The total open geodetic number of a graph

A.P. SANTHAKUMARAN
Department of Mathematics
Hindustan University
Hindustan Institute of Technology and Science
Padur, Chennai-603 103, India.
e-mail: apskumar1953@yahoo.co.in
and
T. KUMARI LATHA
Department of Mathematics
Sri K.G.S. Arts College
Srivaikuntam-628 619, India.
e-mail: rajapaul1962@gmail.com

Abstract

For a connected graph G of order $n \geq 2$, a set S of vertices of G is a geodetic set of G if each vertex v of G lies on a x-y geodesic for some elements x and y in S. The geodetic number $g(G)$ of G is the minimum cardinality of a geodetic set of G. A geodetic set of cardinality $g(G)$ is called a g-set of G. A set S of vertices of a connected graph G is an open geodetic set of G if for each vertex v in G, either v is an extreme vertex of G and $v \in S$; or v is an internal vertex of an x-y geodesic for some $x, y \in S$. An open geodetic set of minimum cardinality is a minimum open geodetic set and this cardinality is the open geodetic number, $og(G)$. A connected open geodetic set of G is an open geodetic set S such that the subgraph $< S >$ induced by S is connected. The minimum cardinality of a connected open geodetic set of G is the connected open geodetic number of G and is denoted by $og_c(G)$. A total open geodetic set of a graph G is an open geodetic set S such that the subgraph $< S >$ induced by S contains no isolated vertices. The minimum cardinality of a total open geodetic set of G is the total open geodetic number of G and is denoted by $og_t(G)$. A total open geodetic set of cardinality $og_t(G)$ is called og_t-*set* of G. Certain general properties satisfied by total open geodetic sets are discussed. Graphs with total open geodetic number 2 are characterized. The total open geodetic numbers of certain standard graphs are determined. It is proved that for positive integers r, d and

$k \geq 4$ with $r \leq d \leq 2r$, there exists a connected graph of radius r, diameter d and total open geodetic number k. It is also proved that for the positive integers a, b, n with $4 \leq a \leq b \leq n$, there exists a connected graph G of order n such that $og_t(G) = a$ and $og_c(G) = b$.

Keywords: Geodetic number, open geodetic number, connected open geodetic number, total open geodetic number.
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1 Introduction

By a graph $G = (V, E)$ we mean a finite, undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by n and m, respectively. Throughout this paper, G denotes a connected graph of order at least two. For basic graph theoretic terminology we refer to Harary [6]. The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest u-v path in G. An u-v path of length $d(u, v)$ is called an *u-v geodesic*. It is known that the distance function d is a metric on the vertex set $V(G)$. For any vertex v of G, the *eccentricity* $e(v)$ of v is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the *radius*, *rad* G and the maximum eccentricity is its *diameter, diam* G of G. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices which are adjacent with v. A vertex v is an *extreme vertex* of G if the subgraph induced by its neighbors is complete. A vertex v of a connected graph G is called a *support vertex* of G if it is adjacent to an end vertex of G. A *geodetic set* of G is a set $S \subseteq V(G)$ such that every vertex of G is contained in a geodesic joining some pair of vertices in S. The *geodetic number* $g(G)$ of G is the minimum cardinality of a geodetic set of G. A vertex x is said to *lie on* a u-v geodesic P if x is a vertex of P and x is called an *internal vertex* of P if x is not a member of $\{u, v\}$. A set S of vertices of a connected graph G is an *open geodetic set* if for each vertex v in G, either v is an extreme vertex of G and $v \in S$; or v is an internal vertex of a x-y geodesic for some $x, y \in S$. An open geodetic set of minimum cardinality is a minimum open geodetic set and this cardinality is the *open geodetic number, $og(G)$*. Certainly, every open geodetic set is a geodetic set and so $g(G) \leq og(G)$. The geodetic number of a graph was introduced in [1, 5, 7] and further studied in [2, 3]. The open geodetic number of a graph was introduced in [4, 8] and further studied in [10, 11]. A *connected open geodetic set* of G is an open geodetic set S such that the subgraph induced by S is connected. The minimum cardinality of a *connected open geodetic set of* G is the *connected open geodetic number of* G and is denoted by $og_c(G)$. The connected open geodetic number was introduced and studied in [12]. In this paper, the total open geodetic number of a graph is introduced and studied.

The following theorems are used in the sequel.

Theorem 1.1. [10] *Every open geodetic set of a connected graph G contains its extreme vertices. Also, if the set of all extreme vertices of G is an open geodetic set, then S is the unique minimum open geodetic set of G.*

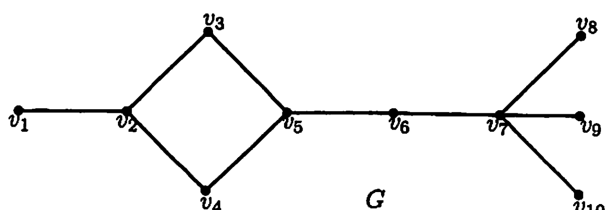
Theorem 1.2. [12] *Each cut-vertex of a connected graph G belongs to every connected open geodetic set of G.*

Theorem 1.3. [10]*If G is a connected graph with a cut-vertex v, then every open geodetic set of G contains at least one vertex from each component of $G - v$.*

2 The total open geodetic number of a graph

Definition 2.1. Let G be a connected graph with at least two vertices. A *total open geodetic set* of a graph G is an open geodetic set S such that the subgraph induced by S contains no isolated vertices. The minimum cardinality of a total open geodetic set of G is the *total open geodetic number* of G and is denoted by $og_t(G)$. A total open geodetic set of cardinality $og_t(G)$ is called *og_t-set* of G.

Example 2.2. For the graph G in Figure 2.1, it is clear that $S = \{v_1, v_8, v_9, v_{10}\}$ is the unique open geodetic set of G so that $og(G) = 4$.



G
Figure 2.1

It is easily verified that the set $S_1 = \{v_1, v_2, v_7, v_8, v_9, v_{10}\}$ is the unique total open geodetic set of G so that $og_t(G) = 6$. Also,it is clear that $S_2 = \{v_1, v_2, v_3, v_5, v_6, v_7, v_8, v_9, v_{10}\}$ is a minimum connected open geodetic set and so $og_c(G) = 9$. Thus the open geodetic number, the total open geodetic number, and the connected open geodetic number of a graph are all different.

Theorem 2.3. *Each extreme vertex and each support vertex of a connected graph G belong to every total open geodetic set of G. If the set of all extreme vertices and support vertices form a total open geodetic set of G, then it is the unique minimum total open geodetic set of G.*

Proof. Since every total open geodetic set is an open geodetic set, by Theorem 1.1, each extreme vertex belongs to every total open geodetic set. Since the total open geodetic set contains no isolated vertices, it follows that each support vertex of G also belongs to every total open geodetic set of G. Let S be the set of all extreme vertices and support vertices of G. Then by the first part of this theorem, every total open geodetic set of G contains S. If S is a total open geodtic set of G, then it follows that S is the unique minimum total open geodetic set of G. □

Corollary 2.4. *For the complete graph $K_n (n \geq 2), og_t(K_n) = n$.*

Theorem 2.5. *For a connected graph G of order $n \geq 2$, $2 \leq og(G) \leq og_t(G) \leq og_c(G) \leq n$.*

Proof. An open geodetic set needs at least two vertices and so $og(G) \geq 2$. Since every total open geodetic set is an open geodetic set, it follows that $og(G) \leq og_t(G)$. Also, since every connected open geodetic set of G is a total open geodetic set, $og_t(G) \leq og_c(G)$. Since $V(G)$ is a connected open geodetic set of G, it is clear that $og_c(G) \leq n$. Hence $2 \leq og(G) \leq og_t(G) \leq og_c(G) \leq n$. □

Corollary 2.6. *Let G be a connected graph. If $og_t(G) = 2$, then $og(G) = 2$.*

Remark 2.7. For the complete graph $K_2, og_t(G) = 2$ and for the complete graph $K_n, og_t(K_n) = n$, so that the total open geodetic number of a graph attains its least value 2 and largest value n. Also, all the inequalities in Theorem 2.5 can be strict.

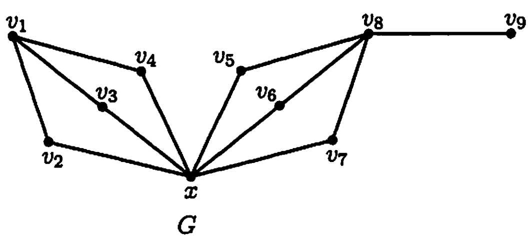


Figure 2.2

For the graph G given in Figure 2.2 of order 10, it is clear that the set $S = \{v_1, v_2, v_4, v_9\}$ is a minimum open geodetic set of G so that $og(G) = 4$.

It is easily verified that the set $S_1 = \{v_1, v_2, v_4, v_8, v_9\}$ is a minimum total open geodetic set of G and so $og_t(G) = 5$. Also, it is clear that the set $S_2 = \{v_1, v_2, v_4, x, v_7, v_8, v_9\}$ is a minimum connected open geodetic set of G so that $og_c(G) = 7$. Thus $2 < og(G) < og_t(G) < og_c(G) < n$.

Also, we notice that for any path of order at least 4, the open geodetic number is 2, whereas the total open geodetic number is 4. This shows that the converse of Corollary 2.6 need not be true.

Theorem 2.8. *For any non-trivial tree T, the set of all end vertices and support vertices of T is the unique minimum total open geodetic set of G.*

Proof. Since the set of all end vertices and support vertices of T forms a total open geodtic set, the result follows from Theorem 2.3. □

Theorem 2.9. *If a connected graph G contains no extreme vertices, then $og_t(G) \geq 4$.*

Proof. First, we observe that if G is a connected graph having no extreme vertices, then the order of G is at least 4. Let S be a total open geodetic set of G. If $u \in S$, then there exist vertices v and w such that u lies in a $v - w$ geodesic. Without loss of generality, assume that $d(v,u) \leq d(u,w)$. Then w does not lie in any $u - v$ geodesic. Since, for some $x, y \in S$, w lies in an $x - y$ geodesic, it follows that at least one of x and y is distinct from all of u, v and w. Thus $|S| \geq 4$ and so $og_t(G) \geq 4$. □

Theorem 2.10. *Let G be a connected graph with cut-vertices and let S be a total open geodetic set of G. If v is a cut-vertex of G, then every component of $G - v$ contains an element of S.*

Proof. Since every total open geodetic set is an open geodetic set, the result follows from Theorem 1.3. □

The following theorem characterize graphs for which the total open geodetic number is 2.

Proposition 2.11. *For any connected graph G, $og_t(G) = 2$ if and only if $G = K_2$.*

Proof. If $G = K_2$, then $og_t(G) = 2$. Conversely, let $og_t(G) = 2$. Let $S = \{u, v\}$ be a minimum total open geodetic set of G. Then uv is an edge. It is clear that a vertex different from u and v cannot lie on a u-v geodesic and so $G = K_2$. □

Proposition 2.12. *For the cycle $G = C_n (n \geq 4)$, $og_t(G) = \begin{cases} 4 & \text{if } n \text{ is even} \\ 5 & \text{if } n \text{ is odd.} \end{cases}$*

Proof. Since G has no extreme vertices, by Theorem 2.9, $og_t(G) \geq 4$.
Case 1. n is even.

Let $n = 2k (k \geq 2)$. Let G be the cycle $C_{2k} : v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_{2k}, v_1$. Then v_{k+1} is the antipodal vertex of v_1. Let $S = \{v_1, v_2, v_{k+1}, v_{k+2}\}$. It is clear that S is an open geodetic set of G and the subgraph induced by S has no isolated vertices so that $og_t(G) = 4$.

Case 2. n is odd.

Let $n = 2k+1$ $(k \geq 2)$. Let G be the cycle $C_{2k+1} : v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_{2k}, v_{2k+1}, v_1$. Then v_{k+1} and v_{k+2} are the antipodal vertices of v_1. Let $S = \{v_1, v_2, v_{k+1}, v_{k+2}, v_{k+3}\}$. It is clear that S is a minimum total open geodetic set of G so that $og_t(G) = 5$. □

Proposition 2.13. *For the wheel $W_n = K_1 + C_{n-1}$ $(n \geq 5)$, $og_t(W_n) = n - 1$.*

Proof. Let $W_n = K_1 + C_{n-1}$ $(n \geq 5)$ with x, the vertex of K_1 and $V(C_{n-1}) = \{v_1, v_2, \ldots, v_{n-1}\}$. It is clear that x does not belong to any minimum total open geodetic set of G. If S is a subset of $V(C_{n-1})$ of cardinality at most $n-2$, let v_i $(1 \leq i \leq n-1)$ be such that $v_i \notin S$ and $v_{i+1} \in S$. Then v_{i+1} is not an internal vertex of any geodesic joining a pair of vertices in S. Hence S is not an open geodetic set of W_n. Since $W = \{v_1, v_2, \ldots, v_{n-1}\}$ is an open geodetic set of W_n, it follows that W is the unique minimum open geodetic set of W_n and so $og(W_n) = n-1$. Since the subgraph induced by W has no isolated vertices, it follows that $og_t(W_n) = n - 1$. □

Proposition 2.14. *For the complete bipartite graph $G = K_{r,s} (2 \leq r \leq s)$, $og_t(K_{r,s}) = 4$.*

Proof. Let $G = K_{r,s}$. Let X and Y be the partite sets of G with $|X| = r$ and $|Y| = s$. Since G contains no extreme vertices, by Theorem 2.9, $og_t(G) \geq 4$. Any open geodetic set S of G must contain at least two vertices from each of X and Y. Since the subgraph induced by S has no isolated vertices, it follows that $og_t(W_n) = 4$. □

Ostrand [9] showed that every two positive integers a and b with $a \leq b \leq 2a$ are realizable as the radius and diameter respectively of some connected graph. Now, Ostrand's theorem can be extended so that the total open geodetic number can be prescribed, when $a \leq b \leq 2a$.

Theorem 2.15. *For positive integers r, d and $k \geq 4$ with $r \leq d \leq 2r$, there exists a connected graph G with rad $G = r$, diam $G = d$ and $og_t(G) = k$*

Proof. If $r = 1$, then $d = 1$ or 2. For $d = 1$, let $G = K_k$. Then $og_t(G) = k$. For $d = 2$, let $G = K_{1,k-1}$. Then $og_t(G) = k$. Now, let $r \geq 2$. We construct a graph G with the desired properties as follows :

Case 1. $r = d$.

Let $C_{2r} : u_1, u_2, \ldots, u_{2r}, u_1$ be a cycle of order $2r$. Let G be the graph in Figure 2.3, obtained from C_{2r} by adding the new vertices $v_1, v_2, \ldots, v_{k-3}$ and joining each $v_i (1 \leq i \leq k-3)$ with u_1 and u_2 of C_{2r} and also join u_r and u_{r+2}. It is easily verified that the eccentricity of each vertex of G is r so that $rad\ G = diam\ G = r$.

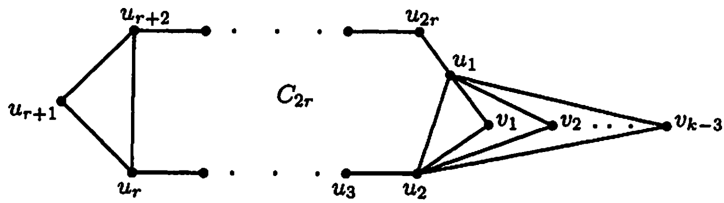


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Figure 2.3

Let $S = \{v_1, v_2, \ldots, v_{k-3}, u_{r+1}\}$ be the set of all extreme vertices of G. By Theorem 2.3, every total open geodetic set of G contains S. It is clear that for any $x \notin S$, $S \cup \{x\}$ is not a total open geodetic set of G. It is easily verified that the set $S_1 = S \cup \{u_1, u_{r+2}\}$ is a minimum total open geodetic set of G so that $og_t(G) = k$.

Case 2. $r < d$.

Let $C_{2r} : u_1, u_2, \ldots, u_{2r}, u_1$ be a cycle of order $2r$ and let $P_{d-r+1} : v_0, v_1, v_2, \ldots, v_{d-r}$ be a path of order $d-r+1$. Let H be the graph obtained from C_{2r} and P_{d-r+1} by identifying v_0 of P_{d-r+1} and u_1 of C_{2r}. Now add the new vertices $w_1, w_2, \ldots, w_{k-4}$ to the graph H and join each vertex $w_i (1 \leq i \leq k-4)$ with the vertex v_{d-r-1} and also, join u_r and u_{r+2} thereby producing the graph G in Figure 2.4. Then $rad\ G = r$ and $diam\ G = d$.

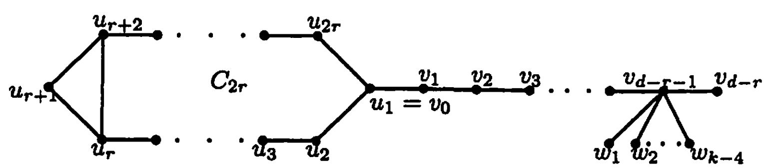

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Figure 2.4

Let $S = \{w_1, w_2, \ldots, w_{k-4}, v_{d-r}, u_{r+1}, v_{d-r-1}\}$ be the set of all extreme vertices and support vertices of G. By Theorem 2.3, every total open geodetic set of G contains S. Since $S \cup \{u_r\}$ is a total open geodetic set of G, it follows that $og_t(G) = k$. □

Remark 2.16. For $k = 2$, by Proposition 2.11, $og_t(G) = 2$ if and only if $G = K_2$. Hence for $k = 2$, a graph exists only when $r = d = 1$.

Theorem 2.17. *For positive integers a, b and n with $4 \le a \le b \le n$, there exists a connected graph G of order n, with $og_t(G) = a$ and $og_c(G) = b$.*

Proof. We prove this theorem by considering four cases.
Case 1. $a = b = n$.
Let $G = K_n$. Then $og_c(G) = og_t(G) = n$.
Case 2. $a < b < n$.
Let $P_{b-a+4} : u_1, u_2, \ldots, u_{b-a+4}$ be a path of order $b - a + 4$. Let G be the graph of order n in Figure 2.5, obtained from P_{b-a+4} by adding the new vertices $w_1, w_2, \ldots, w_{n-b}; v_1, v_2, \ldots, v_{a-4}$ to P_{b-a+4} and joining $w_1, w_2, \ldots, w_{n-b}$ with both u_2 and u_4; and also joining each $v_i (1 \le i \le a-4)$ with u_{b-a+3}.

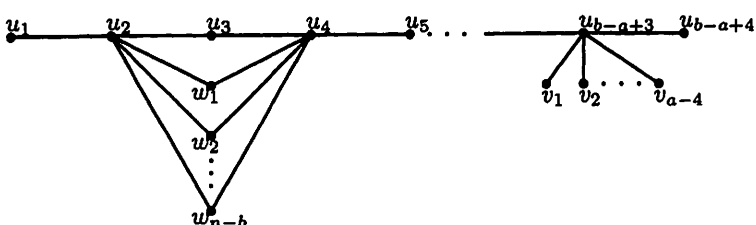

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Figure 2.5

Let $S_1 = \{u_1, u_{b-a+4}, v_1, v_2, \ldots, v_{a-4}\}$, $S_2 = \{u_2, u_{b-a+3}\}$ and $S_3 = \{u_2, u_4, u_5, \ldots, u_{b-a+3}\}$ denote the sets of all extreme vertices, support vertices and cut-vertices, respectively. Since $S_1 \cup S_2$ is a total open geodetic set of G, it follows from Theorem 2.3 that $og_t(G) = a$. Similarly, since $S_1 \cup S_3$ is a connected open open geodetic set of G, it follows from Theorems 1.2 and 2.3 that $og_c(G) = b$.
Case 3. $a = b < n$.
Let $P_3 : u_1, u_2, u_3$ be a path of order 3. Let G be the graph of order n in Figure 2.6, obtained from P_3 by adding the new vertices $v_1, v_2, \ldots, v_{a-4}$ and joining each $v_i (1 \le i \le a-4)$ with u_2; and also, adding the new vertices $w_1, w_2, \ldots, w_{n-a+1}$ and joining each $w_i (1 \le i \le n - a + 1)$ with u_1 and u_3.

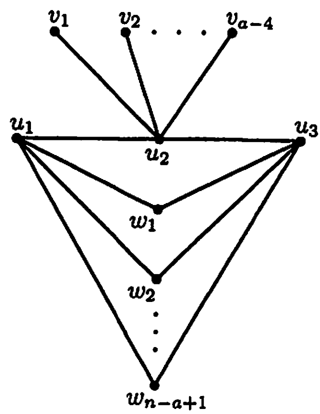

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Figure 2.6

First, let $a > 4$. Let $S = \{v_1, v_2, \ldots, v_{a-4}, u_2\}$. By Theorem 2.3, every total open geodetic set of G contains S. It is easily verified that for any vertex w_i $(1 \le i \le n-a+1)$, $S_i = S \cup \{u_1, u_3, w_i\}$ is a minimum total open geodetic set of G so that $og_t(G) = a$. Since S_i is also a minimum connected open geodetic set of G, we have $og_c(G) = a$. Thus $og_t(G) = og_c(G) = a = b$. Next, let $a = 4$. Then it is clear that for any vertex w_i $(1 \le i \le n-a+1)$, $T_i = \{u_1, u_2, u_3, w_i\}$ is a minimum total open geodetic set as well as a minimum connected open geodetic set of G so that $og_t(G) = og_c(G) = 4 = a = b$.

Case 4. $a < b = n$.

Let $P_{b-a+4} : u_1, u_2, \ldots, u_{b-a+4}$ be a path of order $b-a+4$. Let G be the graph of order n in Figure 2.7, obtained from P_{b-a+4} by adding the new vertices $v_1, v_2, \ldots, v_{a-4}$ and joining each $v_i (1 \le i \le a-4)$ with u_{b-a+3}.

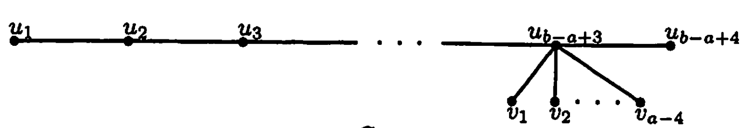

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Figure 2.7

Let $S = \{u_1, u_{b-a+4}, v_1, v_2, \ldots, v_{a-4}, u_2, u_{b-a+3}\}$ be the set of all extreme vertices and support vertices of G, it follows from Theorem 2.3 that $og_t(G) = a$. Let $S_1 = \{u_2, u_3, \ldots, u_{b-a+3}\}$ be the set of all cut-vertices of G. Since $S \cup S_1$ is a connected open geodetic set of G, it follows from Theorems 1.2 and 2.3 that $og_c(G) = b = n$. □

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