On mark sequences in bipartite digraphs

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Abstract

A bipartite r-digraph is an orientation of a bipartite multigraph that is without loops and contains at most r edges between any pair of vertices from distinct parts. In this paper, we obtain necessary and sufficient conditions for a pair of sequences of non-negative integers in non-decreasing order to be a pair of sequences of numbers, called marks (or r-scores), attached to the vertices of a bipartite r-digraph. These characterizations provide algorithms for constructing the corresponding bipartite multi digraph.

Keywords: bipartite multidigraph, mark, mark sequence, oriented graph, transmitter.

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1 Introduction

An r-digraph is an orientation of a multigraph that is without loops and contains at most r edges between any pair of distinct vertices. So, 1-digraph is an oriented graph, and a complete 1-digraph is a tournament. Let D be an r-digraph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$, and let $d^+_{v_i}$ and $d^-_{v_i}$ denote the outdegree and indegree, respectively, of a vertex v_i. Define p_{v_i} (or simply p_i)$= r(n-1) + d^+_{v_i} - d^-_{v_i}$ as the mark (or r-score) of v_i, so that $0 \leq p_{v_i} \leq 2r(n-1)$. Then the sequence $P = [p_i]_1^n$ in non-decreasing order is called the mark sequence of D.

An analogous result to Landau's theorem on tournament scores [5] is the following characterization of marks in r-digraphs and is due to Pirzada [10].

Theorem 1.1. *A sequence $P = [p_i]_1^n$ of non-negative integers in non-decreasing order is the mark sequence of an r-digraph if and only*

$$\sum_{i=1}^{t} p_i \geq rt(t-1),$$

for $1 \leq t \leq n$, with equality when $t = n$.

Various results on mark sequences in digraphs are given in [7, 8, 10] and we can find certain stronger inequalities of marks for digraphs in [6] and for multidigraphs in [17]. Further we can see characterizations of marks for bipartite digraphs in [4, 19] and on mark sets in [18]. Also analogous results for scores in oriented graphs can be found in [1, 11, 14].

A bipartite r-digraph is an orientation of a bipartite multigraph that is without loops and contains at most r edges between any pair of vertices from distinct parts. So bipartite 1-digraph is an oriented bipartite graph and a complete bipartite 1-digraph is a bipartite tournament. Let $D(X,Y)$ be a bipartite r-digraph with $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. For any vertex v_i in $D(X,Y)$, let $d_{v_i}^+$ and $d_{v_i}^-$ be the outdegree and indegree, respectively, of v_i. Define p_{x_i} (or simply p_i) $= rn + d_{x_i}^+ - d_{x_i}^-$ and q_{y_j} (or simply q_j)$= rm + d_{y_j}^+ - d_{y_j}^-$ as the marks (or r-scores) of x_i in X and y_j in Y respectively. Clearly, $0 \leq p_{x_i} \leq 2rn$ and $0 \leq q_{y_j} \leq 2rm$. Then the sequences $P = [p_i]_1^m$ and $Q = [q_j]_1^n$ in non-decreasing order are called the mark sequences of $D(X,Y)$.

One interpretation of a bipartite r-digraph is as follows. It can be interpreted as the result of a competition between two teams in which each player of one team plays with every player of the other team at most r times in which ties(draws) are allowed. A player receives two points for each win, and one point for each tie. With this marking system, player x_i (respectively y_j) receives a total of p_{x_i} (respectively q_{y_j}) points. The sequences P and Q of non-negative integers in non-decreasing order are said to be realizable if there exists a bipartite r-digraph with mark sequences P and Q.

In a bipartite r-digraph $D(X,Y)$, if there are a_1 arcs directed from vertex $x \in X$ to vertex $y \in Y$ and a_2 arcs directed from vertex y to vertex x, with $0 \leq a_1 \leq r$, $0 \leq a_2 \leq r$ and $0 \leq a_1 + a_2 \leq r$, we denote it by $x(a_1 - a_2)y$.

We have one of the following six possibilities between any two vertices $x \in X$ and $y \in Y$ in a bipartite 2-digraph $D(X,Y)$.

(i) Exactly two arcs directed from x to y and no arc directed from y to x, and this is denoted by $x(2-0)y$,

(ii) Exactly two arcs directed from y to x and no arc directed from x to y, and this is denoted by $x(0-2)y$,

(iii) Exactly one arc directed from x to y and exactly one arc directed from y to x, and this is denoted by $x(1-1)y$ and is called a pair of symmetric arcs between x and y,

(iv) Exactly one arc directed from x to y and no arc directed from y to x, and this is denoted by $x(1-0)y$,

(v) Exactly one arc directed from y to x and no arc directed from x to y, and this is denoted by $x(0-1)y$,

(vi) No arc directed from x to y and no arc directed from y to x, and this is denoted by $x(0-0)y$.

The following characterization of mark sequences in bipartite 2-digraphs [19] is analogous to a result on scores in bipartite tournaments due to Beineke and Moon [2].

Theorem 1.2. *Let $P = [p_i]_1^m$ and $Q = [q_j]_1^n$ be the sequences of non-negative integers in non-decreasing order. Then P and Q are the mark sequences of some bipartite 2-digraph if and only if*

$$\sum_{i=1}^{f} p_i + \sum_{j=1}^{g} q_j \geq 4fg,$$

for $1 \leq f \leq m$ and $1 \leq g \leq n$ with equality when $f = m$ and $g = n$.

Analogous results for scores in oriented bipartite graphs can be found in [9].

An oriented tetra in a bipartite r-digraph is an induced 1-subdigraph with two vertices from each part. Define oriented tetras of the form $x(1-0)y(1-0)x'(1-0)y'(1-0)x$ and $x(1-0)y(1-0)x'(1-0)y'(0-0)x$ to be of α-type and all other oriented tetras to be of β-type. A bipartite r-digraph is said to be of α-type or β-type according as all of its oriented tetras are of α-type or β-type respectively. We assume, without loss of generality, that β-type bipartite r-digraphs have no pair of symmetric arcs because symmetric arcs $x(a-a)y$, where $1 \leq a \leq \frac{r}{2}$, can be transformed to $x(0-0)y$ with the same marks. A transmitter is a vertex with indegree zero.

2 Criteria for realizability of marks

We have the following immediate observation about bipartite r-digraphs with given marks.

Lemma 2.1. *Among all bipartite r-digraphs with given mark sequences, those with the fewest arcs are of β-type.*

Proof. Suppose $D(X,Y)$ is a bipartite r-digraph with mark sequences P and Q and let $D(X,Y)$ be not of β-type. Then $D(X,Y)$ has an oriented tetra of α-type, that is, $x(1-0)y(1-0)x'(1-0)y'(1-0)x$ or $x(1-0)y(1-0)x'(1-0)y'(0-0)x$ where $x, x' \in X$ and $y, y' \in Y$. Since $x(1-0)y(1-0)x'(1-0)y'(1-0)x$ can be transformed to $x(0-0)y(0-0)x'(0-0)y'(0-0)x$ with the same mark sequences and four arcs fewer, and $x(1-0)y(1-0)x'(1-0)y'(0-0)x$ can be transformed to $x(0-0)y(0-0)x'(0-0)y'(0-1)x$ with the same mark sequences and two arcs fewer, therefore, in both cases we obtain a bipartite r-digraph having same mark sequences P and Q with fewer arcs. Note that if there are symmetric arcs between x and y, that is $x(a-a)y$, where $1 \le a \le \frac{r}{2}$, then these can be transformed to $x(0-0)y$ with the same mark sequences and a arcs fewer. Hence the result follows. □

Lemma 2.2. *Let $P = [p_i]_1^m$ and $Q = [q_j]_1^n$ be mark sequences of a β-type bipartite r-digraph. Then either the vertex with mark p_m, or the vertex with mark q_n, or both can act as transmitters.*

We now have some observations about bipartite r-digraphs, as these will be required in application of Theorem 2.3. We know if $P = [p_1, p_2, \ldots, p_m]$ and $Q = [q_1, q_2, \ldots, q_n]$ are mark sequences of a bipartite r-digraph, then $p_i \le 4n$, $1 \le i \le m$ and $q_j \le 4m$, $1 \le j \le m$.

1. If $P = [p_1, p_2, \ldots, p_{m-1}, p_m]$ and $Q = [0, 0, \ldots, 0, 0]$ with each $p_i = 2rn$ are mark sequences of some bipartite r-digraph, then $P' = [p_1, p_2, \ldots, p_{m-1}]$ and $Q' = [0, 0, \ldots, 0]$ are also mark sequences of some bipartite r-digraph.

2. If $P = [p_1, p_2, \ldots, p_{m-1}, p_m]$ and $Q = [0, 0, \ldots, 0, q_n]$ with $4n - p_m = 3$, $q_n \ge 3$ are mark sequences of some bipartite r-digraph, then $P' = [p_1, p_2, \ldots, p_{m-1}]$ and $Q' = [0, 0, \ldots, 0, q_n - 3]$ are also mark sequences of some bipartite r-digraph.

3. If $P = [p_1, p_2, \ldots, p_{m-1}, p_m]$ and $Q = [0, 0, \ldots, 0, q_n]$ with $4n - p_m = 4$, $q_n \ge 4$ are mark sequences of some bipartite r-digraph, then $P' = [p_1, p_2, \ldots, p_{m-1}]$ and $Q' = [0, 0, \ldots, 0, q_n - 4]$ are also mark sequences of some bipartite r-digraph.

4. If $P = [p_1, p_2, \ldots, p_{m-1}, p_m]$ and $Q = [0, 0, \ldots, 0, q_n]$ with $4n - p_m = 4$, $q_m \ge 3$ are mark sequences of some bipartite r-digraph, then $P' = [p_1, p_2, \ldots, p_{m-1}]$ and $Q' = [0, 0, \ldots, 0, q_n - 3]$ are also mark sequences of some bipartite r-digraph.

5. If $P = [p_1, p_2, \ldots, p_{m-1}, p_m]$ and $Q = [0, 0, \ldots, 0, 1, 3]$ with $4n - p_m = 4$, are mark sequences of some bipartite r-digraph, then the sequences $P' = [p_1, p_2, \ldots, p_{m-1}]$ and $Q' = [0, 0, \ldots, 0, 0, 0]$ are also mark sequences of some bipartite r-digraph.

6. If $P = [p_1, p_2, \ldots, p_{m-1}, p_m]$ and $Q = [0, 0, \ldots, 0, 1, 1, 2]$ with $4n - p_m = 4$, are mark sequences of some bipartite r-digraph, then the sequences $P' = [p_1, p_2, \ldots, p_{m-1}]$ and $Q' = [0, 0, \ldots, 0, 0, 0]$ are also mark sequences of some bipartite r-digraph.

7. If $P = [p_1, p_2, \ldots, p_{m-1}, p_m]$ and $Q = [0, 0, \ldots, 0, 1, 1, 1, 1]$ with $4n - p_m = 4$, are mark sequences of some bipartite r-digraph, then $P' = [p_1, p_2, \ldots, p_{m-1}]$ and $Q' = [0, 0, \ldots, 0, 0, 0]$ are also mark sequences of some bipartite r-digraph.

We note that the sequences of non-negative integers given by $[p_1]$ and $[q_1, q_2, \ldots, q_n]$, with $p_1 + q_1 + q_2 + \cdots + q_n = 2rn$, are always mark sequences of some bipartite r-digraph. We observe that the bipartite r-digraph $D(X, Y)$, with vertex sets $X = \{x_1\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$, where for q_i even, say $2t$, we have $x_1((r-t)-t)y_i$ and for q_i odd, say $2t+1$, we have $x_1((r-t-1)-t)y_i$, has mark sequences $[p_1]$ and $[q_1, q_2, \ldots, q_n]$. Also the sequences $[0]$ and $[2r, 2r, \ldots, 2r]$ are mark sequences of some bipartite r-digraph.

The next result provides a useful recursive test whether or not a pair of sequences is realizable as marks.

Theorem 2.3. *Let $P = [p_i]_1^m$ and $Q = [q_j]_1^n$ be the sequences of non-negative integers in non-decreasing order with $p_m \geq q_n$, $rn \leq p_m \leq 2rn$.*

(A) If $q_n \leq 2r(m-1)+1$, let P' be obtained from P by deleting the entry p_m, and Q' be obtained as follows.

For $[2r-(i-1)]n \geq p_m \geq (2r-i)n$, $1 \leq i \leq r$, reducing $[2r-(i-1)]n - p_m$ largest entries of Q by i each, and reducing $p_m - (2r-i)n$ next largest entries by $i-1$ each.

(B) In case $q_n > 2r(m-1)+1$, say $q_n = 2r(m-1)+1+h$, where $1 \leq h \leq r-1$, then let P' be obtained from P by deleting the entry p_m, and Q' be obtained from Q by reducing the entry q_n by $h+1$.

Then P and Q are the mark sequences of some bipartite r-digraph if and only if P' and Q' (arranged in non-decreasing order) are the mark sequences of some bipartite r-digraph.

Proof. Let P' and Q' be the mark sequences of some bipartite r-digraph $D'(X', Y')$. First suppose Q' is obtained from Q as in A. Construct a bipartite r-digraph $D(X, Y)$ as follows. Let $X = X' \cup x$ and $Y = Y'$, where x does not belong to X'. Let $x((r-i)-0)y$ for those vertices y of Y' whose marks are reduced by i in going from P and Q to P' and Q', and $x(r-0)y$ for those vertices y of Y' whose marks are not reduced in going from P and Q to P' and Q'. Then $D(X, Y)$ is the bipartite r-digraph with mark sequences P and Q. Now, if Q' is obtained from Q as in B, then construct a bipartite r-digraph $D(X,Y)$ as follows. Let $X = X' \cup x$ and $Y = Y'$, where x does not belong to X'. Let $x((r-h-1)-0)y$ for that vertex y of Y' whose marks are reduced by h in going from P and Q to P' and Q'. Then $D(X,Y)$ is the bipartite r-digraph with mark sequences P and Q.

Conversely, suppose P and Q be the mark sequences of a bipartite r-digraph $D(X,Y)$. Without loss of generality, we choose $D(X,Y)$ to be of β-type. Then by Lemma 2.2, any of the vertex $x \in X$ or $y \in Y$ with mark p_m or q_n respectively can be a transmitter. Let the vertex $x \in X$ with mark p_m be a transmitter. Clearly, $p_m \geq rn$ and because if $p_m < rn$, then by deleting p_m we have to reduce more than n entries from Q, which is absurd.

(A) Now $q_n \leq 2r(m-1)+1$ because if $q_n > 2r(m-1)+1$, then on reduction $q'_n = q_n - 1 > 2r(m-1)+1-1 = 2r(m-1)$, which is impossible.

Let $[2r-(i-1)]n \geq p_m \geq (2r-i)n$, $1 \leq i \leq r$, let V be the set of $[2r-(i-1)]n - p_m$ vertices of largest marks in Y, and let W be the set of $p_m - (2r-i)n$ vertices of next largest marks in Y and let $Z = Y - \{V, W\}$. Construct $D(X,Y)$ such that $x((r-i)-0)v$ for all $v \in V$, $x((r-i-1)-0)w$ for all $w \in W$ and $x(r-0)z$ for all $z \in Z$. Clearly, $D(X,Y) - x$ realizes P' and Q' (arranged in non-decreasing order).

(B) Now in D, let $q_n > 2r(m-1)+1$, say $q_n = 2r(m-1)+1+h$, where $1 \leq h \leq r-1$. This means $y_m(r-0)x_i$, for all $1 \leq i \leq m-1$. Since x_m is a transmitter, there cannot be an arc from y_n to x_m. Therefore $x_m((r-h-1)-0)y_n$, since y_n needs $h+1$ more marks. Now delete x_m, it will decrease the mark of y_n by $h+1$, and the resulting bipartite r-digraph will have mark sequences P' and Q' as desired. □

Theorem 2.3 provides an algorithm of checking whether or not the sequences P and Q of non-negative integers in non-decreasing order are the mark sequences, and for constructing a corresponding bipartite r-digraph. Let $P = [p_1, p_2, \ldots, p_m]$ and $Q = [q_1, q_2, \ldots, q_n]$, where $p_m \geq q_n$, $rn \leq p_m \leq 2rn$ and $q_n \leq 2r(m-1)+1$, be the mark sequences of a bipartite r-digraph with parts $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ respectively. Deleting p_m and performing A of Theorem 2.3 if $[2r-(i-1)]n \geq p_m \geq (2r-i)n$, $1 \leq i \leq r$, we get $Q' = [q'_1, q'_2, \ldots, q'_n]$. If the marks of the vertices y_j were decreased by i in this process, then the construction yielded $x_m((r-i)-0)y_j$, if these were decreased by $i-1$, then the construction yielded $x_m((r-i-1)-0)y_j$. If we perform B of Theorem 2.3, the mark of y_n was decreased by $h+1$, the construction yielded $x_m((r-h-1)-0)y_n$. For vertices y_j whose marks remained unchanged, the construction yielded $x_m(r-0)y_j$. Note that if the conditions $p_m \geq rn$ does not hold, then we delete q_n for which the conditions get satisfied and the same argument is used for defining arcs. If this procedure is applied recursively, then it tests whether or not P and Q are the mark sequences, and if P and Q are the mark sequences, then a bipartite r-digraph with mark sequences P and Q is constructed.

We illustrate this reduction and the resulting construction with the following examples.

Exercise 2.4. Consider the two sequences of non-negative integers given by $P = [14, 14, 15]$ and $Q = [6, 6, 8, 9]$. We check whether or not P and Q are mark sequences of some bipartite 3-digraph.

1. $P = [14, 14, 15]$, $Q = [6, 6, 8, 9]$.

   We delete 15. Clearly $[2r - (i-1)]n = [2.3 - (3-1)]4 = 16 \geq 15 \geq (2r - i)n = (2.3 - 3)4 = 12$. So reduce $[2r - (i-1)]n - p_m = [2.3 - (3 - 1]4 - 15 = 16 - 15 = 1$ largest entry of Q by $i = 3$ and $p_m - (2r - i)n = 15 - (2.3 - 3)4 = 15 - 12 = 3$ next largest entries of Q by $i - 1 = 3 - 1 = 2$ each, we get $P = [14, 14]$, $Q = [4, 4, 6, 6]$, and arcs are defined as $x_3(0-0)y_4$, $x_3(1-0)y_3$, $x_3(1-0)y_2$, $x_3(1-0)y_1$.

2. $P = [14, 14]$, $Q = [4, 4, 6, 6]$.

   We delete 14. Here $[2r - (i-1)]n = [2.3 - (3-1)]4 = 16 \geq 14 \geq (2r - i)n = (2.3 - 3)4 = 12$. Reduce $[2r - (i-1)]n - p_m = [2.3 - (3 - 1]4 - 14 = 16 - 14 = 2$ largest entries of Q by $i = 3$ and $p_m - (2r - i)n = 14 - (2.3 - 3)4 = 14 - 12 = 2$ next largest entries of Q by $i - 1 = 3 - 1 = 2$ each, we get $P = [14]$, $Q = [2, 2, 3, 3]$, and arcs are defined as $x_2(0-0)y_4$, $x_2(0-0)y_3$, $x_2(1-0)y_2$, $x_2(1-0)y_1$.

3. $P = [14]$, $Q = [2, 2, 3, 3]$.

   We delete 14. Here $[2r - (i-1)]n = [2.3 - (3-1)]4 = 16 \geq 14 \geq (2r - i)n = (2.3 - 3)4 = 12$. Reduce $[2r - (i-1)]n - p_m = [2.3 - (3 - 1]4 - 14 = 16 - 14 = 2$ largest entries of Q by $i = 3$ and $p_m - (2r - i)n = 14 - (2.3 - 3)4 = 14 - 12 = 2$ next largest entries of Q by $i - 1 = 3 - 1 = 2$ each, we get ϕ, $Q = [0, 0, 0, 0]$, and arcs are defined as $x_1(0-0)y_4$, $x_1(0-0)y_3$, $x_1(1-0)y_2$, $x_1(1-0)y_1$.

   The resulting bipartite 3-digraph has mark sequences $P = [14, 14, 15]$ and $Q = [6, 6, 8, 9]$ with vertex sets $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3, y_4\}$ and arcs as $x_3(0-0)y_4$, $x_3(1-0)y_3$, $x_3(1-0)y_2$, $x_3(1-0)y_1$, $x_2(0-0)y_4$, $x_2(0-0)y_3$, $x_2(1-0)y_2$, $x_2(1-0)y_1$, $x_1(0-0)y_4$, $x_1(0-0)y_3$, $x_1(1-0)y_2$, $x_1(1-0)y_1$.

The following is the combinatorial criterion [4] for determining whether the sequences of non-negative integers are realizable as marks. This is analogous to Landau's theorem [5] on tournament scores and similar to the result by Beineke and Moon [2] on bipartite tournament scores.

Theorem 2.5. *Let $P = [p_i]_1^m$ and $Q = [q_j]_1^n$ be the sequences of non-negative integers in non-decreasing order. Then P and Q are the mark sequences of some bipartite r-digraph if and only if*

$$\sum_{i=1}^{f} p_i + \sum_{j=1}^{g} q_j \geq 2rfg, \tag{1}$$

for $1 \leq f \leq m$ and $1 \leq g \leq n$, with equality when $f = m$ and $g = n$.

The concept of scores has been extended to oriented hypergraphs [13, 22], hypertournaments [3, 16, 20, 21, 22] and bipartite hypertournaments [12, 15].

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