Distance antimagic graphs

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Abstract

Let $G = (V, E)$ be a graph of order n. Let $f : V \to \{1, 2, \ldots, n\}$ be a bijection. For any vertex $v \in V$, the neighbor sum $\sum_{u \in N(v)} f(u)$ is called the weight of the vertex v and is denoted by $w(v)$. If $w(x) \neq w(y)$ for any two distinct vertices x and y, then f is called a distance antimagic labeling. In this paper we present several results on distance antimagic graphs along with open problems and conjectures.

Keywords: distance magic labeling, (a, d)-distance antimagic labeling, distance antimagic labeling.
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1 Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [4].

In 1994 Vilfred [7] in his doctoral thesis introduced the concept of sigma labelings. B. D. Acharya et al. [1] further studied the concept under the name of neibourhood magic graphs. The same concept was introduced by Miller et al. [5] under the name 1-vertex magic vertex labeling. Sugeng et al. [6] introduced the term distance magic labeling for this concept.

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Definition 1.1. [5] A *distance magic labeling* of a graph G of order n is a bijection $f : V \to \{1, 2, \ldots, n\}$ with the property that there is a positive integer k such that $\sum_{y \in N(x)} f(y) = k$ for every $x \in V$. The constant k is called the *magic constant* of the labeling f.

The sum $\sum_{y \in N(x)} f(y)$ is called the *weight* of the vertex x and is denoted by $w(x)$.

For a recent survey and open problems on distance magic graphs we refer to Arumugam et al. [2].

Let G be a distance magic graph of order n with labeling f and magic constant k. Then $\sum_{u \in N_{G^c}(v)} f(u) = \frac{n(n+1)}{2} - k - f(v)$, and hence the set of all vertex weights in G^c is $\{\frac{n(n+1)}{2} - k - i : 1 \leq i \leq n\}$, which is an arithmetic progression with first term $a = \frac{n(n+1)}{2} - k - n$ and common difference $d = 1$.

Motivated by this observation, in [3] we introduced the following concept of (a, d)-distance antimagic graph.

Definition 1.2. [3] A graph G is said to be *(a, d)-distance antimagic* if there exists a bijection $f : V \to \{1, 2, \ldots, n\}$ such that the set of all vertex weights is $\{a, a + d, a + 2d, ..., a + (n - 1)d\}$ and any graph which admits such a labeling is called an *(a, d)-distance antimagic graph.*

Thus the complement of every distance magic graph is an $(a, 1)$-distance antimagic graph.

We observe that if a graph G is (a, d)-distance antimagic with $d > 0$, then for any two distinct vertices u and v we have $w(u) \neq w(v)$. This observation naturally leads to the concept of distance antimagic labeling and in this paper we present several basic results and open problems on this concept.

We need the following definitions

Definition 1.3. A leaf of a tree T is a vertex of degree one and a support vertex of T is a vertex adjacent to a leaf.

Definition 1.4. The corona $G \odot H$ of two graphs G and H is the graph obtained by taking one copy of G of order n and n copies of H, and then joining the i^{th} vertex of G to every vertex in the i^{th} copy of H.

2 Distance antimagic graphs

Definition 2.1. Let $G = (V, E)$ be a graph of order n. Let $f : V \to \{1, 2, \ldots, n\}$ be a bijection. If $w(x) \neq w(y)$ for any two distinct vertices x

and y in V, then f is called a *distance antimagic labeling*. Any graph G which admits a distance antimagic labeling is called a *distance antimagic graph*.

Example 2.2. A distance antimagic labeling of the Petersen graph is given in Figure 1. The vertex labels are in usual font and weights are in bold font.

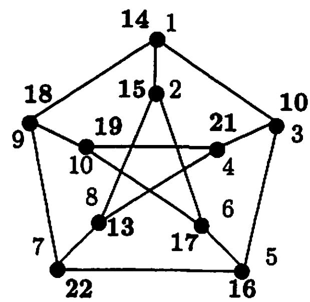

Figure 1. Petersen Graph

Observation 2.3. Obviously if a graph G is distance antimagic, then for any two distinct vertices u and v, we have $N(u) \neq N(v)$.

Hence we have the following

(i) The complete bipartite graph $G = K_{m,n}$ is distance antimagic if and only if $G = K_{1,1} = K_2$.

(ii) If a tree T is distance antimagic, then every support vertex v has exactly one leaf adjacent to v.

Any graph G which is (a, d)-distance antimagic with $d > 0$ is distance antimagic. The converse is not true. For example, the graph P_4 is distance antimagic but not (a, d)-distance antimagic. In [3] we have proved that $G = nK_2$ is $(1,1)$-distance antimagic and the Cartesian product $C_n \square K_2$ is $(n+2, 1)$-distance antimagic and hence these graphs are distance antimagic. Also the complete graph K_n is trivially distance antimagic.

When n is even, the cycle C_n is not (a, d)-distance antimagic([3]) but it is distance antimagic, as shown in the following theorem.

Theorem 2.4. *The cycle C_n is distance antimagic except when $n = 4$.*

Proof. Let $n \neq 4$ and let $C_n = (v_1, v_2, \ldots, v_n, v_1)$.
Case(i) n is odd.
Define $f : V(C_n) \to \{1, 2, \ldots, n\}$ by $f(v_i) = i$ for $1 \leq i \leq n$. Clearly f is a bijection and

$$w(v_i) = \begin{cases} n+2 & \text{if } i = 1 \\ 2i & \text{if } 2 \le i \le n-1 \\ n & \text{if } i = n \end{cases}$$

Case(ii) n is even.
Define $f : V(C_n) \to \{1, 2, \ldots, n\}$ by $f(v_i) = i$ for $2 \le i \le n-1$, $f(v_1) = n$ and $f(v_n) = 1$. Then

$$w(v_i) = \begin{cases} 3 & \text{if } i = 1 \\ n+3 & \text{if } i = 2 \\ 2i & \text{if } 3 \le i \le n-2 \\ n-1 & \text{if } i = n-1 \\ 2n-1 & \text{if } i = n \end{cases}$$

In both cases the vertex weights are distinct and hence C_n is distance antimagic if $n \neq 4$. When $n = 4$, $N(v_1) = N(v_3) = \{v_2, v_4\}$ and hence C_4 is not distance antimagic. □

Theorem 2.5. *The wheel W_n is distance antimagic except when $n = 4$.*

Proof. Let $n \neq 4$ and let $W_n = C_n + K_1$, where $C_n = (v_1, v_2, \ldots, v_n, v_1)$ and the vertex of maximum degree is v_0.
Case(i) n is odd.
Define $f : V(W_n) \to \{1, 2, \ldots, n+1\}$ by $f(v_i) = i$ for $1 \le i \le n$ and $f(v_0) = n+1$. Clearly f is a bijection and

$$w(v_i) = \begin{cases} \frac{n(n+1)}{2} & \text{if } i = 0 \\ 2n+3 & \text{if } i = 1 \\ n+1+2i & \text{if } 2 \le i \le n-1 \\ 2n+1 & \text{if } i = n \end{cases}$$

Case(ii) n is even.
Define $f : V(W_n) \to \{1, 2, \ldots, n\}$ by $f(v_i) = i$ for $2 \le i \le n-1$, $f(v_1) = n$, $f(v_n) = 1$ and $f(v_0) = n+1$. Then

$$w(v_i) = \begin{cases} \frac{n(n+1)}{2} & \text{if } i = 0 \\ n+4 & \text{if } i = 1 \\ 2n+4 & \text{if } i = 2 \\ n+1+2i & \text{if } 3 \le i \le n-2 \\ 2n & \text{if } i = n-1 \\ 3n & \text{if } i = n \end{cases}$$

In both cases the vertex weights are distinct and hence W_n is distance antimagic if $n \neq 4$. When $n = 4$, $N(v_1) = N(v_3) = \{v_2, v_4, v_5\}$ and hence W_4 is not distance antimagic. □

Theorem 2.6. *The path P_n is distance antimagic.*

Proof. Let $P_n = (v_1, v_2, \dots, v_n)$.
Case(i) $n = 2k$.
Define a bijection $f : V(G) \to \{1, 2, \dots, 2k\}$ by $f(v_i) = i$ for $1 \le i \le 2k$. Then the weight of any vertex is given by $w(v_i) = 2i$ where $1 \le i \le 2k-1$, and $w(v_{2k}) = 2k - 1$.
Case(ii) $n = 2k + 1$.
Define a bijection $f : V(G) \to \{1, 2, \dots, 2k+1\}$ by $f(v_i) = i - 1$ for $2 \le i \le 2k+1$ and $f(v_1) = 2k + 1$. Then

$$w(v_i) = \begin{cases} 1 & \text{if } i = 1 \\ 2k+3 & \text{if } i = 2 \\ 2(i-1) & \text{if } 3 \le i \le 2k \\ 2k-1 & \text{if } i = 2k+1 \end{cases}$$

In both cases for any two distnict vertices x and y in $V(G)$, $w(x) \neq w(y)$. Hence P_n is distance antimagic. □

Theorem 2.7. *The graph $G = rK_2 + K_1$ is distance antimagic.*

Proof. Let $V(G) = \{u_1, u_2, \dots, u_r, v_1, v_2, \dots, v_r, u\}$, $u_iv_i \in E(G)$, and u is adjacent to u_i, v_i for all i, $1 \le i \le r$. Now define $f : V(G) \to \{1, 2, \dots, 2r+1\}$ by $f(u_i) = i$ and $f(v_i) = r + i$, where $1 \le i \le r$ and $f(u) = 2r + 1$. Then

$$w(x) = \begin{cases} 3r+1+i & \text{if } x = u_i \\ 2r+1+i & \text{if } x = v_i \\ r(2r+1) & \text{if } x = u \end{cases}$$

Clearly the vertex weights are all distinct and hence G is distance antimagic. □

Theorem 2.8. *For any graph G of order n, the carona $H = G \odot K_1$ is distance antimagic.*

Proof. Let $f : V(G) \to \{1, 2, \dots, n\}$ be a bijection. Let $V(G) = \{v_1, v_2, \dots, v_n\}$ where $w(v_1) \le w(v_2) \le \dots \le w(v_n)$. Let u_i be the leaf which is adjacent to v_i in H. Now $g : V(H) \to \{1, 2, \dots, 2n\}$ defined by

$$\begin{aligned} g(v_i) &= f(v_i) && \text{and} \\ g(u_i) &= n + i, && 1 \le i \le n, \end{aligned}$$

is a distance antimagic labeling of H. □

Theorem 2.9. *The graph $G = S_k(K_{1,n})$, $k \ge 3$ is distance antimagic, where $S_k(K_{1,n})$ is the graph obtained from $K_{1,n}$ by subdividing every edge $k - 2$ times.*

Proof. If $n = 1$ or 2 then G is a path, which is antimagic by Theorem 2.6. Now suppose $n \geq 3$. Let $P_j = (v_{1j}, v_{2j}, \ldots, v_{(k-1)j}, x)$, $1 \leq j \leq n$ be the paths in G obtained by subdividing the n edges of $K_{1,n}$. Now define $f : V(G) \to \{1, 2, \ldots, (k-1)n+1\}$ by $f(v_{ij}) = (i-1)n+j$ for $1 \leq i \leq k-1$, $1 \leq j \leq n$ and $f(x) = (k-1)n+1$. Then

$$w(v_{ij}) = \begin{cases} n+j & \text{if } i = 1 \text{ and } 1 \leq j \leq n \\ 2(i-1)n + 2j & \text{if } 2 \leq i \leq k-2 \text{ and } 1 \leq j \leq n \\ (k-2)n^2 + \frac{n(n+1)}{2} & \text{if } v_{ij} = x \end{cases}$$

Clearly the vertex weights are all distinct and hence G is distance antimagic. □

Example 2.10. The graph $S_5(K_{1,4})$ with distance antimagic labeling is given in Figure 2.

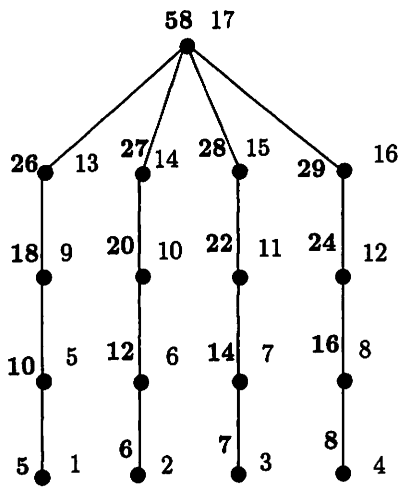

Figure 2. $S_5(K_{1,4})$ with a distance antimagic labeling

3 Conclusion and scope

In this paper we have introduced the concept of distance antimagic labeling and we have presented several families of graphs which admit such a labeling. If a graph G is distance antimagic, then $N(u) \neq N(v)$ for any two distinct vertices u, v in $V(G)$. We do not have an example of a graph to show that the converse is not true. Hence we take the risk of making the following conjecture.

Conjecture 3.1. *A graph G is distance antimagic if and only if $N(u) \neq N(v)$ for any two distinct vertices u, v in $V(G)$.*

For trees the above conjecture takes the following form.

Conjecture 3.2. *A tree T is distance antimagic if and only if every support vertex v has exactly one leaf adjacent to v.*

Problem 3.3. *If G is distance antimagic, is it true that the graphs $G+K_1$, $G+K_2$, the cartesian product $G\Box K_2$ are distance antimagic?*

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