Pairwise distance similar sets in graphs

R. ANANTHA KUMAR AND S. ARUMUGAM*
National Centre for Advanced Research in Discrete Mathematics (n-CARDMATH)
Kalasalingam University, Anand Nagar, Krishnankoil-626 126, India.
e-mail: ananthgraph@yahoo.com, s.arumugam.klu@gmail.com

Abstract

Let $G = (V, E)$ be a connected graph. Two vertices u and v are said to be *distance similar* if $d(u, x) = d(v, x)$ for all $x \in V - \{u, v\}$. A nonempty subset S of V is called a *pairwise distance similar set* (in short '*pds*-set') if either $|S| = 1$ or any two vertices in S are distance similar. The maximum (minimum) cardinality of a maximal pairwise distance similar set in G is called the *pairwise distance similar number* (*lower pairwise distance similar number*) of G and is denoted by $\Phi(G)$ ($\Phi^-(G)$). The maximal *pds*-set with maximum cardinality is called a Φ-set of G. In this paper we initiate a study of these parameters.

Keywords: distance similar vertices, pairwise distance similar set, pairwise distance similar number.
2010 Mathematics Subject Classification Number: 05C12.

1 Introduction

By a graph $G = (V, E)$ we mean a finite, undirected and connected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [1]. The distance $d(u, v)$ between two vertices u and v in G is the length of a shortest u-v path in G. The open neighborhood $N(v)$ of a vertex v consists of the set of all vertices adjacent to v, that is, $N(v) = \{w \in V : vw \in E\}$, and the closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood $N(S)$, is defined to be $\bigcup_{v \in S} N(v)$. For any two disjoint subsets A, $B \subseteq V$, let $[A, B]$ denote the set of all edges with one end in A and the other end in B.

*Also at School of Electrical Engineering and Computer Science, The University of Newcastle, NSW 2308, Australia; Department of Computer Science, Liverpool Hope University, Liverpool, UK; Department of Computer Science, Ball State University, USA.

Saenpholphat et al. [2] while studying the concept of *connected resolving sets*, introduced the concept of distance similar vertices.

Definition 1.1. Two vertices u and v of a connected graph G are *distance similar* if $d(u,x) = d(v,x)$ for all $x \in V(G) - \{u,v\}$.

We observe that u and v are distance similar vertices if and only if $N(u) = N(v)$ if $uv \notin E(G)$ and $N[u] = N[v]$ if $uv \in E(G)$. Hernando et al. [3] called a pair of vertices satisfying the above equivalent conditions as *twins* and introduced the concept of the *twin graph* of a graph G. The relation $\equiv$ on $V(G)$ defined by $u \equiv v$ if and only if $u = v$ or u, v are twins is an equivalence relation on $V(G)$. For each vertex $v \in V(G)$, let v^* be the set of vertices of G that are equivalent to v under $\equiv$. Let $\{v_1^*, v_2^*, \ldots, v_k^*\}$ be the partition of $V(G)$ induced by $\equiv$. Each $\langle v_i^* \rangle$ is either independent or a complete graph in G. Further $\langle [v_i^*, v_j^*] \rangle$ is a complete bipartite graph if $v_iv_j \in E(G)$ and is an empty graph if $v_iv_j \notin E(G)$. The *twin graph* of G, denoted by G^*, is the graph with vertex set $V(G^*) = \{v_1^*, v_2^*, \ldots, v_k^*\}$, and $v_i^*v_j^* \in E(G^*)$ if and only if $v_iv_j \in E(G)$. We observe that each equivalence class v_i^* has the property that any two vertices of v_i^* are pairwise distance similar and v_i^* is maximal with respect to this property.

Motivated by this observation, in this paper we introduce the concept of pairwise distance similar set and pairwise distance similar number of a graph. We present basic results on this parameter.

We need the following definition.

Definition 1.2. [4] Let G_0 be a graph with $V(G_0) = \{v_1, v_2, \ldots, v_n\}$ and let $G_1, G_2, \ldots, G_n$ be n disjoint graphs. The *composition graph* $G = G_0[G_1, G_2, \ldots, G_n]$ is formed as follows: We replace each vertex v_i in G_0 with the graph G_i and make each vertex of G_i adjacent to each vertex of G_j whenever v_i is adjacent to v_j in G_0. In particular the graph $P_n[G_1, G_2, \ldots, G_n]$ is called the *sequential join* of the graphs $G_1, G_2, \ldots, G_n$.

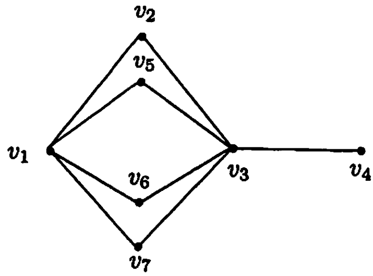
2 Main Results

Definition 2.1. Let $G = (V, E)$ be a connected graph. A nonempty subset S of V is called a *pairwise distance similar set* (*pds*-set) if either $|S| = 1$ or any two vertices in S are distance similar. The maximum (minimum) cardinality of a maximal *pds*-set in G is called the *pairwise distance similar number* (*lower pairwise distance similar number*) of G and is denoted by $\Phi(G)$ ($\Phi^-(G)$). Any maximal *pds*-set with $|S| = \Phi(G)$ is called a Φ-set of G.

Clearly the equivalence classes $v_1^*, v_2^*, \ldots, v_k^*$ with respect to the relation $\equiv$ are precisely the set of all maximal *pds*-sets of a graph G. Hence

$\Phi(G) = \max_{1 \le i \le k} |v_i^*|$ and $\Phi^-(G) = \min_{1 \le i \le k} |v_i^*|$. Hence it follows that both $\Phi(G)$ and $\Phi^-(G)$ can be computed with complexity $O(n^2)$.

Example 2.2. For the graph G_1 given in Figure 1, $v_1^* = \{v_1\}$, $v_2^* = \{v_2, v_5, v_6, v_7\}$, $v_3^* = \{v_3\}$ and $v_4^* = \{v_4\}$ are the maximal *pds*-sets. Hence $\Phi^-(G_1) = 1$ and $\Phi(G_1) = 4$. The twin graph G_1^* is P_4 and $G_1 = G_1^*[K_1, \overline{K_4}, K_1, K_1]$.



G_1
Figure 1

Example 2.3. For the graph G_2 given in Figure 2, $V_1 = \{v_1, v_2, v_3\}$, $V_2 = \{v_4, v_5\}$, $V_3 = \{v_6, v_7\}$ and $V_4 = \{v_8, v_9\}$ are maximal *pds*-sets. Also, $\Phi^-(G_2) = 2$ and $\Phi(G_2) = 3$. The twin graph $G_2^* = P_4$ and $G_2 = G_2^*[\overline{K_3}, K_2, K_2, K_2]$. For the graph G_3 given in Figure 2, $\{v_1, v_2\}$, $\{v_3, v_4\}$ and $\{v_5, v_6\}$ are maximal *pds*-sets. Hence $\Phi^-(G_3) = \Phi(G_3) = 2$. Also $G_3^* = P_3$ and $G_3 = G_3^*[K_2, K_2, K_2]$.

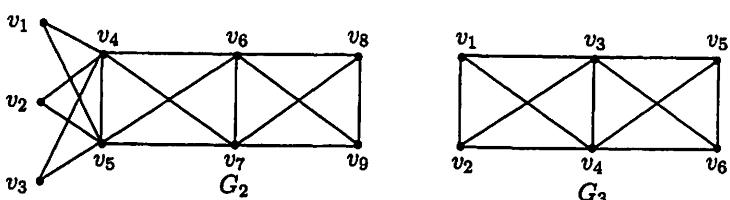


Figure 2

Observation 2.4.

(1) If a proper subset S of $V(G)$ is a *pds*-set of G, then the edge induced subgraph of $[S, N(S) - S]$ is a complete bipartite graph.

(2) Let G be any connected graph of order n which is not complete. Then there exist at least two disjoint maximal *pds*-sets in $V(G)$ and hence $\Phi^-(G) \le \lfloor \frac{n}{2} \rfloor$.

(3) Let G be any nontrivial connected graph of order n which is not complete and let k be any nonnegative integer with $k \leq \lfloor \frac{n}{2} \rfloor$. Then $\Phi^-(G) = k$ if and only if $G = G^*[G_1, G_2, \ldots, G_t]$ where G^* is the twin graph of G and each G_i is complete or independent with $k = \min\{|V(G_i)| : i = 1, 2, \ldots, t\}$.

(4) Let G be any connected graph. Then $\Phi(G) = 1$ if and only if $|v_i^*| = 1$ for each i and hence it follows that $G = G^*$.

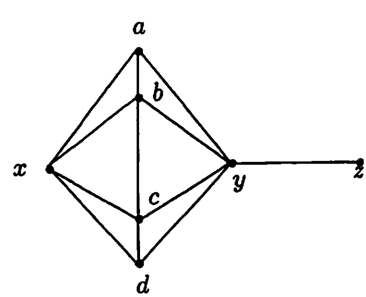
(5) Let T be any tree. Then $\Phi(T^*) = 1$ where T^* is the twin graph of T.

Lemma 2.5. *If S is a pds-set of G, then for each $u \in V(G) - S$, $|\{d(u, v) : v \in S\}| = 1$.*

Proof. Let $S = \{v_1, v_2, \ldots, v_k\}$ be a Φ-set of G. Let $u \in V(G) - S$ and $d(u, v_1) = t$. Since v_1 and v_i are distance similar for all $i = 2, 3, \ldots, k$, we have $d(u, v_i) = t$ for all $i = 2, 3, \ldots, k$. Hence $|\{d(u, v_i) : v_i \in S\}| = 1$. □

Corollary 2.6. *Let G be any connected graph which is not complete. Let S be any Φ-set of G. Then $S \subseteq N(u)$ for any vertex u in $N(S) - S$. Hence $1 \leq \Phi^-(G) \leq \Phi(G) \leq \Delta(G)$.*

Remark 2.7. The converse of Lemma 2.5 is not true. For the graph G_4 given in Figure 3, $S = \{a, b, c, d\}$ is not a Φ-set of G since the vertices a and b are not distance similar. However, $|\{d(u, v) : v \in S\}| = 1$ for all $u \in V(G_4) - S$.



G_4

Figure 3

We now proceed to determine $\Phi(G)$ and $\Phi^-(G)$ for some standard graphs.

Observation 2.8.

(1) Let G be any connected graph of order n. Then $\Phi(G) = n$ if and only if $G = K_n$, and $\Phi^-(G) = n$ if and only if $G = K_n$.

(2) Let G be any connected graph of order n. Then $\Phi(G) = n - 1$ if and only if $G = K_{1,n-1}$.

(3) For the cycle C_n, we have $\Phi(C_n) = \Phi^-(C_n) = \begin{cases} 3 & \text{if} \quad n = 3 \\ 2 & \text{if} \quad n = 4 \\ 1 & \text{if} \quad n \geq 5. \end{cases}$

Theorem 2.9. *Let G be any graph with $\delta(G) \geq 2$ and $g(G) \geq 5$ where $g(G)$ is the girth of G. Then $\Phi(G) = 1$.*

Proof. Suppose G has a Φ-set S with $|S| \geq 2$. Then $S \subseteq N(u)$ for some $u \in V(G) - S$. Since $g(G) \geq 5$, it follows that S is independent and since $\delta(G) \geq 2$, there exists a vertex $v \in N(S) - S$ with $v \neq u$. Now, both u and v are adjacent to all the vertices in S. Thus G contains a cycle C_4, a contradiction. Hence $\Phi(G) = 1$. □

Theorem 2.10. *Let T be any tree. For any vertex v of T, let $l(v)$ denote the number of leaves adjacent to v. Then $\Phi(T) = \max\{l(v) : v$ is a support vertex of $T\}$ and $\Phi^-(T) = 1$.*

Proof. Suppose $S = \{v_1, v_2, \ldots, v_k\}$ is a Φ-set of T with $|S| \geq 2$ and $S \subseteq N(u)$ for some $u \in V(T)$. Since $\langle S \rangle$ is independent, $\langle S \cup \{u\} \rangle$ is a star. Let $w \in V(T) - (S \cup \{u\})$. If w is adjacent to some $v_i \in S$, then w is adjacent to all the vertices of S and hence $\langle S \cup \{u, w\} \rangle$ contains a cycle, a contradiction. Therefore v_i, $i = 1, 2, \ldots, k$ are pendent vertices in T. Thus $\Phi(T) = \max\{l(v) : v$ is a support vertex of $T\}$. Further, if x is any support vertex in T, then $\{x\}$ is a maximal *pds*-set of T and hence $\Phi^-(T) = 1$. □

Theorem 2.11. *Let G be a graph of order n with maximum degree $\Delta(G) > 0$. Then $\Phi(G) = \Delta(G)$ if and only if G is isomorphic to the complete bipartite graph $K_{\Delta,n-\Delta}$.*

Proof. Let S be any Φ-set of G with $|S| = \Delta(G)$ and let $S = N(u)$ for some $u \in V(G)$. If $\langle S \rangle$ is complete, then $H = \langle S \cup \{u\} \rangle = K_{\Delta+1}$. Hence it follows that $G = H$ and $\Phi(G) = \Delta(G) + 1$, a contradiction. Hence $\langle S \rangle$ is independent. Now, let $v \in V(G) - N[u]$. If $d(v, S) = k \geq 2$, let $P = (v, v_1, \ldots, v_t)$ be a geodesic joining v and S. Then v_{t-1} is adjacent to all the vertices of S and also v_{t-2}. Hence $d(v_{t-1}) \geq \Delta(G) + 1$, a contradiction. Hence $d(v, S) = 1$ for all $v \in V(G) - N[u]$. Also since $d(v) = \Delta(G)$ for all $v \in V(G) - S$, it follows that $V(G) - S$ is independent. Hence G is isomorphic to the complete bipartite graph with bipartition S, $V(G) - S$.

The converse is obvious. □

Corollary 2.12. *For any graph G of order n, $\Phi^-(G) = \Delta(G)$ if and only if n is even and $G = K_{\frac{n}{2},\frac{n}{2}}$.*

Proof. If $\Phi^-(G) = \Delta(G)$, then $\Phi(G) = \Delta(G)$. By Theorem 2.11, we have $\Delta = n - \Delta$ and $G = K_{\frac{n}{2},\frac{n}{2}}$. The converse is obvious. $\square$

Theorem 2.13. *Let G be any connected graph of order n at least four. Then $\Phi(G) = n - 2$ if and only if G is isomorphic to one of the graphs $P_3[K_{n-2}, K_1, K_1]$, $P_2[K_{n-2}, 2K_1]$, $P_2[\overline{K_{n-2}}, 2K_1]$, $P_2[\overline{K_{n-2}}, K_2]$.*

Proof. Let S be any Φ-set of G with $|S| = n-2$ and let $S \subseteq N(u)$ for some $u \in V(G)$. Clearly $G^* = P_3$ or P_2. If $G^* = P_3$, then $G = P_3[K_{n-2}, K_1, K_1]$ and if $G^* = P_2$, then $G = P_2[\overline{K_{n-2}}, K_2]$ or $P_2[K_{n-2}, 2K_1]$ or $P_2[\overline{K_{n-2}}, 2K_1]$. Hence it follows that G is isomorphic to one of the graphs given in the theorem.

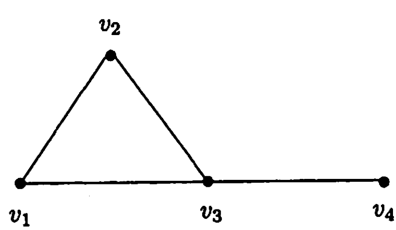
The converse is obvious. $\square$

Theorem 2.14. *Let G be any connected graph of order $n \geq 6$ and let $H = K_{n-3}$ or $\overline{K_{n-3}}$. Then $\Phi(G) = n - 3$ if and only if G is isomorphic to one of the following.*

(i) $P_2[H, 3K_1]$, $P_2[\overline{K_{n-3}}, K_3]$.

(ii) $P_3[K_1, H, K_2]$, $P_3[H, K_2, K_1]$, $P_3[H, K_1, K_2]$, $P_3[K_{n-3}, K_1, 2K_1]$, $P_3[K_{n-3}, 2K_1, K_1]$, $K_3[\overline{K_{n-3}}, 2K_1, K_1]$.

(iii) $P_4[H, K_1, K_1, K_1]$, $P_4[K_1, H, K_1, K_1]$, $G_1[\overline{K_{n-3}}, K_1, K_1, K_1]$ *where G_1 is the graph given in Figure 4.*



G_1

Figure 4

Proof. Let S be any Φ-set of G with $|S| = n - 3$ and let G^* be the twin graph of G. Then $2 \leq |V(G^*)| \leq 4$. If $|V(G^*)| = 2$, then G is isomorphic to one of the graphs given in (i). If $|V(G^*)| = 3$ and $G^* = K_3$, then $G = K_3[\overline{K_{n-3}}, 2K_1, K_1]$. Now suppose $G^* = P_3 = (v_1^*, v_2^*, v_3^*)$. We may assume without loss of generality that the vertex v^* of G^* corresponding to S is either v_1^* or v_2^*. If $v = v_2^*$, then G is isomorphic to $P_3[K_1, H, K_2]$. If $v^* = v_1^*$, then G is isomorphic to one of the graphs $P_3[H, K_2, K_1]$, $P_3[H, K_1, K_2]$, $P_3[K_{n-3}, K_1, 2K_1]$ or $P_3[K_{n-3}, 2K_1, K_1]$.

Now, suppose $|V(G^*)| = 4$. Since there exist three singleton subsets of $V(G)$ which are maximal *pds*-sets, G^* is not isomorphic to $K_4 - e$, $K_{1,3}$ or C_4. Hence G^* is isomorphic to P_4 or $K_{1,3} + e$. If $G^* = P_4$, then $G = P_4[H, K_1, K_1, K_1]$ or $P_4[K_1, H, K_1, K_1]$. If $G^* = K_{1,3} + e = G_1$, then $G = G_1[\overline{K_{n-3}}, K_1, K_1, K_1]$.

The converse is obvious. □

Theorem 2.15. *Let G_i, $0 \le i \le n$, be any nontrivial connected graphs with $V(G_0) = \{v_1, v_2, \ldots, v_n\}$ and let $G = G_0[G_1, G_2, \ldots, G_n]$. Let $k = \max\{\Phi(G_i) : 1 \le i \le n\}$. Then $k \le \Phi(G) \le k\Phi(G_0)$ and the bounds are sharp.*

Proof. Clearly any Φ-set of G_i, $1 \le i \le n$, is a *pds*-set of G and hence it follows that $\Phi(G) \ge k$. Also if S is any Φ-set of G and $V(G_i) \cap S \ne \emptyset$, then $V(G_i) \cap S$ is a *pds*-set of G_i and $\{v_i \in V(G_0) : V(G_i) \cap S \ne \emptyset\}$ is a *pds*-set of G_0. Hence $\Phi(G) \le k\Phi(G_0)$. The upper bound is attained for the graph $G = C_4[K_3, K_4, K_3, K_4]$ and the lower bound is attained for the graph $G = P_4[K_4, K_4, K_4, K_4]$. Thus the bounds are sharp. □

Corollary 2.16. *Let G and H be any two nontrivial connected graphs. Then* $\max\{\Phi(G), \Phi(H)\} \le \Phi(G + H) \le \Phi(G) + \Phi(H)$.

Corollary 2.17. *Let $G = P_k[G_1, G_2, \ldots, G_k]$, $k \ge 4$ and each G_i is a nontrivial graph. Then* $\Phi(G) = \max\{\Phi(G_i) : i = 1, 2, \ldots, k\}$.

3 Conclusion and Scope

In this paper we have initiated a study of *pds*-set and the two parameters $\Phi(G)$ and $\Phi^-(G)$. The following are some interesting problems for further investigation.

Problem 3.1. *Characterize graphs G for which $\Phi^-(G) = 1$.*

Problem 3.2. *Characterize graphs G for which $\Phi(G) = 1$.*

Problem 3.3. *For which graphs G and H, we have*

(a) $\Phi(G + H) = \max\{\Phi(G), \Phi(H)\}$

(b) $\Phi(G + H) = \Phi(G) + \Phi(H)$?

Ackowledgement

The authors are thankful to the Department of Science and Technology, Govt. of India, New Delhi for its support through *n*-CARDMATH Project No. SR/S4/MS:427/07.

References

[1] G. Chartrand and L. Lesniak, *Graphs & Digraphs*, Chapman and Hall, CRC, Fourth Edition, 2005.

[2] V. Saenpholphat and P. Zhang, Connected Resolvability of Graphs, *Czech. Math.*, **53**(128) (2003), 827–840.

[3] C. Hernando, M. Mora, I. M. Pelayo and C. Seara, Extremal Graph Theory for Metric Dimension and Diameter, *Electron. J. Combin.*, **17** (2010), #R30.

[4] M. C. Golumbic, *Algorithmic Graph Theory and Perfect Graphs*, Elsevier, Second Edition, 2004.