Hamilton-Waterloo problem: Existence of larger order from smaller order

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Abstract

In this paper, we provide a powerful technique for the existence of Hamilton-Waterloo Problem from lower order to higher order.

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1 Introduction

An *r-factor* in a graph G is an r-regular spanning subgraph of G. Partition of G into edge-disjoint 2-factors is called *2-factorization* of G. A 2-factor is called a C_r-factor, if all its components are of order r. The well known *Oberwolfach Problem* formulated by Ringel [10] in 1967 asks for a 2-factorization of the complete graph K_n, in which each 2-factor is isomorphic to a given 2-factor of K_n. For survey of results see [2].

Hamilton-Waterloo Problem is one among many variations of the Oberwolfach Problem. Let R and S be given 2-factors of the complete graph K_n. The *Hamilton-Waterloo Problem* (HWP) [8] asks for a 2-factorization of K_n (respectively $K_n - I$, when n even, where I is a 1-factor of K_n) in which α 2-factors are isomorphic to R, β 2-factors are isomorphic to S such that $\alpha + \beta = \frac{n-1}{2}$ (respectively, $\alpha + \beta = \frac{n-2}{2}$, when n even) for all possible $\alpha, \beta \in Z_{\frac{n+1}{2}}$ (respectively, $\alpha, \beta \in Z_{\frac{n}{2}}$, when n even). If such a factorization exists we say that $(\alpha, \beta) \in HWP(n; R, S)$ or $HWP(n; R, S)$ exists. We say that $HWP(n; r, s)$ exists or $(\alpha, \beta) \in HWP(n; r, s)$ if α C_r-factors and β C_s-factors exist in K_n (respectively, $K_n - I$, when n even) for all possible values of $\alpha, \beta \in Z_{\frac{n+1}{2}}$ (respectively $\alpha, \beta \in Z_{\frac{n}{2}}$, when n even) such that $\alpha + \beta = \frac{n-1}{2}$ (respectively, $\alpha + \beta = \frac{n-2}{2}$, when n even).

In 2002, Peter Adams et al. [1] have shown that $HWP(n; r, s)$ exists when $(r, s) \in \{(4,6), (4,8), (4,16), (8,16), (3,5), (3,15), (5,15)\}$ and for all possible cycle lengths when $n \leq 17$. This was the first remarkable result in this topic. Horak et al. [11], Dinitz and Ling [5, 6] and Lei and Shen [13] have studied the existence of $HWP(n; 3, n)$. Existence of $HWP(n; 3, 4)$ is shown by Danziger et al. [4]. There are many results by Fu et al. [9, 12], Bryant and Danziger [3], if all the components of 2-factors under consideration are of even order. But very few results are known for HWP, if the 2-factors consist of components of odd order. In this paper, we give some constructions to show the existence of $HWP(n; r, s)$ for higher values of n, if it exists for lower values of n, which in turn reduce the domain of unknown cases.

Let $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be the partite sets of the complete bipartite graph $K_{n,n}$. We define the set of edges of distance k in $K_{n,n}$ as below.

Definition 1.1. Define $E_k = \{\{x_i, y_j\} \in E(K_{n,n}) : (j - i) \equiv k (\text{mod n}), 1 \leq i, j \leq n\}$, i.e., E_k, $0 \leq k \leq n-1$ is the set of edges of distance k in $K_{n,n}$, which is also an 1-factor of $K_{n,n}$.

From the definition of E_k, it is clear that $\{E_0, E_1, \ldots, E_{n-1}\}$ is an 1-factorization of $K_{n,n}$. The distance between the vertices x_i and y_j in $K_{n,n}$ is denoted by $d_{K_{n,n}}(x_i, y_j)$.

Definition 1.2. Let $i, j \in Z_n$. Define $|i-j|_n = min\{|i-j|, n-|i-j|\}$. If $D = \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$, then $K_n \cong \langle D \rangle_n$, where $\langle D \rangle_n$ is a graph with vertex set Z_n and edge set $\{\{i, j\} : |i-j|_n \in D, i, j \in Z_n\}$. We call $\{i, j\}$ is an edge of difference $|i-j|_n$. The set D is called connection set of K_n.

The *wreath product* of two graphs G and H is a graph $G \otimes H$ with vertex set $V(G) \times V(H)$, there is an edge between the vertices (u, v) and (u', v') if and only if (i) $\{u, u'\}$ is an edge of G, or (ii) $u = u'$ and $\{v, v'\}$ is an edge of H. The notation pK_n denotes p copies of K_n.

To prove our results, we require the following:

Theorem 1.3. [7] *$K_{m,n}$ has a C_k-factorization if and only if (i) $m = n \equiv 0 (mod\ 2)$, (ii) $k \equiv 0 (mod\ 2)$, $k \geq 4$ and (iii) $2n \equiv 0 (mod\ k)$ with precisely one exception, namely $m = n = k = 6$.*

Theorem 1.4. [3] *Let R and S be bipartite 2-factors of K_{4n}. Then $(\alpha, \beta) \in HWP(4n; R, S)$, except when $\alpha \in \{1, 2n-2\}$.*

2 Main Results

Lemmas 2.1 and 2.4 give the construction for the existence of $HWP(n;r,s)$ for larger n, provided $HWP(l;r,s)$ exists, where $l = lcm(r,s)$ and $r,s \geq 4$ are even. Without loss of generality, we assume that $r \neq s$.

Lemma 2.1. *Let $r \equiv 0 (mod\ 4)$, s be even and $l = lcm(r,s)$. If $HWP(l;r,s)$ exists, then $HWP(n;r,s)$ exists, where $n = pl$ for any integer p, except the case when $(r,s) = (4,6)$.*

Proof. By the hypothesis $r|n$ and $s|n$ and $l \equiv 0 (\text{mod } 4)$. We write $K_n = K_{2p} \otimes K_{l/2} = (F_0 \otimes K_{l/2}) \oplus (F_1 \otimes \overline{K}_{l/2}) \oplus (F_2 \otimes \overline{K}_{l/2}) \oplus \ldots \oplus (F_{2p-2} \otimes \overline{K}_{l/2}) = pK_l \oplus \underbrace{pK_{l/2,l/2} \oplus \cdots \oplus pK_{l/2,l/2}}_{2p-2\ times}$ where $\{F_0, F_1, \ldots, F_{2p-2}\}$ is an 1-factorization of K_{2p}. By the hypothesis, $K_l - I'$ has required number of C_r-factors and C_s-factors, where I' is a 1-factor of K_l. The union of all I' from p copies of K_l gives a 1-factor I of K_n. Now consider $K_n - I$. Let α, β be two integers satisfying $\alpha + \beta = \frac{n-2}{2}$, $0 \leq \alpha, \beta \leq \frac{n-2}{2}$. For the given α, β and l there exist α', β', i and j, $0 \leq \alpha', \beta' \leq (l-2)/2$, $0 \leq i \leq 2p-2$, $j = 2p-2-i$ and $\alpha' + \beta' = (l-2)/2$, such that $\alpha = \alpha' + i(l/4)$ and $\beta = \beta' + j(l/4)$. By Theorem 1.3, we obtain $l/4$ C_r-factors from $pK_{l/2,l/2}$ which correspond to each $F_j \otimes \overline{K}_{l/2}$, $1 \leq j \leq i$ and $l/4$ C_s-factors from $pK_{l/2,l/2}$ which correspond to each of the remaining $F_j \otimes \overline{K}_{l/2}$, $i+1 \leq j \leq 2p-2$. By the hypothesis, we have α' C_r-factors and β' C_s-factors from pK_l. Hence $(\alpha, \beta) \in HWP(n;r,s)$. □

Remark 2.2. When $(r,s) = (4,6)$, the construction given in Lemma 2.1 will not work, since C_6-factorization of $K_{6,6}$ does not exist.

Remark 2.3. By Theorem 1.4 and Lemma 2.1, we conclude that to show the existence of $HWP(4n;r,s)$, when $r \equiv 0 (\text{mod } 4)$ and s is even, it is enough to show that $(\alpha, \beta) \in HWP(l;r,s)$ when $\alpha \in \{1, \frac{l-4}{2}\}$ where $l = lcm(r,s)$.

Lemma 2.4. *Let $r,s \geq 4$ be even integers and $l = lcm(r,s)$. If $HWP(l;r,s)$ exists, then $HWP(n;r,s)$ exists, where $n = pl$ for any even integer p.*

Proof. By the hypothesis $r|n$, $s|n$ and l is even. We write $K_n = K_p \otimes K_l = (\overline{K}_p \otimes K_l) \oplus (F_1 \otimes \overline{K}_l) \oplus (F_2 \otimes \overline{K}_l) \oplus \ldots \oplus (F_{p-1} \otimes \overline{K}_l) = pK_l \oplus \underbrace{\frac{p}{2}K_{l,l} \oplus \cdots \oplus \frac{p}{2}K_{l,l}}_{p-1\ times}$, where $\{F_1, F_2, \ldots, F_{p-1}\}$ is an 1-factorization of K_p. By the hypothesis, $K_l - I'$ has required number of C_r-factors and C_s-factors, where I' is a 1-factor of K_l. The union of all I' from p copies of K_l gives a 1-factor I of K_n. Now consider $K_n - I$. Let α, β be integers such that $\alpha+\beta=\frac{n-2}{2}$, $0\leq\alpha,\beta\leq\frac{n-2}{2}$. For the given α,β and l there exist α',β',i and j, $0\leq\alpha',\beta'\leq(l-2)/2$, $0\leq i\leq p-1$, $j=p-1-i$ and $\alpha'+\beta'=(l-2)/2$, such that $\alpha=\alpha'+i(l/2)$ and $\beta=\beta'+j(l/2)$. By Theorem 1.3, we obtain $l/2$ C_r-factors of K_n-I from $\frac{p}{2}K_{l,l}$ which correspond to each $F_j\otimes\overline{K}_l$, $1\leq j\leq i$ and $l/2$ C_s-factors from $\frac{p}{2}K_{l,l}$ which correspond to each of the remaining $F_j\otimes\overline{K}_l$, $i+1\leq j\leq p-1$. By the hypothesis, we have α' C_r-factors and β' C_s-factors from pK_l. Hence $(\alpha,\beta)\in HWP(n;r,s)$. □

Note 2.5. Let n be even. In any 2-factor of $K_{n/2}$, there are at least $\frac{n+4}{8}$ pairs of edges such that each pair of edges are of same difference. For, let D_i, $1\leq i\leq n/4$ be the number of edges of difference i, in $K_{n/2}$. In any 2-factor of $K_{n/2}$, we have $\sum_{i=1}^{n/4}D_i=n/2$, $0\leq D_i\leq n/2$. By pairing the edges of same difference, we can get a maximum of $n/4$ pairs, if all D_i are even. We get minimum number of pairs, if each D_i is odd, say $D_i=1,3,5,\ldots,n/4$ or $n/4-1$. The number of D_i such that D_i is odd, can be at most $\frac{n}{4}-1$. The remaining edges of same difference can be paired to get at least $(\frac{n}{2}-(\frac{n}{4}-1))/2=\frac{n+4}{8}$ pairs.

The next theorem deals the HWP if the 2-factor R consists of uniform odd length cycles and the 2-factor S is a hamilton cycle.

Theorem 2.6. *Let $r\geq 3$ be an odd integer. If $HWP(\frac{n}{2};r,\frac{n}{2})$ exists, then $(\alpha,\beta)\in HWP(n;r,n)$, when $n/4\leq\beta\leq\frac{n-2}{2}$ and $n\equiv 0(mod\ 4)$.*

Proof. We write $K_n=2K_{n/2}\oplus K_{n/2,n/2}=K_{n/2}\cup K'_{n/2}\oplus K_{n/2,n/2}$, where $K'_{n/2}\cong K_{n/2}$. By the hypothesis, $K_{n/2}-I'$ has required number of C_r-factors and $C_{n/2}$-factors, where I' is a 1-factor of $K_{n/2}$. The union of I' from the 2 copies of $K_{n/2}$ gives a 1-factor I of K_n. Now consider K_n-I. If $\beta=\frac{n-2}{2}$ then $\alpha=0$ and the problem is nothing but the well known Oberwolfach Problem whose solution is known. If $\beta=n/4$, we can obtain β C_n-factors from $K_{n/2,n/2}$, given by $\{E_{2i}\cup E_{2i+1},\ 0\leq i\leq n/4-1\}$. Then, we have $\alpha(=\frac{n-4}{4})$ C_r-factors from $K_{n/2}$ and hence in K_n. If $n/4<\beta<(n-2)/2$, then $\beta=n/4+\beta'$, $1\leq\beta'<\frac{n-4}{4}$ and $\alpha=\alpha'$, where $\alpha'+\beta'=\frac{n-4}{4}$. By the hypothesis, we get α' C_r-factors and β' $C_{n/2}$-factors from $K_{n/2}\cup K'_{n/2}$ and $n/4$ C_n-factors from $K_{n/2,n/2}$ as above. By combining the β' $C_{n/2}$-factors with appropriate C_n-factors of $K_{n/2,n/2}$, we get the required β C_n-factors as follows: Let C^j be the jth $C_{n/2}$-factor of $K_{n/2}$ and let $C^{j'}$ be the copy of C^j in $K'_{n/2}$. From Note 2.5, any 2-factor of $K_{n/2}$ contains at least $\frac{n+4}{8}$ pairs of edges such that edges in each pair have the same difference. Let $\{u,v\}$ and $\{w,z\}$ be two edges of same difference in the 2-factor C^j of $K_{n/2}$ i.e., $|u-v|_{n/2}=|w-z|_{n/2}$. Define $x_1=min\{u,v\}$, $x_2=max\{u,v\}$, $y_1=min\{w,z\}$ and $y_2=max\{w,z\}$. Without loss of generality, we can take the edge

$\{u,v\} = \{x_1,x_2\}$ and $\{w,z\} = \{y_1,y_2\}$ in $K_{n/2}$. Let x_1', x_2', y_1' and y_2' be copies of x_1, x_2, y_1 and y_2 respectively, in $C^{j'}$. By the choice of edges $\{u,v\}$ and $\{w,z\}$, we have $d_{K_{n/2,n/2}}(x_1,y_1') = d_{K_{n/2,n/2}}(x_2,y_2') = d^{j'}$, say. We may choose either the distance $d^{j'}$ or the distance $\frac{n}{2} - d^{j'}$. First, let us consider the distance $d^{j'}$. Define $d^j = d^{j'}$, when $d^{j'}$ is even; $d^j = d^{j'} - 1$, when $d^{j'}$ is odd. It is clear that, the subgraph induced by the edges of distances d^j and $d^j + 1$, (i.e., $E_{d^j} \cup E_{d^j+1}$) is a C_n-factor of $K_{n/2,n/2}$. Now consider $C^j \cup C^{j'} \oplus E_{d^j} \cup E_{d^j+1}$ in $K_{n/2} \cup K'_{n/2} \oplus K_{n/2,n/2}$. By the choice of C^j and $C^{j'}$, the edges $\{x_1,x_2\}, \{y_1,y_2\} \in C^j$ and $\{x_1',x_2'\}, \{y_1',y_2'\} \in C^{j'}$. Fix $\{x_1,x_2\}$ in C^j and $\{y_1',y_2'\}$ in $C^{j'}$. Now we construct the two C_n-factors, H_1^j and H_2^j of K_n as follows: $H_1^j = C^j \cup C^{j'} \cup \{x_1,y_1'\} \cup \{x_2,y_2'\} - \{x_1,x_2\} - \{y_1',y_2'\}$ see Figure 2.1 and $H_2^j = E_{d^j} \cup E_{d^j+1} - \{x_1,y_1'\} - \{x_2,y_2'\} \cup \{x_1,x_2\} \cup \{y_1',y_2'\}$, see Figure 2.2. Clearly H_1^j and H_2^j are C_n-factors of K_n.

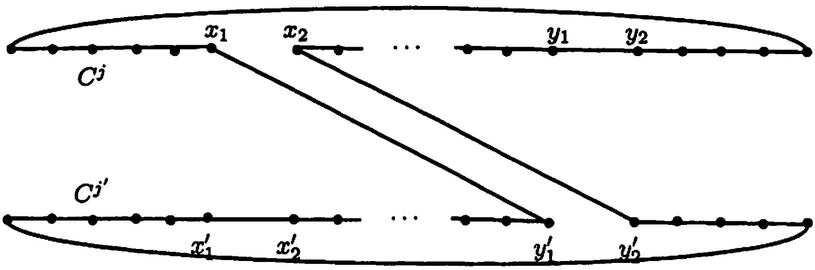

Figure 2.1. H_1^j

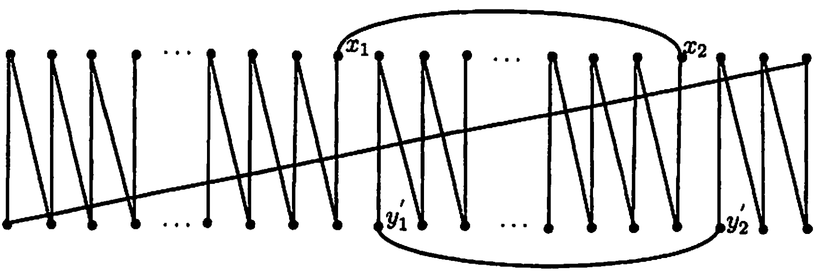

Figure 2.2. H_2^j

If we choose the distance $\frac{n}{2} - d^{j'}$, then define $d^j = \frac{n}{2} - d^{j'}$, when $d^{j'}$ is even; $d^j = \frac{n}{2} - d^{j'} - 1$, when $d^{j'}$ is odd. Fix $\{y_1,y_2\}$ in C^j and $\{x_1',x_2'\}$ in $C^{j'}$. In this case the C_n-factors are given by $H_1^j = C^j \cup C^{j'} - \{y_1,y_2\} -$

$\{x_1', x_2'\} \cup \{y_1, x_1'\} \cup \{y_2, x_2'\}$ and $H_2^j = E_{d^j} \cup E_{d^j+1} - \{y_1, x_1'\} - \{y_2, x_2'\} \cup \{y_1, y_2\} \cup \{x_1', x_2'\}$. In each time while we are choosing the pair of edges of same difference in C^j, a suitable distance $d^{j'}$ or $\frac{n}{2} - d^{j'}$ of $K_{n/2,n/2}$ may be considered so that the C_n-factor induced by $E_{d^j} \cup E_{d^j+1}$ is disjoint from the C_n-factors chosen previously in $K_{n/2,n/2}$ for the construction of β' C_n-factors of K_n. This is possible since in a 2-factor of $K_{n/2}$, there are at least $\frac{n+4}{8}$ pairs of edges, in which each pair has same difference and each difference is associated with two distances. □

Remark 2.7. We know that $HWP(10; 5, 10)$ and $HWP(14; 7, 14)$ exists [1]. By Theorem 2.6, $HWP(2^t 10; 5, 2^t 10)$ exists when $\frac{n-10}{2} \leq \beta \leq \frac{n-2}{2}$ and $HWP(2^t 14; 7, 2^t 14)$ exists when $\frac{n-14}{2} \leq \beta \leq \frac{n-2}{2}$ for all positive integer t.

3 Conclusion and scope

Theorem 2.6 provides a technique to find the existence of HWP for larger order provided the smaller order exists. In fact it reduces the domain of unknown cases, with which one can try to settle HWP by solving the unknown cases of HWP for smaller order in the case 2-factors consisting of uniform odd length cycles and hamilton cycles.

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