The detour monophonic number of a graph*

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Abstract

For a connected graph $G = (V, E)$ of order at least two, a *chord* of a path P is an edge joining two non-adjacent vertices of P. A path P is called a *monophonic path* if it is a chordless path. A longest x-y monophonic path is called an *x-y detour monophonic path*. A set S of vertices of G is a *monophonic set* of G if each vertex v of G lies on an x-y monophonic path for some elements x and y in S. The minimum cardinality of a monophonic set of G is the *monophonic number* of G, denoted by $m(G)$. A set S of vertices of G is a *detour monophonic set* of G if each vertex v of G lies on an x-y detour monophonic path for some x and y in S. The minimum cardinality of a detour monophonic set of G is the *detour monophonic number* of G and is denoted by $dm(G)$. We determine bounds for it and characterize graphs which realize these bounds. Also for each pair a, b of integers with $2 \leq a \leq b$, we prove that there is a connected graph G with $m(G) = a$ and $dm(G) = b$.

Keywords: monophonic set, monophonic number, monophonic distance, detour monophonic set, detour monophonic number.
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1 Introduction

By a graph $G = (V, E)$ we mean a simple connected graph of order at least two. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology we refer to Harary [6]. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with

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v. The *closed neighborhood* of a vertex v is the set $N[v] = N(v) \bigcup \{v\}$. A vertex v is an *extreme vertex* if the subgraph induced by its neighbors is complete. For a cut-vertex v in a connected graph G and a component H of $G - v$, the subgraph H and the vertex v together with all edges of G joining v to $V(H)$ is called a *branch* of G at v.

The *closed interval* $I[x, y]$ consists of all vertices lying on some x-y geodesic of G, while for $S \subseteq V$, $I[S] = \bigcup\limits_{x,y \in S} I[x, y]$. A set S of vertices is a *geodetic set* if $I[S] = V$, and the minimum cardinality of a geodetic set is the *geodetic number* $g(G)$. A geodetic set of cardinality $g(G)$ is called a *g-set*. The geodetic number of a graph was introduced in [1, 7] and further studied in [3]. The *detour distance* $D(u, v)$ between two vertices u and v in G is the length of a longest u-v path in G. An u-v path of length $D(u, v)$ is called an *u-v detour*. It is known that D is a metric on the vertex set V of G. The closed detour interval $I_D[x, y]$ consists of x, y, and all the vertices in some x-y detour of G. For $S \subseteq V$, $I_D[S]$ is the union of the sets $I_D[x, y]$ for all $x, y \in S$. A set S of vertices is a detour set if $I_D[S] = V$, and the minimum cardinality of a detour set is the detour number $dn(G)$. The concept of detour distance, detour number was introduced and studied in [2, 4] and further studied in [5].

A *chord* of a path P is an edge joining two non-adjacent vertices of P. A path P is called *monophonic* if it is a chordless path. A set S of vertices of a graph G is a *monophonic set* if each vertex v of G lies on an x-y monophonic path for some elements x and y in S. The minimum cardinality of a monophonic set of G is the *monophonic number* of G, denoted by $m(G)$ [10]. For any two vertices u and v in a connected graph G, the *monophonic distance* $d_m(u, v)$ from u to v is defined as the length of a longest u-v monophonic path in G. The *monophonic eccentricity* $e_m(v)$ of a vertex v in G is $e_m(v) = \max \{d_m(v, u) : u \in V(G)\}$. The *monophonic radius*, $rad_m G$ of G is $rad_m G = \min \{e_m(v) : v \in V(G)\}$ and the *monophonic diameter*, $diam_m G$ of G is $diam_m G = \max \{e_m(v) : v \in V(G)\}$. A vertex u in G is a *monophonic eccentric vertex* of a vertex v in G if $e_m(u) = d_m(u, v)$. The monophonic distance was introduced in [8] and further studied in [9].

For the graph G given in Figure 1.1, it is clear that $S = \{v_1, v_5, v_8, v_{10}\}$ is a minimum geodetic set of G and so $g(G) = 4$; $S_1 = \{v_1, v_5, v_{10}\}$ is a minimum detour set of G and so $dn(G) = 3$; and $S_2 = \{v_1, v_5\}$ is a minimum monophonic set of G and so $m(G) = 2$. Thus the geodetic number, detour number and the monophonic number of a graph are all different.

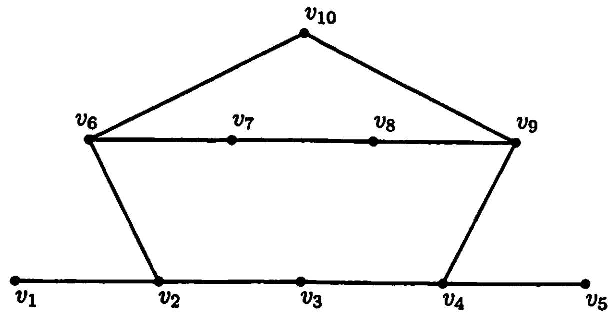

Figure 1.1: G

The following theorems will be used in the sequel.

Theorem 1.1. [6] *Let G be a connected graph with at least three vertices. The following statements are equivalent.*

(i) G is a block.

(ii) Every two vertices of G lie on a common cycle.

Theorem 1.2. [10] *Each extreme vertex of a connected graph G belongs to every monophonic set of G.*

Throughout this paper G denotes a connected graph with at least two vertices.

2 Detour monophonic number

Definition 2.1. A set S of vertices of a graph G is a *detour monophonic set* if each vertex v of G lies on an x-y detour monophonic path for some $x, y \in S$. The minimum cardinality of a detour monophonic set of G is the *detour monophonic number* of G and is denoted by $dm(G)$.

Example 2.2. For the graph G given in Figure 2.1, $S_1=\{x, y, z\}$, $S_2=\{x, w, z\}$, $S_3=\{u, z, y\}$, $S_4=\{x, u, z\}$, $S_5=\{y, w, z\}$ and $S_6 = \{u, w, z\}$ are the minimum detour monophonic sets of G and so $dm(G)$=3.

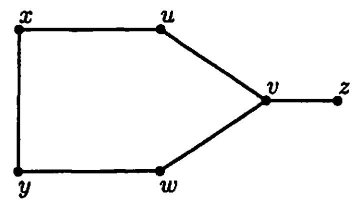

Figure 2.1: A graph G with $dm(G) = 3$.

A vertex v in a graph G is a *detour monophonic vertex* if v belongs to every minimum detour monophonic set of G. If G has a unique minimum detour monophonic set S, then every vertex in S is a detour monophonic vertex. In the next theorem, we show that there are certain vertices in a nontrivial connected graph G that are detour monophonic vertices of G.

Theorem 2.3. *Each extreme vertex of a connected graph G belongs to every detour monophonic set of G. Moreover, if the set S of all extreme vertices of G is a detour monophonic set, then S is the unique minimum detour monophonic set of G.*

Proof. Let u be an extreme vertex and let S be a detour monophonic set of G. Suppose that $u \notin S$. Then u is an internal vertex of an x-y detour monophonic path, say P, for some x, $y \in S$. Let v and w be the neighbors of u on P. Then v and w are not adjacent and so u is not an extreme vertex, which is a contradiction. Therefore u belongs to every detour monophonic set of G. Thus, if S is the set of all extreme vertices of G, then $m(G) \geq |S|$. On the other hand, if S is a detour monophonic set of G, then $m(G) \leq |S|$. Therefore $m(G) = |S|$ and S is the unique minimum detour monophonic set of G. □

Corollary 2.4. *For the complete graph $K_p (p \geq 2)$, $dm(K_p) = p$.*

Theorem 2.5. *Let G be a connected graph with a cut vertex v and let S be a detour monophonic set of G. Then every component of $G - v$ contains an element of S.*

Proof. Suppose that there is a component B of $G - v$ such that B contains no vertex of S. Let u be any vertex in B. Since S is a detour monophonic set, there exists a pair of vertices x and y in S such that u lies in some x-y detour monophonic path P: $x = u_0, u_1, u_2, \ldots, u, \ldots, u_n = y$ in G with $u \neq x, y$. Since v is a cut vertex of G, the x-u subpath P_1 of P and the u-y subpath P_2 of P both contain v, it follows that P is not a path, which is a contradiction. □

Theorem 2.6. *No cut vertex of a connected graph G belongs to any minimum detour monophonic set of G.*

Proof. Let v be a cut vertex of G and let S be a minimum detour monophonic set that contains v of G. Then by Theorem 2.5, every component of $G - v$ contains an element of S. Let U and W be two distinct components of $G - v$ and let $u \in U$ and $w \in W$. Then v is an internal vertex of an u-w detour monophonic path. Let $S' = S - \{v\}$. It is clear that every vertex that lies on an u-v detour monophonic path also lies on an u-w detour monophonic path. Hence it follows that S' is a detour monophonic set of G, which is a contradiction to S a minimum detour monophonic set of G. □

Corollary 2.7. *If T is a tree with k end vertices, then* $dm(T) = k$.

Proof. It follows from Theorem 2.3 and Theorem 2.6. □

Since every end-block B is a branch of G at some cut-vertex, it follows by Theorem 2.5 and Theorem 2.6 that every minimum detour monophonic set of G contains at least one vertex from B that is not a cut vertex. Thus the following corollaries are consequences of Theorems 2.5 and 2.6.

Corollary 2.8. *If G is a connected graph with* $k \geq 2$ *end-blocks, then* $dm(G) \geq k$.

Corollary 2.9. *If k is the maximum number of blocks to which a vertex in a graph G belongs, then* $dm(G) \geq k$.

We denote the vertex connectivity of a connected graph G by $\kappa(G)$ or κ.

Theorem 2.10. *If G is a non-complete connected graph such that it has a minimum cut set consisting of* κ *vertices, then* $dm(G) \leq p - \kappa$.

Proof. Since G is a non-complete connected graph, it is clear that $1 \leq \kappa \leq p - 2$. Let $U = \{u_1, u_2, u_3, \ldots, u_\kappa\}$ be a minimum cut set of G. Let G_1, G_2, ..., $G_r (r \geq 2)$ be the components of $G - U$ and let $S = V(G) - U$. Then every vertex $u_i (1 \leq i \leq \kappa)$ is adjacent to at least one vertex of G_j for each j $(1 \leq j \leq r)$. It is clear that S is a detour monophonic set of G and so $dm(G) \leq |S| = p - \kappa$. □

Remark 2.11. The bound in Theorem 2.10 is sharp. For the cycle C_4, $dm(C_4) = 2$, $\kappa = 2$ and $p - \kappa = 2$. Thus $dm(C_4) = p - \kappa$.

Theorem 2.12. *For the cycle* $C_n (n \geq 3)$,

$$dm(C_n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases}$$

Proof. Let $C_n : v_1, v_2, v_3, ..., v_n, v_1$ be a cycle of length n. If n is even, then clearly $S = \{v_1, v_{\frac{n}{2}+1}\}$ is a minimum detour monophonic set of C_n and so $dm(C_n) = 2$. If n is odd, then clearly $S = \{v_1, v_2, v_3\}$ is a minimum detour monophonic set of C_n and so $dm(C_n) = 3$. □

Theorem 2.13. *For the wheel $W_n = K_1 + C_{n-1} (n \geq 5)$,*

$$dm(W_n) = \begin{cases} 2 \text{ if } n = 5 \\ 3 \text{ if } n \text{ is odd and } n \geq 7 \\ 4 \text{ if } n \text{ is even} \end{cases}$$

Proof. Let $W_n = K_1 + C_{n-1} (n \geq 5)$ be the wheel with $V(C_{n-1}) = \{v_1, v_2, v_3, ..., v_{n-1}, v_1\}$ and $V(K_1) = \{x\}$. If $n = 5$, then $S = \{v_1, v_3\}$ is a minimum detour monophonic set of W_n and so $dm(W_n) = 2$. If n is odd and $n \geq 7$, then $S = \{v_1, v_{\frac{n+1}{2}}, x\}$ is a minimum detour monophonic set of W_n and so $dm(W_n) = 3$. If n is even, then $S = \{v_1, v_{\frac{n}{2}}, v_{\frac{n}{2}+1}, x\}$ is a minimum detour monophonic set of W_n and so $dm(W_n) = 4$. □

Theorem 2.14. *For the complete bipartite graph $K_{m,n} (m, n \geq 2)$, $dm(K_{m,n}) = min\{4, m, n\}$.*

Proof. Let $V_1 = \{x_1, x_2, ..., x_m\}$ and $V_2 = \{y_1, y_2, ..., y_n\}$ be the partite sets of $K_{m,n}$ where $m \leq n$. When $m = 2$ or 3, $S = V_1$ is the minimum detour monophonic set of G and so $dm(K_{m,n}) = |S| = m$.

Now, let $m \geq 4$ and let $S = \{x_1, x_2, y_1, y_2\}$. Clearly S is a detour monophonic set of G, it follows that $dm(G) \leq 4$. It remains to show that if X is a 3-element subset of $V(G)$, then X is not a detour monophonic set of G. First, assume that X is a subset of V_1 or of V_2, say the former. Since X contains three elements from V_1, there exists an element $u \in V_1$ and $u \notin X$. It is clear that u is not an internal vertex of any $v - w$ detour monophonic path, $v, w \in X$. Hence X is not a detour monophonic set of G. Therefore, we may take that $X \cap V_1 = \{x_i, x_j\}$ and $X \cap V_2 = \{y_k\}$. Then it is clear that X is not a detour monophonic set of G. □

Theorem 2.15. *For any connected graph G, $2 \leq dm(G) \leq p$.*

Proof. Since $V(G)$ is a detour monophonic set of G, it follows that $dm(G) \leq p$. Also it is clear that $dm(G) \geq 2$ and so $2 \leq dm(G) \leq p$. □

Theorem 2.16. *For any integer n such that $2 \leq n \leq p$, there is a minimal(with respect to edges) graph G of order p such that $dm(G) = n$.*

Proof. For $n = p$, the theorem follows from Corollary 2.4 by taking $G = K_p$. For $2 \leq n \leq p-1$, the tree T given in Figure 2.2 has p vertices and it follows from Corollary 2.7 that $dm(G) = n$. As the graph G is a tree, it is minimal with respect to edges. □

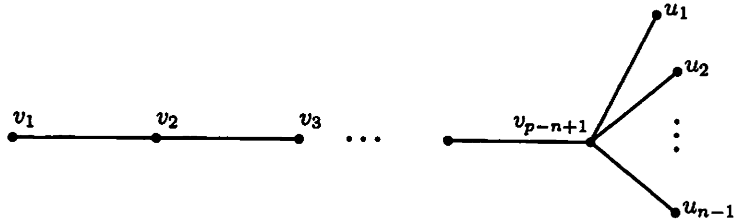

Figure 2.2: A graph G in Theorem 2.16 with $dm(G) = n$.

Now we proceed to characterize graphs G for which the bounds in Theorem 2.15 are attained.

Theorem 2.17. *For any connected graph G, $dm(G) = p$ if and only if G is complete.*

Proof. Let $dm(G) = p$. Suppose that G is not a complete graph. Then there exist two vertices u and v such that u and v are not adjacent in G. Since G is connected, there is a detour monophonic path from u to v, say P, with length at least 2. Clearly, $(V(G) - V(P))\bigcup\{u, v\}$ is a detour monophonic set of G and hence $dm(G) \leq p - 1$, which is a contradiction. Conversely, if G is complete, then by Corollary 2.4, $dm(G) = p$. □

Definition 2.18. Let x be any vertex in G. A vertex y in G is said to be an x-detour monophonic superior vertex if for any vertex z with $d_m(x, y) < d_m(x, z)$, z lies on an x-y detour monophonic path.

Example 2.19. For the graph G given in Figure 2.1, the vertex detour monophonic superior vertices are given in Table 2.1.

Vertex	x	y	z	u	v	w
Vertex detour monophonic superior vertices	$\{z\}$	$\{z\}$	$\{x, y\}$	$\{w, y\}$	$\{x, y\}$	$\{x, u\}$

Table 2.1

We give below a property related with monophonic eccentric vertex of x and x-detour monophonic superior vertex in a graph G.

Theorem 2.20. *Let x be any vertex in G. Then every monophonic eccentric vertex of x is an x-detour monophonic superior vertex.*

Proof. Let y be a monophonic eccentric vertex of x so that $e_m(x) = d_m(x, y)$. If y is not an x - detour monophonic superior vertex, then there exists a vertex z in G such that $d_m(x, y) < d_m(x, z)$ and z does not lie on any x-y detour monophonic path and hence $e_m(x) < d_m(x, z)$, which is a contradiction. □

Note 2.21. *The converse of Theorem 2.20 is not true. For the even cycle $C_{2n}(n \geq 3)$, the eccentric vertex of x is an x-detour monophonic superior vertex but it is not a monophonic eccentric vertex of x.*

Theorem 2.22. *Let G be a connected graph. Then $dm(G) = 2$ if and only if there exist two vertices x and y such that y is an x-detour monophonic superior vertex and every vertex of G is on an x-y detour monophonic path.*

Proof. Let $dm(G) = 2$ and let $S = \{x, y\}$ be a minimum detour monophonic set of G. If y is not an x - detour monophonic superior vertex, then there is a vertex z in G with $d_m(x, y) < d_m(x, z)$ and z does not lie an any x-y detour monophonic path. Thus S is not a minimum detour monophonic set of G, which is a contradiction. The converse is clear from the definition. □

Theorem 2.23. *Let G be a connected graph of order $p \geq 3$. Then $dm(G) = p - 1$ if and only if $G = K_1 + \bigcup m_j K_j$, where $\sum m_j \geq 2$.*

Proof. Let $G = K_1 + \bigcup m_j K_j$, where $\sum m_j \geq 2$. Then G has exactly one cut vertex and all the remaining vertices are extreme vertices and hence by Theorems 2.3 and 2.6, $dm(G) = p - 1$.

Conversely, let $dm(G) = p - 1$. If G has two or more cut vertices, then by Theorem 2.6, $dm(G) \leq p-2$, which is a contradiction. Thus, the number of cut vertices k of G is at most one. We consider two cases.

Case (i) $k = 0$.

If G is complete, then by Corollary 2.4, $dm(G) = p$, which is a contradiction. If G is not complete, then there exist two vertices x and y in G such that $d(x, y) \geq 2$. By Theorem 1.1, x and y lie on a common cycle, say C. If $d_m(x, y) = 2$, then the length of C is 4. Now $S = (V(G) - V(C)) \cup \{x, y\}$ is a detour monophonic set of G and so $dm(G) \leq p - 2$, which is a contradiction. If $d_m(x, y) \geq 3$, then let P be an x-y detour monophonic path. Then $S = (V(G) - V(P)) \cup \{x, y\}$ is a detour monophonic set of G and so $dm(G) \leq p - 2$, which is a contradiction.

Case (ii) $k = 1$.

Let x be the cut vertex of G. If $p = 3$, then $G = P_3 = K_1 + \bigcup m_j K_1$, where $\sum m_j = 2$. If $p \geq 4$, we claim that $G = K_1 + \bigcup m_j K_j$, $\sum m_j \geq 2$. It is enough to prove that every block of G is complete. Suppose that there exists a block B, which is not complete. Let u and v be two vertices in B such that $d(u, v) \geq 2$. Then by Theorem 1.1, both u and v lie on a common cycle. Then as in Case(i) $dm(G) \leq p - 2$, which is a contradiction. Thus every block of G is complete so that $G = K_1 + \bigcup m_j K_j$, where K_1 is the vertex x and $\sum m_j \geq 2$. □

Since every detour monophonic set is a monophonic set of a connected graph G, we have $m(G) \leq dm(G)$. Next theorem gives a realization for this result.

Theorem 2.24. *For every pair a, b of positive integers with $2 \leq a \leq b$, there is a connected graph G with $m(G) = a$ and $dm(G) = b$.*

Proof. For $2 \leq a = b$, any tree with a end vertices has the desired property. Then assume that $2 \leq a < b$. Let $P_i : x_i, y_i, z_i (1 \leq i \leq 3)$ be 3 copies of a path of length 2. Let G be the graph obtained by adding $b + 1$ new vertices $x, z, v_1, v_2, ..., v_{a-1}, w_1, w_2, ..., w_{b-a}$ and (i) joining each $u \in \{x_1, x_2, x_3, v_1, v_2, ...v_{a-1}, w_1, w_2, ..., w_{b-a}\}$ to x, (ii) joining each $v \in \{z_1, z_2, z_3, w_1, w_2, ..., w_{b-a}\}$ to z. The graph G is shown in Figure 2.3. Let $S = \{v_1, v_2, \ldots, v_{a-1}\}$ be the set of all extreme vertices of G. Clearly, By Theorem 1.2, every monophonic set of G contains S. Clearly, S is not a monophonic set of G. Let $S' = S \cup \{z\}$. It is easily verified that S' is a monophonic set of G and so $m(G) = a$.

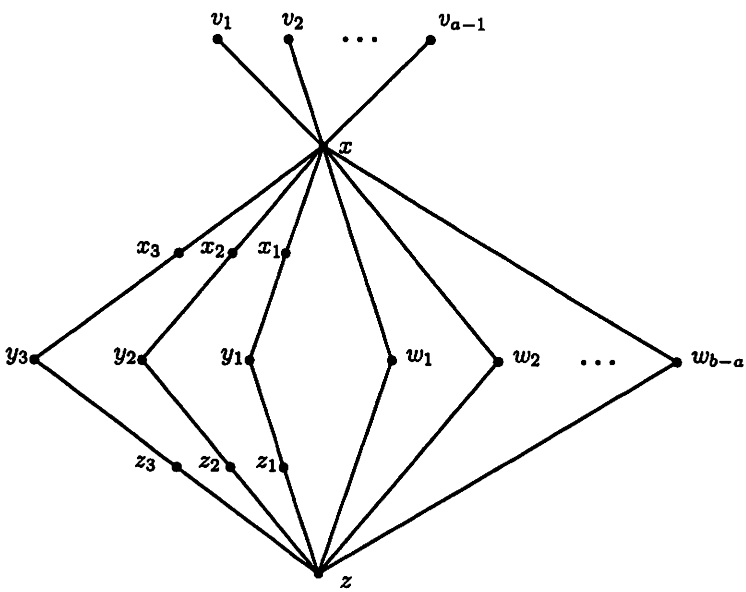

Figure 2.3:G

By Theorem 2.3, every detour monophonic set of G contains S. Clearly, S is not a detour monophonic set of G. It is easily verified that $w_i (1 \leq i \leq b - a)$ is not an internal vertex of any detour monophonic path in G. It follows that every detour monophonic set of G contains $S' = S \cup$

$\{w_1, w_2, ..., w_{b-a}\}$. Clearly S' is not a detour monophonic set of G and so $dm(G) > |S'| = b - 1$. It is easily verified that $S'' = S' \cup \{z\}$ is a detour monophonic set of G and so $dm(G) = b$. □

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