$\widehat{S}_k$-Factorization of complete multipartite symmetric digraphs

P.HEMALATHA
Department of Mathematics
Kongu Engineering College, Erode 638 052, Tamilnadu, India.
e-mail: latha_saroja@yahoo.co.in
and
A.MUTHUSAMY
Department of Mathematics
Periyar University, Salem, Tamilnadu, India.
e-mail: ambdu@yahoo.com

Abstract

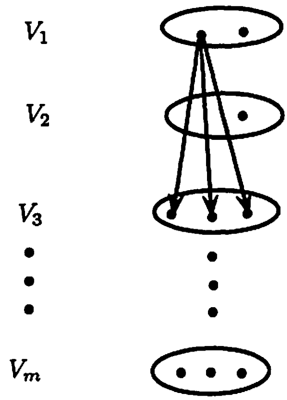
In this paper we focus our study on finding necessary and sufficient conditions required for the existence of an $\widehat{S}_k$ -factorization of $(K_m \circ \overline{K}_n)^*$ and $(C_m \circ \overline{K}_n)^*$. In particular, we show that the necessary conditions for the existence of an $\widehat{S}_k$-factorization of $(K_m \circ \overline{K}_n)^*$ are sufficient except when none of m, n is a multiple of k. In fact our results deduce some of the results of Ushio on $\widehat{S}_k$-factorizations of complete bipartite and tripartite symmetric digraphs.

Keywords: $\widehat{S}_k$-factorization, complete multipartite symmetric digraphs, wreath product of graphs.
2010 Mathematics Subject Classification Number: 05C70.

1 Introduction

Let C_m, K_m and $\overline{K}_m$ denote a cycle, a complete graph and the complement of a complete graph on m vertices respectively. For a graph G, the *symmetric digraph of* G denoted by G^*, is obtained by replacing each edge of G by a pair of symmetric arcs and rG denotes r disjoint copies of G. The graph with vertex set V having partite sets $V_1, V_2, \ldots, V_m$ such that $|V_i| = n_i$ and edge set $E = \{uv | u \in V_i,\ v \in V_j, \text{ and } i, j \in \{1, 2, \ldots, m\} \text{ with } i \neq j\}$ is called *complete m-partite graph* and is denoted by $K_{n_1, n_2, \ldots, n_m}$.

The *directed star* $\overrightarrow{K}_{1,k-1}$ having vertices in two partite sets V_i and V_j of the m - partite digraph with all arcs having tail at the center and head at the end vertices is denoted by $\widehat{S}_k$.



$\widehat{S_4} = \overrightarrow{K}_{1,3}$

For a digraph $\overrightarrow{G}$, the spanning subdigraph F of $\overrightarrow{G}$ is called an $\widehat{S}_k$ *-factor*, if each component of F is isomorphic to $\widehat{S}_k$. If $\overrightarrow{G}$ can be expressed as an arc - disjoint sum of $\widehat{S}_k$- factors, then we say that $\widehat{S}_k$*-factorizes* $\overrightarrow{G}$ or $\overrightarrow{G}$ has an $\widehat{S}_k$- *factorization* and we denote it by $\widehat{S}_k \| \overrightarrow{G}$. The *wreath product* of two graphs G and H denoted by $G \circ H$, has the vertex set $V(G) \times V(H)$ and the edge set $E(G \circ H) = \{(a,b)(c,d) | ac \in E(G)$ or $a = c$ and $bd \in E(H)\}$. For the definitions and notations not defined here we refer [1].

The existence of the star factorization of the complete bipartite and tripartite digraphs have been studied by many authors [4], [5],[6],[7] due to its wide range of applications in many fields. Ushio [3] gives an application of $K_{1,k}$ - factorization of complete bipartite graphs to combinatorial multiple-valued index-file organisation schemes of order two in database management systems. Recently, Ushio [4], [5] has obtained some necessary or sufficient conditions for the existence of an $\widehat{S}_k$ - factorization of complete bipartite and tripartite symmetric digraphs, but the same is unknown for the m-partite symmetric digraphs. Recently, the authors [2] have obtained some necessary conditions and some sufficient conditions for the existence of $\widehat{S}_k$-factorization of symmetric digraphs of tensor product of graphs.

In this paper, we have obtained some necessary conditions and some sufficient conditions for the existence of an $\widehat{S}_k$-factorization of $K^*_{n_1,n_2,\ldots,n_m}$ and $(C_m \circ \overline{K}_n)^*$. Infact, our results deduce some results of Ushio [4, 5] on $\widehat{S}_k$-factorizations of $K^*_{n,n}$ and $K^*_{n,n,n}$.

2 Main Results

2.1 Necessary Conditions

In this section we establish some necessary conditions for the existence of an $\widehat{S}_k$-factorization in $K^*_{n_1,n_2,\ldots,n_m}$ and $(C_m \circ \overline{K}_n)^*$.

Theorem 2.1. *If $K^*_{n_1,n_2,\ldots,n_m}$ has an $\widehat{S}_k$- factorization, then*

(a) $n_1 = n_2 = \ldots = n_m = n \equiv 0(mod\ k-1)$

(b) $mn \equiv 0(mod\ k)$

(c) $(m-1)n \geq (k-1)^2$.

Proof. Assume that $K^*_{n_1,n_2,\ldots,n_m}$ has an $\widehat{S}_k$-factorization. Let r be the total number of $\widehat{S}_k$-factors, s be the number of components of each $\widehat{S}_k$-factor and b be the total number of components in the factorization. By applying counting technique, we have $s = \frac{n_1+n_2+\ldots+n_m}{k}$, $b = \frac{\sum_{i,j=1,i\neq j}^{m} n_i n_j}{k-1}$ and $r = \frac{b}{s} = \frac{k}{k-1}(\frac{\sum_{i,j=1,i\neq j}^{m} n_i n_j}{\sum_{i=1}^{m} n_i})$. For a vertex $x \in V_i$, $i = 1, 2, \ldots, m$, let $c(x)$ and $t(x)$ denote number of components having x as a center and an end vertex respectively. Then, $r = c(x) + t(x)$, $c(x) = \frac{d^+(x)}{k-1} = \frac{\sum_{j=1,j\neq i}^{m} n_j}{k-1}$, $t(x) = d^-(x) = \sum_{j=1,j\neq i}^{m} n_j$. From the above equalities we have $r = \frac{k}{k-1}\sum_{j=1,j\neq i}^{m} n_j$, for $i = 1, 2, \ldots, m$ and hence $n_1 = n_2 = \ldots = n_m = n$ say. Now since $|V_i| = n$ for all $i = 1, 2, \ldots, m$, it is clear that $K^*_{n_1,n_2,\ldots,n_m} \cong (K_m \circ \overline{K}_n)^*$. Therefore,

$$r = \frac{k(m-1)n}{k-1}$$

and

$$s = \frac{mn}{k}. \tag{1}$$

By the definition of $\widehat{S}_k$, number of arcs from the vertex $x \in V_i$ to V_j must be a multiple of $k-1$. Therefore

$$n \equiv 0(mod\ k-1). \tag{2}$$

This proves (a). Again by (1), $mn \equiv 0(mod\ k)$. This proves (b). To complete the proof it remains to show that $(m-1)n \geq (k-1)^2$. Among the s components of an $\widehat{S}_k$-factor, let s_{ij} be the number of components with center vertices at V_i and end vertices at V_j. Then, $0 \leq s_{ij} \leq \frac{n}{k-1}$. W. l. o. g. n can be written as,

$n = s_{12} + (k-1)s_{21} + s_{13} + (k-1)s_{31} + \ldots + s_{1m} + (k-1)s_{m1}$.
i.e., $s_{12} + s_{13} + \ldots + s_{1m} = n - (k-1)(s_{21} + s_{31} + \ldots + s_{m1})$.
$\sum_{j=2}^{m} s_{1j} = (k-1)p - (k-1)\sum_{j=2}^{m} s_{j1}$

for some integer $p \geq 1$ (using equation (2)). Hence,

$$\sum_{j=2}^{m} s_{1j} = (k-1)(p - \sum_{j=2}^{m} s_{j1}) = (k-1)q,$$

for some $q \geq 0$. In general,

$$\sum_{j=1, j\neq i}^{m} s_{ij} = (k-1)q, \tag{3}$$

for $i = 1, 2, ..., m$. But

$$\sum_{j=1, j\neq i}^{m} s_{ij} \leq \frac{(m-1)n}{k-1},\ i = 1, 2, ..., m. \tag{4}$$

From the equations (3) and (4) we have, $(k-1)q \leq \frac{(m-1)n}{k-1}$. i.e., $(m-1)n \geq (k-1)^2$. Hence (c) holds. □

Theorem 2.2. *If $(C_m \circ \overline{K}_n)^*$ has an $\widehat{S}_k$–factorization, then*

(a) $mn \equiv 0(mod\ k)$

(b) $n \equiv 0(mod\ k-1)$

(c) $2n \geq (k-1)^2$.

Proof. Let $V_1, V_2, ..., V_m$ be the m- partite sets of the vertex set of $(C_m \circ \overline{K}_n)^*$. Assume that $(C_m \circ \overline{K}_n)^*$ has an $\widehat{S}_k$-factorization. Let r be the number of $\widehat{S}_k$-factors, s be the number of components in each $\widehat{S}_k$-factor and b be the total number of components in the $\widehat{S}_k$-factorization. We know that $(C_m \circ \overline{K}_n)^*$ has mn vertices and $2mn^2$ arcs. Therefore, $s = \frac{mn}{k}$. This proves (a). By the definition of $\widehat{S}_k$, the set of arcs from the vertex $x \in V_i$ to V_j must be exhausted by a collection of $\widehat{S}_k$'s with center at x. Therefore the number of arcs from x to V_j must be a multiple of $k-1$. i.e., $n \equiv 0(mod\ k-1)$. This proves (b). Among the s components of each $\widehat{S}_k$-factor, let s_{ij} be the number of components having center at V_i and end vertices are in V_j. Therefore, $0 \leq s_{ij} \leq \frac{n}{k-1}$.
i.e.,

$$\sum_{j=i-1, i\neq j}^{i+1} s_{ij} \leq \frac{2n}{k-1},\ i, j = 1, 2, ..., m \tag{5}$$

W. l. o. g. n can be written as,

$$\begin{aligned} n &= s_{12} + (k-1)s_{21} + s_{1m} + (k-1)s_{m1} \\ i.e., s_{12} + s_{1m} &= n - (k-1)(s_{21} + s_{m1}) \\ &= (k-1)q - (k-1)(s_{21} + s_{m1}) \text{ (using (b))} \\ &= (k-1)(q - (s_{21} + s_{m1})) \\ &= (k-1)p, \text{ for some } p \geq 0. \end{aligned}$$

In general,

$$\sum_{j=i-1, i\neq j}^{i+1} s_{ij} = (k-1)p,$$

$i = 1, 2, ..., m$. Using this value in (5), we get $(k-1)p \leq \frac{2n}{k-1}$. i.e., $2n \geq ((k-1)^2)p$. i.e., $2n \geq (k-1)^2$. This proves (c). □

Notation: We denote an $\widehat{S}_k$ with center vertex u and end vertices $v_1, v_2, ...,$ v_{k-1} by $(u; v_1, v_2, ..., v_{k-1})$.

2.2 Sufficient conditions

Lemma 2.3. *If $(C_m \circ \overline{K}_n)^*$ and $(K_m \circ \overline{K}_n)^*$ have $\widehat{S}_k$-factorizations, then so does $(C_m \circ \overline{K}_{sn})^*$ and $(K_m \circ \overline{K}_{sn})^*$, for every positive integer s.* □

Lemma 2.4. *If $n \equiv 0 (mod\ k(k-1))$ then $\widehat{S}_k \| (C_m \circ \overline{K}_n)^*$, for all m, $k \geq 2$.*

Proof. Let $n = k(k-1)s$. If $s = 1$, then $n = k(k-1)$. Let the m - partite sets of $(C_m \circ \overline{K}_n)^*$ be $V_1 = \{1^1, 2^1, ..., (k(k-1))^1\}$, $V_2 = \{1^2, 2^2, ..., (k(k-1))^2\}, ..., V_m = \{1^m, 2^m, ..., (k(k-1))^m\}$. Now for $j = 0, 1, 2, ..., k-1$, we can construct k^2 $\widehat{S}_k$ - factors $F_{1j}, F_{2j}, ..., F_{kj}$ of $(C_m \circ \overline{K}_n)^*$ as follows:

$$\begin{aligned} F_{1j} = \bigoplus_{i=0}^{(k-2)} \{ & (((i+1)^1; ((k-1)(j+i)+1)^2, \\ & ((k-1)(j+i)+2)^2, ..., ((k-1)(j+i+1))^2), \\ & (((k-1)(k-1+j)+i+1)^2; ((k-1)(k-1+j+i)+1)^3, \\ & ((k-1)(k-1+j+i)+2)^3, ..., ((k-1)(k+j+i))^3), \\ & (((k-1)(k-2+j)+i+1)^3; ((k-1)(k-2+j+i)+1)^4, \\ & ((k-1)(k-2+j+i)+2)^4, ..., ((k-1)(k-1+j+i))^4), \\ & ... \\ & (((k-1)(k-(m-1)+j)+i+1)^m; ((k-1)(i+1)+1)^1, \\ & ((k-1)(i+1)+2)^1, ..., ((k-1)(i+2))^1)\}, \\ & j = 0, 1, 2, ..., (k-1) \end{aligned}$$

$$
\begin{aligned}
F_{2j} &= \bigoplus_{i=0}^{(k-2)} \{(((k-1)+i+1)^1; ((k-1)(j+i+1)+1)^2, \\
&\quad ((k-1)(j+i+1)+2)^2, ..., ((k-1)(j+i+2))^2), \\
&\quad (((k-1)j+i+1)^2; ((k-1)(j+i+1)+1)^3, \\
&\quad ((k-1)(j+i+1)+2)^3, ..., ((k-1)(j+i+2))^3), \\
&\quad (((k-1)j+i+1)^3; ((k-1)(j+i+1)+1)^4, \\
&\quad ((k-1)(j+i+1)+2)^4, ..., ((k-1)(j+i+2))^4), \\
&\quad ... \\
&\quad (((k-1)j+i+1)^m; ((k-1)(i+2)+1)^1, \\
&\quad ((k-1)(i+2)+2)^1, ..., ((k-1)(i+3))^1)\}, \\
&\quad j = 0, 1, 2, ..., (k-1) \\
F_{3j} &= \bigoplus_{i=0}^{(k-2)} \{(2(k-1)+i+1)^1; ((k-1)(j+i+2)+1)^2, \\
&\quad ((k-1)(j+i+2)+2)^2, ..., ((k-1)(j+i+3))^2), \\
&\quad (((k-1)(k+1+j)+i+1)^2; ((k-1)(k+3+j+i)+1)^3, \\
&\quad ((k-1)(k+3+j+i)+2)^3, ..., ((k-1)(k+4+j+i))^3), \\
&\quad (((k-1)((k+2+j)+i+1)^3; ((k-1)(k+4+j+i)+1)^4, \\
&\quad ((k-1)(k+4+j+i)+2)^4, ..., ((k-1)(k+5+j+i))^4), \\
&\quad ... \\
&\quad (((k-1)(k+(m-1)+j)+i+1)^m; ((k-1)(i+3)+1)^1, \\
&\quad ((k-1)(i+3)+2)^1, ..., ((k-1)(i+4))^1)\}, \\
&\quad j = 0, 1, 2, ..., (k-1) \\
F_{4j} &= \bigoplus_{i=0}^{(k-2)} \{((3(k-1)+i+1)^1; ((k-1)(j+i+3)+1)^2, \\
&\quad ((k-1)(j+i+3)+2)^2, ..., ((k-1)(j+i+4))^2), \\
&\quad (((k-1)(k+2+j)+i+1)^2; ((k-1)(k+5+j+i)+1)^3, \\
&\quad ((k-1)(k+5+j+i)+2)^3, ..., ((k-1)(k+6+j+i))^3), \\
&\quad (((k-1)((k+4+j)+i+1)^3; ((k-1)(k+7+j+i)+1)^4, \\
&\quad ((k-1)(k+7+j+i)+2)^4, ..., ((k-1)(k+8+j+i)^4), \\
&\quad ... \\
&\quad (((k-1)(k+2m-2)+j)+i+1)^m; ((k-1)(i+4)+1)^1, \\
&\quad ((k-1)(i+4)+2)^1, ..., ((k-1)(i+5))^1)\}, \\
&\quad j = 0, 1, 2, ..., (k-1) \\
&\quad \cdots
\end{aligned}
$$

$$
\begin{aligned}
F_{kj} &= \bigoplus_{i=0}^{(k-2)} \{(((k-1)(k-1)+i+1)^1; ((k-1)(k-1+j+i)+1)^2, \\
&\quad ((k-1)(k-1+j+i)+2)^2, ..., ((k-1)(k+j+i))^2), \\
&\quad (((k-1)(2k-2+j)+i+1)^2; ((k-1)(2k-3+j+i)+1)^3, \\
&\quad ((k-1)(2k-3+j+i)+2)^3, ..., ((k-1)(2k-2+j+i))^3), \\
&\quad (((k-1)((2k-4+j)+i+1)^3; ((k-1)(2k-5+j+i)+1)^4, \\
&\quad ((k-1)(2k-5+j+i)+2)^4, ..., ((k-1)(2k-4+j+i)^4), \\
&\quad \dots \\
&\quad (((k-1)(2(k-(m-1))+j)+i+1)^m; ((k-1)(j+i)+1)^1, \\
&\quad ((k-1)(j+i)+2)^1, ..., ((k-1)(j+i+1))^1)\}, \\
&\quad j = 0, 1, 2, ..., (k-1),
\end{aligned}
$$

where the additions are taken modulo $k(k-1)$ with residues $1, 2, ..., k(k-1)$. Clearly each F_{pj}, $1 \le p \le k$, $0 \le j \le k-1$ is an $\widehat{S}_k$-factor. When p and j varies we have k^2 $\widehat{S}_k$-factors of $(C_m \circ \overline{K}_{k(k-1)})^*$. Due to symmetry of arcs, we have another k^2 $\widehat{S}_k$-factors as above. All these $2k^2$ $\widehat{S}_k$-factors together comprise an $\widehat{S}_k$-factorization of $(C_m \circ \bar{K}_{k(k-1)})^*$. By Lemma 2.3, $\widehat{S}_k \| (C_m \circ \bar{K}_{sk(k-1)})^*$ and hence $\widehat{S}_k \| (C_m \circ \overline{K}_n)^*$. □

Theorem 2.5. *If $n \equiv 0 (mod\ k-1)$, then $\widehat{S}_k \| (K_k \circ \overline{K}_n)^*$.*

Proof. Let $n = (k-1)s$. If $s = 1$, then $n = k-1$. Let the k - partite sets of $(K_k \circ \overline{K}_n)^*$ be $V_1 = \{1^1, 2^1, ..., (k-1)^1\}$, $V_2 = \{1^2, 2^2, ..., (k-1)^2\}, ...,$ $V_k = \{1^k, 2^k, ..., (k-1)^k\}$. We now construct $k(k-1)$ $\widehat{S}_k$-factors F_p^r, $r = 1, 2, ..., k$, $p = 0, 1, 2, ..., k-2$ of $(K_k \circ \overline{K}_{k-1})^*$ having the vertices of V_r, $r \in \{1, 2, ..., k\}$ as its center vertices as follows:

$$F_p^r = \{((i+p+1)^r; 1^{(i+r+1)}, 2^{(i+r+1)}, ..., (k-1)^{(i+r+1)}),\ i = 0, 1, 2, ..., k-2\},$$

where the additions in the superscripts are taken modulo k with residues $1, 2, ..., k$ and $(i+p+1)$ is taken modulo $k-1$ with residues $1, 2, ..., k-1$. Arc - disjoint sum of F_p^r, $r = 1, 2, ..., k$, $p = 0, 1, 2, ..., k-2$ gives all the $\widehat{S}_k$-factors of $(K_k \circ \overline{K}_{k-1})^*$. All these $k(k-1)$ $\widehat{S}_k$ -factors of $(K_k \circ \overline{K}_{k-1})^*$ comprise an $\widehat{S}_k$ -factorization of $(K_k \circ \overline{K}_{k-1})^*$. Therefore, by Lemma 2.3, $\widehat{S}_k \| \left((K_k \circ \overline{K}_{(k-1)s})^* \cong (K_k \circ \overline{K}_n)^*\right)$. □

Theorem 2.6. $\widehat{S}_k \| (K_{sk} \circ \overline{K}_n)^*$ *if and only if* $n \equiv 0 (mod\ k-1)$.

Proof. Assume that $n \equiv 0 (mod\ k-1)$. Let $n = r(k-1)$. If $r = 1$, then $n = k-1$. Therefore by the definition of wreath product we have,

$$(K_{sk} \circ \overline{K}_{k-1})^* \cong s(K_k \circ \overline{K}_{k-1})^* \oplus (K_s \circ \overline{K}_{k(k-1)})^*. \qquad (6)$$

Case (i) s even.

Then $(K_s \circ \overline{K}_{k(k-1)})^* \cong \bigoplus_{i=1}^{s-1}(F_i \circ \overline{K}_{k(k-1)})^*$, where F_i is a 1-factor of K_s. Now each $(F_i \circ \overline{K}_{k(k-1)})^* \cong \frac{s}{2}K_{k(k-1),k(k-1)}$. But by Lemma 2.4, $\widehat{S}_k \| K_{k(k-1),k(k-1)}$ and hence
$\widehat{S}_k \| \left(\frac{s}{2}K_{k(k-1),k(k-1)} \cong (F_i \circ \overline{K}_{k(k-1)})^*\right)$. Therefore,

$$\widehat{S}_k \| \bigoplus_{i=1}^{s-1}(F_i \circ \overline{K}_{k(k-1)})^*. \tag{7}$$

Case (ii) s odd.

Then $(K_s \circ \overline{K}_{k(k-1)})^* \cong \bigoplus_{i=1}^{\frac{s-1}{2}}(H_i \circ \overline{K}_{k(k-1)})^*$, where H_i is a Hamilton cycle of K_s. By Lemma 2.4 we have, $\widehat{S}_k \| (H_i \circ \overline{K}_{k(k-1)})^*$ and hence

$$\widehat{S}_k \| \left(\bigoplus_{i=1}^{\frac{s-1}{2}}(H_i \circ \overline{K}_{k(k-1)})^* \cong (K_s \circ \overline{K}_{k(k-1)})^* \right). \tag{8}$$

(7) and (8) shows that $\widehat{S}_k \| (K_s \circ \overline{K}_{k(k-1)})^*$ for all s. By Theorem 2.5 we have,

$$\widehat{S}_k \| s(K_k \circ \overline{K}_{k-1})^*. \tag{9}$$

Hence $\widehat{S}_k$-factorization of $(K_{sk} \circ \overline{K}_{k-1})^*$ follows from (6), (7), (8) and (9). Again by Lemma 2.3, $\widehat{S}_k \| ((K_{sk} \circ \overline{K}_{r(k-1)})^* \cong (K_{sk} \circ \overline{K}_n)^*)$. Necessity follows from Theorem 2.1. □

Note 2.7. If $n \equiv 0 (mod\ k)$, then by Theorem 2.1, $n \equiv 0(mod\ k(k-1))$. The $\widehat{S}_k$-factorization of $(K_m \circ \overline{K}_n)^*$ follows from the proof of cases (i) and (ii) of Theorem 2.6.

Theorem 2.8. *Suppose m is a prime and k is not a multiple of m, then $\widehat{S}_k \| (K_m \circ \overline{K}_n)^*$ if and only if $n \equiv 0(mod\ k(k-1))$.*

Proof. Assume that $\widehat{S}_k \| (K_m \circ \overline{K}_n)^*$. If m is a prime and k is not a multiple of m, then $(m, k) = 1$. Then by Theorem 2.1, $n \equiv 0(mod\, k(k-1))$. Hence the necessity follows.
Conversely, assume that $n \equiv 0(mod\ k(k-1))$. Now,

$$(K_m \circ \overline{K}_n)^* \cong \begin{cases} \bigoplus_{i=1}^{\frac{m-1}{2}}(H_i \circ \overline{K}_n)^*, & \text{if m is odd;} \\ \bigoplus_{i=1}^{m-1}(F_i \circ \overline{K}_n)^*, & \text{if m is even} \end{cases} \tag{10}$$

where H_i and F_i are respectively a Hamilton cycle and a 1-factor of K_m. By Lemma 2.4,

$$\widehat{S}_k \| (H_i \circ \overline{K}_n)^* \text{ and } \widehat{S}_k \| (F_i \circ \overline{K}_n)^*. \tag{11}$$

Hence $\widehat{S}_k$-factorization of $(K_m \circ \overline{K}_n)^*$ follows from (10) and (11). □

Remark 2.9. The necessary conditions given in Theorem 2.2 are sufficient in the following cases:

(1) When $m = 2$, $\widehat{S}_k \| ((C_2 \circ \overline{K}_n)^* = K^*_{n,n})$ if and only if $n \equiv 0(mod\ k(k-1))$.
Proof follows from Theorem 2.2 and Lemma 2.4.

(2) When $m = 3$, $\widehat{S}_k \| (C_3 \circ \overline{K}_n)^*$, if $k \equiv 0(mod\ 3)$, $n \equiv 0(mod\ \frac{2k}{3}(k-1))$, $2n \geq (k-1)^2$ and $\widehat{S}_k \| (C_3 \circ \overline{K}_n)^*$, if $k \equiv 1, 2(mod\ 3)$, $n \equiv 0(mod\ k(k-1))$.
Proof follows from the result of Ushio [5].

(3) When $m = 4$, $(C_4 \circ \overline{K}_n)^* \cong K^*_{2n,2n}$. Hence for even $k \geq 2$, $\widehat{S}_k \| (C_4 \circ \overline{K}_n)^*$ if $n \equiv 0(mod\ \frac{k}{2}(k-1))$ and for odd $k \geq 3$, $\widehat{S}_k \| (C_4 \circ \overline{K}_n)^*$ if $n \equiv 0(mod\ k(k-1))$, by Lemma 2.4.

(4) When $m \geq 5$, $\widehat{S}_k \| (C_m \circ \overline{K}_n)^*$ if $n \equiv 0(mod\ k(k-1))$, $k \geq 2$.
Proof follows from Lemma 2.4.

(5) For all $m \geq 2$, $\widehat{S}_3 \| (C_m \circ \overline{K}_n)^*$, if $3|mn$ and n is even.

Proof. Since 3 is a prime, $3|mn$ implies at least one of m, n is a multiple of 3.

Case (i) $m \equiv 0(mod\ 3)$.

Let $m = 3s$ and $n = 2r$ for some positive integer s and r. Let $r = 1$ and hence $n = 2$. For $j = 1, 2, 3$ and $p = 0, 1, 2, ..., s-1$, we get $6s$ $\widehat{S}_3$-factors F'_{jp} and F''_{jp} of $(C_{3s} \circ \overline{K}_2)^*$ as follows:

$$\begin{aligned} F'_{jp} &= \bigoplus_{p=0}^{(s-1)} \{((1)^{j+3p}; (1)^{j+3p+1}, (2)^{j+3p+1}), \\ &\quad ((2)^{j+3p}; (1)^{j+3p+m-1}, (2)^{j+3p+m-1})\}, \\ F''_{jp} &= \bigoplus_{p=0}^{(s-1)} \{((1)^{j+3p}; (1)^{j+3p+m-1}, (2)^{j+3p+m-1}), \\ &\quad ((2)^{j+3p}; (1)^{j+3p+1}, (2)^{j+3p+1})\} \end{aligned}$$

where the additions in superscripts are taken modulo m with residues $1, 2, 3$. Thus, when p and j varies the $6s$ $\widehat{S}_3$-factors F'_{jp} and F''_{jp} together comprise an $\widehat{S}_3$-factorization of $(C_{3s} \circ \overline{K}_2)^*$. Hence by Lemma 2.3, $\widehat{S}_3 \| ((C_{3s} \circ \overline{K}_{2r})^* \cong (C_{3s} \circ \overline{K}_n)^*)$.

Case (ii) $n \equiv 0(mod\ 3)$.

Since $3|n$ and n is even, $n = 6r$ for some positive integer r. Then by Lemma 2.4, $\widehat{S}_3 \| (C_m \circ \overline{K}_n)^*$ for all $m \geq 2$. □

3 Conclusion

Theorems 2.6, 2.8 and Note 2.7 show that the necessary conditions given in Theorem 2.1 are sufficient for the existence of an $\widehat{S}_k$ -factorization of $(K_m \circ \overline{K}_n)^*$, if at least one of m, $n \equiv 0 (mod\ k)$. Further, Lemma 2.4 deduce some of the results of Ushio [4], [5] when $m = 2, 3$.

References

[1] J. A. Bondy and U. S. R. Murty, *Graph theory with applications*, Elsevier, New York (1976).

[2] P. Hemalatha and A. Muthusamy, $\widehat{S}_k-$factorization of symmetric Digraphs of Tensor product of Graphs, *Ramanujan Math. Soc. Lect. Notes Ser.*, **7** (2008), 233–236.

[3] K. Ushio, G-designs and related designs, *Discrete Math.*, **116** (1993), 299–311.

[4] K. Ushio, Star-factorization of symmetric complete bipartite digraphs, *Discrete Math.*, **167/168** (1997), 593–596.

[5] K. Ushio, $\widehat{S}_k-$factorization of symmetric complete tripartite digraphs, *Discrete Math.*, **197/198** (1999), 791–797.

[6] K. Ushio, Star-factorization of symmetric complete bipartite multi-digraphs, *Discrete Math.*, **215** (2000), 293–297.

[7] H. Wang, On $K_{1,k}$-factorization of complete bipartite graph, *Discrete Math.*, **126** (1994),359–364.