Characterizations of k-ctrees and graph valued functions

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Abstract

We introduce k-ctrees, which are a natural generalization of trees. A k-ctree can be constructed by recursion as follows: Any set of k independent vertices is a k-ctree, and a k-ctree of order $n + 1$ is obtained by inserting an $(n + 1)^{th}$-vertex, and joining it to each of any k independent vertices in a k-ctree of order n. We obtain basic properties and characterizations of k-ctrees involving k-degeneracy, triangle-free properties, and number of edges. Further, we determine the conditions under which k-ctrees are line, middle, or total graphs. Finally we pose some open problems, all of them related with the characterization of k-ctree.

Keywords: k-trees, k-degenerate graph, total graph, line graph, middle graph, and graph valued function.
2010 Mathematics Subject Classification Number: 05C75.

1 Introduction

All graphs considered here are finite, undirected, without loops and without multiple edges. We follow the terminology of Harary [3]. Given a graph G, $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. The *order* of G is the number of vertices of G, and its *size* is $|E(G)|$, the number of its edges. The *neighbourhood* of a vertex u in G, is the set consisting of the vertices u_i of G which are adjacent to u and each u_i is called *neighbouring vertex* of u. A subset S of $V(G)$ is called an *independent*

*Research supported by CSIR, New Delhi, India

set of G, if no two vertices in S are adjacent in G. A graph G is said to be *n-connected* (respectively, *n-edge connected*), if the removal of any m vertices (respectively, m edges) from G, (where $0 \leq m < n$), results in neither a disconnected graph nor a trivial graph.

Multidimensional trees were first introduced by Harary and Palmer [4]. Later, various generalizations of tree-characterization theorems are developed in a natural way for these multidimensional tree -structures (see, Dewdney [1], Patil [6]). While trees are usually defined as those graphs which are connected and acyclic, this class of graphs can be equivalently defined by the following recursive construction rule: A single vertex is a tree, and any tree T of order $n \geq 2$, can be constructed from a tree T' of order $(n-1)$ by inserting an nth - vertex, and joining it to any vertex of T'. Generalizing this construction rule by allowing the base of the recursive growth to be a totally disconnected graph of order k (ie., $\overline{K_k}$) yields a new class of graphs, which is certainly a new class of higher dimensional trees. Next, we introduce the definition of these class of graphs.

Definition 1.1. The class of k-ctrees (for $k \geq 1$) is the set of all graphs that can be obtained by the following recursive construction rule.

1. A totally disconnected graph of order k (ie., $\overline{K_k}$) is a k-ctree.
2. To a k-ctree Q' of order $n-1$ (where $n > k$), insert a new nth - vertex, and join it to any set of k independent vertices of Q'.

In this construction of a k-ctree, the origin $\overline{K_k}$- subgraph is called the *base* of k-ctree. According to the recursive definition of k-ctrees, we have the following facts:

1. 1-ctrees are simply trees.
2. For any $k \geq 3$, k-ctrees of order at least $(k+3)$ are non-planar graphs, because they possess an induced subgraph isomorphic to $K_{3,3}$.

2 Basic properties of k-ctrees

Theorem 2.1. *A graph G of order $p \geq k+1$, is a k-ctree if and only if $V(G)$ can be labelled $v_1, v_2, v_3, \ldots v_p$ so that for each integer i, (where $k+1 \leq i \leq p$) there exist k distinct unordered labels $i_1, i_2, \ldots i_k$ such that $\langle\{v_{i_1}, v_{i_2}, v_{i_3}, \ldots v_{i_k}, v_i\}\rangle = K_{1,k}$ and deg $v_i = k$ in $\langle\{v_1, v_2, v_3, \ldots v_i\}\rangle$.*

Roughly speaking, a graph G is a k-ctree of order $\geq k+1$ if and only if G can be reduced to the base (ie., a totally disconnected graph $\overline{K_k}$) by repeated removal of a vertex of degree k.

Definition 2.2. A vertex v of a graph G is called a *star-vertex* if all its neighbouring vertices are independent.

Notice that a graph is triangle-free if and only if each of its vertex is a star-vertex. The immediate consequence of Theorem 2.1 is the following result.

Theorem 2.3. *A graph G of order $\geq k+1$ is a k-ctree if and only if G has a star-vertex v of degree k and $G-v$ is a k-ctree.*

By repeated application of Theorem 2.3 to k-ctrees we obtain the following
corollaries.

Corollary 2.4. *Every k-ctree of order $p \geq k$ has $k(p-k)$ edges.*

Corollary 2.5. *If G is a k-ctree of order $p \geq 2k+1$, then the set of all vertices of degree k in G forms an independent set.*

Corollary 2.6. *If G is a k-ctree of order $\leq 2k+2$, then G is a bipartite graph.*

Corollary 2.7. *Every k-ctree G of order $p \geq 2k$ has $\delta(G) = k$. Moreover, G is both k-connected and k-edge connected.*

Proof. We prove the result by induction on p. If $p = 2k$, then by Turan's theorem $G = K_{k,k}$. If $p = 2k+1$, then also $G = K_{k,k+1}$. Hence, the result is trivial in either case. Assume the result is true for all k-ctrees of order $\leq n$ (where $n \geq 2k+1$). Let G be a k-ctree of order $n+1$. In view of Theorem 2.3, G contains a star -vertex v of degree k, and $G-v$ is a k-ctree of order n. Hence, by the inductive hypothesis, $\delta(G-v) = k$, and $(G-v)$ is both k-connected and k-edge connected. Consequently, the result follows by the principle of induction. □

Lick and White [5] introduced the concept of n-degenerate graphs. A graph G is said to be *n-degenerate* if every subgraph of G has a vertex of degree at most n. Next, we develop the interrelationships between k-ctrees and n-degenerate graphs.The following result follows from the recursive construction of k-ctrees.

Theorem 2.8. *Every k-ctree is a k-degenerate, triangle-free graph.*

Corollary 2.9. *Every k-ctree G of order $p \geq 2k+1$ contains an induced subgraph isomorphic to $K_{k,k+1}$. Moreover, G has no subgraph isomorphic to $K_{k+1,k+1}$.*

Proof. We prove the result by induction on p. Suppose G is a k-ctree of order $p = 2k+1$. Then G contains $k(k+1)$ edges, and it is a triangle-free graph. Therefore by Turan's theorem $G = K_{k,k+1}$. Assume the result is true for all k-ctrees of order $\leq p-1$, where $p \geq 2k+2$. Let G be a k-ctree of order p. By Theorem 2.3, G has a star-vertex v of degree k, and $G - v$ is a k-ctree. By induction, $G - v$ has an induced subgraph isomorphic to $K_{k,k+1}$. In view of Theorem 2.8, G is k-degenerate. Hence, G has no subgraph isomorphic to $K_{k+1,k+1}$. However, it contains an induced subgraph $K_{k,k+1}$ □

3 Characterizations of k-ctrees

Theorem 3.1. *A graph G is a k-ctree of order p, where $(k+1) \leq p \leq (2k+1)$ if and only if $G = K_{k,p-k}$.*

Proof. The proof follows by the repeated application of Theorem 2.3. □

First, we establish two lemmas, to develop one more characterization of k-ctrees.

Lemma 3.2. *Every k-degenerate, triangle-free graph of order $p \geq 2k$, contains at most $k(p-k)$ edges.*

Proof. We prove the result by induction on p. Every triangle-free graph of order $p = 2k$, contains at most $\frac{p^2}{4} = k^2 = k(p-k)$ edges. Assume that the result is true for all such graphs of order $< p$. Let G be a k-degenerate, triangle-free graph of order p. Suppose the result is not true. Then $|E(G)| > k(p-k) = k(p-k-1)+k$. Since G is k-degenerate, it follows that $\delta(G) \leq k$. Consequently, there exists a vertex v of degree at most k in G. Hence, $|E(G-v)| > k(p-k-1)$. This is a contradiction to the induction hypothesis that $(G-v)$ has at most $k(p-k-1)$ edges. Hence, the result follows by the principle of induction. □

Lemma 3.3. *Every k-degenerate, triangle-free graph G of order $p \geq 2k$, and size $k(p-k)$, has $\delta(G) = k$.*

Proof. If $p = 2k$, then obviously $G = K_{k,k}$ and hence $\delta(G) = k$. If $p > 2k$, and G is a k-degenerate, triangle-free graph having $k(p-k)$ edges, then we prove the result by contradiction. Suppose $\delta(G) < k$. Then there exists a vertex v of degree $< k$ in G. Moreover, $|E(G-v)| > k(p-k-1)$. In view of Lemma 3.2, we have $|E(G-v)| \leq k(p-k-1)$. This is a contradiction to the fact that $G - v$ has at most $k(p-k-1)$ edges. Hence, $\delta(G) \geq k$. Since G is k-degenerate, it follows that $\delta(G) \leq k$. Thus, $\delta(G) = k$. □

Theorem 3.4. *Let G be a graph of order $p \geq 2k$. Then G is a k-ctree if and only if G is a k-degenerate, triangle-free graph of size $k(p-k)$.*

Proof. The necessity follows directly from Corollary 2.4 and Theorem 2.8. We prove the sufficiency by induction on p. If $p = 2k$, then $G = K_{k,k}$ and it is obviously a k-ctree. Next, we assume that any k-degenerate, triangle-free graph of order m (where $2k \leq m < p$) and size $k(m-k)$ is a k-ctree. Let G be a k-degenerate, triangle-free graph of order $p > 2k$ and size $k(p-k)$. In view of Lemma 3.3, $\delta(G) = k$. Consequently, there exists a star-vertex v of degree k in G. By the inductive hypothesis, $G - v$ is a k-ctree. Therefore by Theorem 2.3, G is a k-ctree. □

Definition 3.5. A graph G is a maximal k-degenerate, triangle-free graph if for every edge $e \in E(\bar{G})$, either $G+e$ is not k-degenerate or $G+e$ contains a triangle.

Corollary 3.6. *Every k-ctree is a maximal k-degenerate, triangle-free graph.*

4 Applications of k-ctrees to line, middle and total graphs

Definition 4.1. The *line graph* $L(G)$ of a graph G is the graph whose vertex set coincides with the edge set of G and in which two vertices are adjacent if the corresponding edges are adjacent in G, (see [3]). The n^{th} *-iterated line graph* $L^n(G)$ is defined in a natural way as follows:
$L^1(G) = L(G)$, and $L^n(G) = L(L^{n-1}(G))$ for $n \geq 2$.

In this section, we determine all graphs whose n^{th}-iterated line graphs (for $n \geq 1$) are k-ctrees. Beineke ([3], p.75) has shown that a graph is a line graph if and only if it has none of nine specific graphs as induced subgraphs, and this includes $K_{1,3}$.

Theorem 4.2. *For any non-trivial graph G, the line graph $L(G)$ is a k-ctree if and only if when*

1. $k = 1$; $G = P_m$ *(for $m \geq 2$).*
2. $k = 2$; G *is one of the graphs:* $2K_2, P_4$ *and* C_4.
3. $k \geq 3$; $G = kK_2$.

Proof. Suppose $L(G)$ is a k-ctree of order $p \geq k$. We discuss three cases depending on k:

Case 1. $k = 1$.

Then $L(G)$ is a tree. Assume G has a vertex u of degree ≥ 3. Then any three edges of G incident with u form $K_{1,3}$. Consequently, $L(G)$ contains a triangle K_3. This is impossible due to the fact that $L(G)$ is a tree. Hence, $\Delta(G) \leq 2$. This shows that each component of G is either a cycle C_n (for $n \geq 3$) or a path P_m (for $m \geq 1$). Assume a component of G is a cycle C_n. Then $L(G)$ has a component isomorphic to C_n itself. This is impossible because $L(G)$ is a tree. Consequently, each component of G must be a path. In this situation, G cannot contain more than one component, each of which is a path. Otherwise, $L(G)$ cannot be a tree. Therefore, G must be a path.

Case 2. $k = 2$.

If $p \geq 5$, then it is easy to check that $L(G)$ contains a forbidden subgraph isomorphic to $K_{1,3}$. This proves that $p \leq 4$. In this case, $L(G)$ is isomorphic to one of the graphs: $\overline{K_2}$, $K_{1,2}$ and C_4. Consequently, G is isomorphic to one of the graphs: $2K_2$, P_4 and C_4.

Case 3. $k \geq 3$.

If $p \geq k+1$, then by Theorem 3.1, $L(G)$ contains an induced subgraph isomorphic to $K_{1,k}$. This is impossible, since $k \geq 3$. Therefore, $p = k$. Immediately, $L(G) = \overline{K_k}$ and hence $G = kK_2$.

It is easy to prove the converse of all three cases. □

By repeated application of the above theorem to the iterated line graphs, we obtain the following result.

Corollary 4.3. *For any nontrivial graph G, the n^{th}- iterated line graph $L^n(G)$ (for $n \geq 2$) is a k-ctree if and only if when*

1. $k = 1$; $G = P_{m+n-1}$ *(for $m \geq 2$).*
2. $k = 2$; *G is one of the graphs: $2P_{n+1}$, P_{n+3}, and C_4.*
3. $k \geq 3$; $G = kP_{n+1}$.

Definition 4.4. The *middle graph* $M(G)$ of a graph G (introduced in [2]), is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices of $M(G)$ are adjacent if they are adjacent edges of G or one is a vertex and the other is an edge of G incident with it. The n^{th} *-iterated middle graph* $M^n(G)$, is defined in the following way:

$M^1(G) = M(G)$ and $M^n(G) = M(M^{n-1}(G))$ for $n \geq 2$.

Definition 4.5. The *total graph* [3] of G, denoted by $T(G)$ is defined in the following way. The vertex set of $T(G)$ is $V(G) \cup E(G)$. Two vertices x, y in the vertex set of $T(G)$ are adjacent in $T(G)$ in case one of the following holds:

1. x, y are in $V(G)$ and x is adjacent to y in G.
2. x, y are in $E(G)$ and x, y are adjacent in G.
3. x is in $V(G)$, y is in $E(G)$, and x, y are incident in G.

Finally, we determine all graphs, whose nth-iterated middle graphs (for $n \geq 1$) or total graphs are k-ctrees. Hamada and Yoshimura [2] showed that for any graph G, $M(G) = L(G^+)$, where G^+ is the graph obtained from G by adjoining a pendant edge uu' at every vertex u of G. In view of Theorem 4.2, we have the following observations:

When $k = 1$. If $L(G^+)$ is a tree then G^+ must be a path, which implies that G is either K_1 or K_2.

When $k = 2$. If $L(G^+)$ is a 2-ctree then G^+ must be isomorphic to $2K_2$ or P_4, which implies that G is either $\overline{K_2}$ or K_2.

When $k \geq 3$. There is only one graph $G^+ = kK_2$, whose line graph $L(G^+)$ is a k-ctree, and hence $G = \overline{K_k}$. Since $M(G) = L(G^+)$, the above discussion proves the following result.

Theorem 4.6. *For any nontrivial graph G, the middle graph $M(G)$ is a k-ctree if and only if when*

1. *$k = 1$; G is either K_1 or K_2.*

2. *$k = 2$; G is either $\overline{K_2}$ or K_2.*

3. *$k \geq 3$; $G = \overline{K_k}$.*

An immediate consequence of the above theorem, are the following results.

Proposition 1. *For any graph G, the total graph $T(G)$ is a k-ctree if and only if $G = \overline{K_k}$.*

Proof. Notice that for any non-trivial graph G, $T(G)$ has a triangle, so the result follows from Theorem 2.8. □

Corollary 4.7. *There is only one graph, whose n^{th} middle graph (for $n \geq 2$) is a k-ctree for $1 \leq k \leq 3$. This graph is $\overline{K_k}$.*

5 Open problems

We now pose four problems, which all are related with the characterization of k-ctrees.

1. Let G be a k connected, triangle-free graph with $p \geq 2k$ vertices, and $\delta(G) = k$ with $|E(G)| = k(p - k)$. Then, is G a k-degenerate graph?

2. Let G be a triangle-free graph with $p \geq 2k$ vertices, $k(p-k)$ edges and $K_{k+1,k+1}$ free. Then, is $\delta(G) = k$?

3. Let G be a k-connected, triangle-free graph with $p \geq 2k$ vertices, $k(p-k)$ edges and which is a $K_{k+1,k+1}$ free graph. Then, is G a k-ctree?

4. Let G be a graph, and let k be the smallest integer for which G is a maximal k-degenerate, triangle-free graph. Is then G a k-ctree?

References

[1] A.K. Dewdney, Higher-dimensional tree structures, *J. Combin. Theory Ser. B*, **17** (1974), 160–169.

[2] T. Hamada and I. Yoshimura, Traversability and connectivity of the middle graph of a graph, *Discrete Math.*, **14** (1976), 247–255.

[3] F. Harary, *Graph Theory*, Addison -Wesley, Reading Mass, 1969.

[4] F. Harary and E.M. Palmer, On acyclic simplicial complexes, *Mathematika*, **15** (1968), 115–122.

[5] D.R. Lick and A.T. White, k-degenerate graphs, *Canad.J. Math.*, **22** (1970), 1082–1096.

[6] H.P. Patil, On the structure of k-trees, *J. Comb. Inf. Syst. Sci.*, **11**(2-4), (1986), 57–64.