_n)$ *if and only if* $mn \equiv 0 \, mod \, k$ *and* $2(k-1) | k(m+n-2)$.

Recently, Chitra and Muthusamy [2] have partially settled this conjecture for $k = 3$. For $k = 4$, above conjecture is:

Conjecture 1.2. $P_4 \| (K_m \Box K_n)$ *if and only if* $mn \equiv 0 \, mod \, 4$ *and* $3 | (m+n-2)$.

If $mn \equiv 0 \, mod \, 4$ and $3|(m+n-2)$, then $(m \, mod \, 12, n \, mod \, 12) \in \{(0,2), (2,0), (0,5), (5,0), (0,8), (8,0), (0,11), (11,0), (1,4), (4,1), (2,6), (6,2), (3,8), (8,3), (4,7), (7,4), (4,10), (10,4), (6,8), (8,6), (8,9), (9,8), (4,4), (10,10)\}$. By symmetry, assume that $(m \, mod \, 12, n \, mod \, 12) \in \{(0,2), (0,5), (0,8), (0,11), (1,4), (2,6), (3,8), (7,4), (10,4), (6,8), (8,9), (4,4), (10,10)\}$. In this paper, we prove Conjecture 2 for $(m \, mod \, 12, n \, mod \, 12) \in \{(0,2), (0,8), (2,6), (6,8), (4,4)\}$.

2 Results

Let G_1 and G_2 be graphs with n_1 and n_2 vertices, respectively. Consider the Cartesian product $G_1 \Box G_2$. For $u \in V(G_1)$, the subgraph induced by $\{(u, v) : v \in V(G_2)\}$, in $G_1 \Box G_2$, is called a G_2*-layer* of $G_1 \Box G_2$; and for $v \in V(G_2)$, the subgraph induced by $\{(u, v) : u \in V(G_1)\}$, in $G_1 \Box G_2$, is called a G_1*-layer* of $G_1 \Box G_2$. Then, in $G_1 \Box G_2$, we have n_1 disjoint G_2-layers and n_2 disjoint G_1-layers. Let H_1 and H_2 be spanning subgraphs of G_1 and G_2, respectively; and let $F_1 = G_1 - E(H_1)$ and $F_2 = G_2 - E(H_2)$. Then, in $G_1 \Box G_2$, we have n_1 disjoint copies of H_2 (each belonging to a G_2-layer), n_2 disjoint copies of H_1 (each belonging to a G_1-layer), and the removal of the edges of these copies from $G_1 \Box G_2$ results in $F_1 \Box F_2$. Hence, $G_1 \Box G_2 = n_1 H_2 \oplus n_2 H_1 \oplus (F_1 \Box F_2)$. Observe that if $F \| H_2$, $F \| H_1$, and $F \| (F_1 \Box F_2)$, then $F \| (G_1 \Box G_2)$. We use this observation and the following Theorem A in the proof of Theorems 1 and 2. The removal of the edges of n_1 disjoint copies of H_2 (each belonging to a G_2-layer) from $G_1 \Box G_2$ results in $G_1 \Box F_2$. Hence, $G_1 \Box G_2 = n_1 H_2 \oplus (G_1 \Box F_2)$. We also use this observation in the proofs.

Theorem A. [4] $P_4 \| K_m(n)$ *if and only if* $mn \equiv 0 \, mod \, 4$ *and* $2(m-1)n \equiv 0 \, mod \, 3$.

Theorem 2.1. *If* $(m \bmod 12, n \bmod 12) = (4,4)$, *then* $P_4 \| (K_m \square K_n)$.

Proof. As $(m \bmod 12, n \bmod 12) = (4,4)$, $m = 12r+4$ and $n = 12s+4$ for some nonnegative integers r and s. Observe that

$$\begin{aligned} K_m \square K_n &= K_{12r+4} \square K_{12s+4} \\ &= (12r+4)K_{3s+1}(4) \oplus (12s+4)K_{3r+1}(4) \oplus \\ &\qquad (3r+1)(3s+1)(K_4 \square K_4). \end{aligned}$$

As $(3s+1)(4) \equiv 0 \bmod 4 \equiv (3r+1)(4)$ and $2(3s)(4) \equiv 0 \bmod 3 \equiv 2(3r)(4)$, by Theorem A, $P_4 \| K_{3s+1}(4)$ and $P_4 \| K_{3r+1}(4)$. Hence $P_4 \| (12r+4)K_{3s+1}(4)$ and $P_4 \| (12s+4)K_{3r+1}(4)$.

As $K_4 \square K_4 = 4K_4 \oplus 4K_4$, and as $P_4 \| K_4$, we have $P_4 \| (K_4 \square K_4)$. Hence, $P_4 \| (3r+1)(3s+1)(K_4 \square K_4)$.

This completes the proof. □

Lemma 2.2. $P_4 \| (K_6 \square K_2)$.

Proof.

$$\begin{aligned} G_1 &= (1,1)(2,1)(2,2)(1,2) \oplus (3,1)(4,1)(4,2)(3,2) \oplus (5,1)(6,1)(6,2)(5,2), \\ G_2 &= (6,1)(1,1)(1,2)(6,2) \oplus (3,1)(5,1)(2,1)(4,1) \oplus (3,2)(5,2)(2,2)(4,2), \\ G_3 &= (4,1)(5,1)(5,2)(4,2) \oplus (1,1)(3,1)(6,1)(2,1) \oplus (1,2)(3,2)(6,2)(2,2), \\ G_4 &= (2,1)(3,1)(3,2)(2,2) \oplus (6,1)(4,1)(1,1)(5,1) \oplus (6,2)(4,2)(1,2)(5,2) \end{aligned}$$

is a P_4-factorization of $K_6 \square K_2$. □

Theorem 2.3. *If* $(m \bmod 12, n \bmod 12) = (0,8)$, *then* $P_4 \| (K_m \square K_n)$.

Proof. As $(m \bmod 12, n \bmod 12) = (0,8)$, $m = 12r$ and $n = 12s+8$ for some positive integer r and some nonnegative integer s. Observe that

$$\begin{aligned} K_m \square K_n &= K_{12r} \square K_{12s+8} \\ &= (12r)K_{6s+4}(2) \oplus (12s+8)K_r(12) \oplus (r)(6s+4)(K_{12} \square K_2). \end{aligned}$$

As $(6s+4)(2) \equiv 0 \bmod 4 \equiv (r)(12)$ and $2(6s+3)(2) \equiv 0 \bmod 3 \equiv 2(r-1)(12)$, by Theorem A, $P_4 \| K_{6s+4}(2)$ and $P_4 \| K_r(12)$. Hence $P_4 \| (12r)K_{6s+4}(2)$ and $P_4 \| (12s+8)K_r(12)$.

Note that $K_{12} \square K_2 = (2K_6 \oplus K_2(6)) \square K_2 = 2(K_6 \square K_2) \oplus 2K_2(6)$. By Lemma 2.2, $P_4 \| (K_6 \square K_2)$. Since $2(6) \equiv 0 \bmod 4$ and $2(1)(6) \equiv 0 \bmod 3$, we have by Theorem A, $P_4 \| K_2(6)$. Hence $P_4 \| (K_{12} \square K_2)$. Consequently, $P_4 \| (r)(6s+4)(K_{12} \square K_2)$.

This completes the proof. □

A *near 1-factor* of a graph G with $2k+1$ vertices is a set of k edges that cover all but one vertex. A *near 1-factorization* of G is a decomposition of G into near 1-factors. For every k, the complete graph K_{2k+1} has a near 1-factorization.

Lemma 2.4. *For any nonnegative integer* r, $P_4 \| (K_{12r+6} \Box K_2)$.

Proof. By Lemma 2.2, assume that $r \geq 1$. For $p \in \{1, 2, \ldots, 2r+1\}$, identify six vertices $6p-5, 6p-4, 6p-3, 6p-2, 6p-1, 6p$ of K_{12r+6} into a single vertex v_p, and consider the complete graph K_{2r+1} with vertex set $\{v_1, v_2, \ldots, v_{2r+1}\}$. Let $\mathbb{F}$ be a near 1-factorization of K_{2r+1}. If $F = \{v_{2i}v_{2i+1} : i \in \{1, 2, \ldots, r\}\}$ is a near 1-factor of K_{2r+1} belonging to $\mathbb{F}$, then in $K_{12r+6} \Box K_2$ we associate $2r$ disjoint subgraphs of K_{12r+6}-layers each isomorphic to $K_{6,6}$ with bipartition $(\{(12i-5, k), (12i-4, k), (12i-3, k), (12i-2, k), (12i-1, k), (12i, k)\}, \{(12i+1, k), (12i+2, k), (12i+3, k), (12i+4, k), (12i+5, k), (12i+6, k)\})$, $i \in \{1, 2, \ldots, r\}$, $k \in \{1, 2\}$, and one subgraph isomorphic to $K_6 \Box K_2$ with vertex set $\{(1,1), (2,1), (3,1), (4,1), (5,1), (6,1), (1,2), (2,2), (3,2), (4,2), (5,2), (6,2)\}$ and prism edges $(j,1)(j,2)$, $j \in \{1,2,3,4,5,6\}$. As both the graphs $K_{6,6}$ $(= K_2(6))$ and $K_6 \Box K_2$ (see Lemma 2.2) are P_4-factorable into four P_4-factors, the above associated spanning subgraph of $K_{12r+6} \Box K_2$ is P_4-factorable into four P_4-factors. Since the number of near 1-factors in $\mathbb{F}$ is $2r+1$, we have obtained $4(2r+1)$ edge-disjoint P_4-factors in $K_{12r+6} \Box K_2$. Hence, $P_4 \| (K_{12r+6} \Box K_2)$. □

Theorem 2.5. *If* $(m \bmod 12, n \bmod 12) = (6, 8)$, *then* $P_4 \| (K_m \Box K_n)$.

Proof. As $(m \bmod 12, n \bmod 12) = (6, 8)$, $m = 12r + 6$ and $n = 12s + 8$ for some nonnegative integers r and s. Observe that

$$\begin{aligned} K_m \Box K_n &= K_m \Box K_{12s+8} \\ &= K_m \Box ((6s+4)K_2 \oplus K_{6s+4}(2)) \\ &= (6s+4)(K_m \Box K_2) \oplus mK_{6s+4}(2). \end{aligned}$$

By Lemma 2.4, $P_4 \| (K_m \Box K_2)$.

$(6s+4)(2) \equiv 0 \bmod 4$ and $2(6s+3)(2) \equiv 0 \bmod 3$ implies by Theorem A that $P_4 \| K_{6s+4}(2)$.

This completes the proof. □

Theorem 2.6. *If* $(m \bmod 12, n \bmod 12) = (2, 6)$, *then* $P_4 \| (K_m \Box K_n)$.

Proof. As $(m \bmod 12, n \bmod 12) = (2, 6)$, $m = 12r + 2$ and $n = 12s + 6$ for some nonnegative integers r and s.

For $p \in \{1, 2, \ldots, 2s+1\}$, identify six vertices $6p-5, 6p-4, 6p-3, 6p-2, 6p-1, 6p$ of K_{12s+6} into a single vertex v_p, and consider the complete graph K_{2s+1} with vertex set $\{v_1, v_2, \ldots, v_{2s+1}\}$. Let $\mathbb{F}$ be a near 1-factorization of K_{2s+1}. If $F = \{v_{2i}v_{2i+1} : i \in \{1,2,\ldots,s\}\}$ is a near 1-factor of K_{2s+1} belonging to $\mathbb{F}$, then in $K_{12r+2}\Box K_{12s+6}$ we associate:

(i) ms disjoint subgraphs of K_{12s+6}-layers each isomorphic to $K_{6,6}$ with bipartition $(\{(k,12j-5), (k,12j-4), (k,12j-3), (k,12j-2), (k,12j-1), (k,12j)\}, \{(k,12j+1), (k,12j+2), (k,12j+3), (k,12j+4), (k,12j+5), (k,12j+6)\})$, $k \in \{1,2,\ldots,m\}$, $j \in \{1,2,\ldots,s\}$, (note that first coordinate is from 1 to m and second coordinate is from 7 to n)

(ii) $6r$ disjoint subgraphs of K_{12r+2}-layers each isomorphic to $K_{6,6}$ with bipartition $(\{(12i-9,k), (12i-8,k), (12i-7,k), (12i-6,k), (12i-5,k), (12i-4,k)\}, \{(12i-3,k), (12i-2,k), (12i-1,k), (12i,k), (12i+1,k), (12i+2,k)\})$, $i \in \{1,2,\ldots,r\}$, $k \in \{1,2,\ldots,6\}$, (note that first coordinate is from 3 to m and second coordinate is from 1 to 6) and

(iii) one subgraph isomorphic to $K_6\Box K_2$ with vertex set $\{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)\}$ and prism edges $(1,j)(2,j)$, $j \in \{1,2,3,4,5,6\}$.

As both the graphs $K_{6,6}$ and $K_6\Box K_2$ are P_4-factorable into four P_4-factors, the above associated spanning subgraph of $K_{12r+2}\Box K_{12s+6}$ is P_4-factorable into four P_4-factors. Since the number of near 1-factors in $\mathbb{F}$ is $2s+1$, we have obtained $4(2s+1)$ edge-disjoint P_4-factors in $K_{12r+2}\Box K_{12s+6}$.

Identify the two vertices $1,2$ of K_{12r+2} into a single vertex x and for $p \in \{1,2,\ldots,2r\}$, identify six vertices $6p-3, 6p-2, 6p-1, 6p, 6p+1, 6p+2$ of K_{12r+2} into a single vertex y_p. Consider the complete graph K_{2r+1} with vertex set $\{x, y_1, y_2, \ldots, y_{2r}\}$. $\{y_1y_2, y_3y_4, \ldots, y_{2r-1}y_{2r}\}$ is a near 1-factor of K_{2r+1} and K_{2r+1} is near 1-factorable implies that $K_{2r+1} - \{y_1y_2, y_3y_4, \ldots, y_{2r-1}y_{2r}\}$ is near 1-factorable. Let $\mathbb{H}$ be a near 1-factorization of $K_{2r+1} - \{y_1y_2, y_3y_4, \ldots, y_{2r-1}y_{2r}\}$. If H is a near 1-factor of K_{2r+1} belonging to $\mathbb{H}$, then in $K_{12r+2}\Box K_{12s+6}$ we associate the following:

(i) for an edge of the form y_iy_j in H, n disjoint subgraphs of K_{12r+2}-layers each isomorphic to $K_{6,6}$ with bipartition $(\{(6i-3,k), (6i-2,k), (6i-1,k), (6i,k), (6i+1,k), (6i+2,k)\}, \{(6j-3,k), (6j-2,k), (6j-1,k), (6j,k), (6j+1,k), (6j+2,k)\})$, $k \in \{1,2,\ldots,n\}$,

(ii) for the edge of the form xy_q in H, n disjoint subgraphs of K_{12r+2}-layers each isomorphic to $K_4(2)$ with partite sets $\{(1,k),(2,k)\}, \{(6q-3,k), (6q-2,k)\}, \{(6q-1,k),(6q,k)\}, \{(6q+1,k),(6q+2,k)\}$, $k \in \{1,2,\ldots,n\}$, and

(iii) for the vertex y_t ($t \in \{1,2,\ldots,2r\}$) not covered by H, $3(2s+1)$ subgraphs each isomorphic to $K_2\Box K_6$ with

(a) vertex set $\{(6t-3,6\ell-5), (6t-3,6\ell-4), (6t-3,6\ell-3), (6t-3,6\ell-2), (6t-3,6\ell-1), (6t-3,6\ell), (6t-2,6\ell-5), (6t-2,6\ell-4), (6t-2,6\ell-3), (6t-2,6\ell-2), (6t-2,6\ell-1), (6t-2,6\ell)\}$ and prism edges $(6t-3,6\ell-5)(6t-2,6\ell-5)$, $(6t-3,6\ell-4)(6t-2,6\ell-4)$, $(6t-3,6\ell-3)(6t-2,6\ell-3)$, $(6t-3,6\ell-2)(6t-2,6\ell-2)$, $(6t-3,6\ell-1)(6t-2,6\ell-1)$, $(6t-3,6\ell)(6t-2,6\ell)$,

(b) vertex set $\{(6t-1,6\ell-5), (6t-1,6\ell-4), (6t-1,6\ell-3), (6t-1,6\ell-2),$

$(6t-1,6\ell-1),(6t-1,6\ell),(6t,6\ell-5),(6t,6\ell-4),(6t,6\ell-3)$, $(6t,6\ell-2)$, $(6t,6\ell-1),(6t,6\ell)\}$ and prism edges $(6t-1,6\ell-5)(6t,6\ell-5)$, $(6t-1,6\ell-4)(6t,6\ell-4)$, $(6t-1,6\ell-3)(6t,6\ell-3)$, $(6t-1,6\ell-2)(6t,6\ell-2)$, $(6t-1,6\ell-1)(6t,6\ell-1)$, $(6t-1,6\ell)(6t,6\ell)$,

(c) vertex set $\{(6t+1,6\ell-5),(6t+1,6\ell-4),(6t+1,6\ell-3),(6t+1,6\ell-2),(6t+1,6\ell-1),(6t+1,6\ell),(6t+2,6\ell-5),(6t+2,6\ell-4),(6t+2,6\ell-3),(6t+2,6\ell-2),(6t+2,6\ell-1),(6t+2,6\ell)\}$ and prism edges $(6t+1,6\ell-5)(6t+2,6\ell-5)$, $(6t+1,6\ell-4)(6t+2,6\ell-4)$, $(6t+1,6\ell-3)(6t+2,6\ell-3)$, $(6t+1,6\ell-2)(6t+2,6\ell-2)$, $(6t+1,6\ell-1)(6t+2,6\ell-1)$, $(6t+1,6\ell)(6t+2,6\ell)$, where $\ell\in\{1,2,\ldots,2s+1\}$.

As all the three graphs $K_{6,6}$, $K_2\Box K_6$ and $K_4(2)$ (see Theorem A and Lemma 2.2) are P_4-factorable into four P_4-factors, the above associated spanning subgraph of $K_{12r+2}\Box K_{12s+6}$ is P_4-factorable into four P_4-factors. Since the number of near 1-factors in $\mathbb{H}$ is $2r$, we have obtained $4(2r)$ edge-disjoint P_4-factors in $K_{12r+2}\Box K_{12s+6}$.

Thus we have obtained $4(2r+2s+1)$ edge-disjoint P_4-factors in $K_{12r+2}\Box K_{12s+6}$. This completes the proof. □

Theorem 2.7. *If* $(m \bmod 12, n \bmod 12)=(0,2)$, *then* $P_4\|(K_m\Box K_n)$.

Proof. As $(m \bmod 12, n \bmod 12)=(0,2)$, $m=12r$ and $n=12s+2$ for some positive integer r and some nonnegative integer s. Observe that

$$\begin{aligned}K_m\Box K_n &= K_{12r}\Box K_{12s+2}\\ &= (K_{2r}(6)\oplus 2rK_6)\Box K_{12s+2}\\ &= (12s+2)K_{2r}(6)\oplus(2r)(K_6\Box K_{12s+2}).\end{aligned}$$

$(2r)(6)\equiv 0 \bmod 4$ and $2(2r-1)(6)\equiv 0 \bmod 3$ implies by Theorem A that $P_4\|K_{2r}(6)$. By Theorem 2.6, $P_4\|(K_6\Box K_{12s+2})$.

This completes the proof. □

3 Observations

1. If $m \bmod 12\in\{1,7,10\}$, $n \bmod 12=4$, and $P_4\|(K_m\Box K_4)$, then $P_4\|(K_m\Box K_n)$.

Proof. As $n \bmod 12=4$, $n=12s+4$ for some nonnegative integer s. Observe that

$$\begin{aligned}K_m\Box K_n &= K_m\Box K_{12s+4}\\ &= K_m\Box((3s+1)K_4\oplus K_{3s+1}(4))\\ &= (3s+1)(K_m\Box K_4)\oplus mK_{3s+1}(4).\end{aligned}$$

$(3s+1)(4) \equiv 0 \, mod\, 4$ and $2(3s)(4) \equiv 0 \, mod\, 3$ implies by Theorem A that $P_4 \| K_{3s+1}(4)$. This completes the proof. □

2. If $n \, mod\, 12 = 8$, then $P_4 \| (K_3 \square K_n)$.

Proof. As $n \, mod\, 12 = 8$, $n = 12s+8$ for some nonnegative integer s.

$$F_1 = \bigoplus_{i=1}^{3s+2} [(1,4i-3)(2,4i-3)(3,4i-3)(3,4i-2) \oplus (2,4i-2)(1,4i-2)(1,4i-1)(2,4i-1) \oplus (3,4i-1)(3,4i)(2,4i)(1,4i)]$$

and

$$F_2 = \bigoplus_{i=1}^{3s+2} [(3,4i-3)(1,4i-3)(1,4i)(3,4i) \oplus (2,4i-3)(2,4i-2)(3,4i-2)(1,4i-2) \oplus (1,4i-1)(3,4i-1)(2,4i-1)(2,4i)]$$

are edge-disjoint P_4-factors of $K_3 \square K_n$. The removal of the edges of $F_1 \oplus F_2$ from $K_3 \square K_n$ has three components and each component is isomorphic to $K_{6s+4}(2)$ with partite sets: $\{(1,4i-3),(1,4i)\}$, $\{(1,4i-2),(1,4i-1)\}$, $i \in \{1,2,\ldots,3s+2\}$, for the first copy; $\{(2,4i-3),(2,4i-2)\}$, $\{(2,4i-1),(2,4i)\}, i \in \{1,2,\ldots,3s+2\}$, for the second copy; and $\{(3,4i-3),(3,4i-2)\}$, $\{(3,4i-1),(3,4i)\}$, $i \in \{1,2,\ldots,3s+2\}$ for the third copy.

$(6s+4)(2) \equiv 0 \, mod\, 4$ and $2(6s+3)(2) \equiv 0 \, mod\, 3$ implies by Theorem A that $P_4 \| K_{6s+4}(2)$. This completes the proof. □

3. If $m(mod\, 12) = 8$, then $P_4 \| (K_m \square K_9)$.

Proof. As $m(mod\, 12) = 8$, $m = 12r+8$ for some nonnegative integer r. In order to factorize $K_{12r+8} \square K_9$ into P_4-factors, we require $8r+10$ edge-disjoint P_4-factors of $K_{12r+8} \square K_9$. Observe that

$$\begin{aligned} K_{12r+8} \square K_9 &= ((6r+4)K_2 \oplus K_{6r+4}(2)) \square K_9 \\ &= (6r+4)(K_2 \square K_9) \oplus 9K_{6r+4}(2). \end{aligned}$$

$(6r+4)(2) \equiv 0(mod\, 4)$ and $2(6r+3)(2) \equiv 0(mod\, 3)$ implies by Theorem A that $P_4 \| K_{6r+4}(2)$. Hence, there is a P_4-factorization of $K_{6r+4}(2)$ into P_4-factors $F_1, F_2, \ldots, F_{8r+4}$. Each P_4-factor F_i, $i \in \{1,2,\ldots,8r+4\}$, of $K_{6r+4}(2)$ yields a P_4-factor $F_i^* = F_i^{(1)} + F_i^{(2)} + F_i^{(3)} + F_i^{(4)} + F_i^{(5)} + F_i^{(6)} + F_i^{(7)} + F_i^{(8)} + F_i^{(9)}$ of $9K_{6r+4}(2)$ in $K_{12r+8} \square K_9$, where $F_i^{(j)}$ is a P_4-factor in the j-th K_{12r+8}-layer of $K_{12r+8} \square K_9$.

First take the $8r+1$ P_4-factors $F_4^*, F_5^*, \ldots, F_{8r+4}^*$. Next, we factorize the remaining subgraph $(6r+4)(K_2 \square K_9) \oplus (F_1^* \cup F_2^* \cup F_3^*)$ into the following nine P_4-factors: For $j \in \{0,1,2\}$,

1. $(F_1^{(3j+1)} + F_1^{(3j+2)} + F_1^{(3j+3)})$
$\oplus\ (\bigoplus_{i=1}^{6r+4} [(2i-1, 4+3j)(2i-1, 7+3j)(2i, 7+3j)(2i, 4+3j)$
$\oplus (2i-1, 5+3j)(2i-1, 8+3j)(2i-1, 9+3j)(2i-1, 6+3j)$
$\oplus (2i, 5+3j)(2i, 8+3j)(2i, 9+3j)(2i, 6+3j)]),$
2. $(F_2^{(3j+1)} + F_2^{(3j+2)} + F_2^{(3j+3)})$
$\oplus\ (\bigoplus_{i=1}^{6r+4} [(2i-1, 4+3j)(2i-1, 8+3j)(2i, 8+3j)(2i, 4+3j)$
$\oplus (2i-1, 5+3j)(2i-1, 9+3j)(2i-1, 7+3j)(2i-1, 6+3j)$
$\oplus (2i, 5+3j)(2i, 9+3j)(2i, 7+3j)(2i, 6+3j)]),$
3. $(F_3^{(3j+1)} + F_3^{(3j+2)} + F_3^{(3j+3)})$
$\oplus\ (\bigoplus_{i=1}^{6r+4} [(2i-1, 4+3j)(2i-1, 9+3j)(2i, 9+3j)(2i, 4+3j)$
$\oplus (2i-1, 5+3j)(2i-1, 7+3j)(2i-1, 8+3j)(2i-1, 6+3j)$
$\oplus (2i, 5+3j)(2i, 7+3j)(2i, 8+3j)(2i, 6+3j)]),$

where the addition involving j is taken modulo 9 with residues $1, 2, \ldots, 9$.

This completes the proof. □

4 Conclusion

In conclusion, left over cases are:

- P_4-factorization of $K_m \Box K_n$ for
  - (i) $(m \bmod 12, n \bmod 12) \in \{(0,5), (0,11), (10,10)\}$,
  - (ii) $(m \bmod 12, n \bmod 12) = (3,8)$ and $m \neq 3$,
  - (iii) $(m \bmod 12, n \bmod 12) = (8,9)$ and $n \neq 9$; and
- P_4-factorization of $K_m \Box K_4$ for $m \bmod 12 \in \{1, 7, 10\}$.

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References

[1] R. Balakrishnan and K. Ranganathan, *A textbook of graph theory*, Springer-Verlag, New York, 2000.

[2] V. Chitra and A. Muthusamy, P_3-factorization of Cartesian product of complete graphs, *Bull. Inst. Combin. Appl.*, **63** (2011), 60–72.

[3] A. Muthusamy and P. Paulraja, Path factorization of product graphs, *Proceedings of the National Workshop on Graph Theory and its Applications*, Manonmaniam Sundaranar University, Tirunelveli, Feb. 21-27, 1996, 77–88.

[4] A. Muthusamy and P. Paulraja, Path factorization of complete multipartite graphs, *Discrete Math.*, **195** (1999), 181–201.