On Color Frames of Claws in Graphs

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Dedicated to Kieka Mynhardt in celebration of her birthday

Abstract

A red-blue coloring of a graph G is an edge coloring of G in which every edge of G is colored red or blue. Let F be a connected graph of size 2 or more with a red-blue coloring, at least one edge of each color, where some blue edge of F is designated as the root of F. Such an edge-colored graph F is called a color frame. An F-coloring of a graph G is a red-blue coloring of G in which every blue edge of G is the root edge of a copy of F in G. The F-chromatic index $\chi'_F(G)$ of G is the minimum number of red edges in an F-coloring of G. It has been shown that these concepts generalize both edge domination and matchings in graphs. In this paper, we consider the two color frames Y_1 and Y_2 that result from the claw $K_{1,3}$, where Y_1 has exactly one red edge and Y_2 has exactly two red edges. An edge e in a graph G is a non-claw edge if e belongs to no claw in G. It is shown that if G is a connected graph containing ℓ non-claw edges, then $\chi'_{Y_1}(G) \le \chi'_{Y_2}(G) \le 3\chi'_{Y_1}(G) - 2\ell$ and $\chi'_{Y_1}(G) = \chi'_{Y_2}(G)$ if and only if G is a path or cycle. Furthermore, a pair a, b of positive integers can be realized as the Y_1-chromatic index and Y_2-chromatic index for some connected graph of order at least 4 if and only if $a \le b \le 3a$ and $b \ge 2$.

1 Introduction

An area of graph theory that has received increased attention during recent decades is that of domination. Two books [9, 10] by Haynes, Hedetniemi and Slater are devoted to this subject. Although initiated by Berge [1] and Ore [20] in 1958 and 1962, respectively, it was a paper by Cockayne and Hedetniemi [6] in 1977 that brought popularity to the subject and then led to a theory. This subject is based on a simple definition. A vertex v *dominates* a vertex u in a graph G if either $u = v$ or u is adjacent to v. Over the years a large number of variations of domination have surfaced. Each type of domination is based on a condition under which a vertex v dominates a vertex u in a graph.

In 1999 a new way of looking at domination was introduced by Chartrand, Haynes, Henning and Zhang [3] that encompassed several of the best known domination parameters in the literature. This new view of domination was based on a concept introduced by Rashidi [21] in 1994. A graph G whose vertex set $V(G)$ is partitioned is a *stratified graph.* If $V(G)$ is partitioned into k subsets, then G is *k-stratified.* In particular, the vertex set of a 2-stratified graph is partitioned into two subsets. Typically, the vertices of one subset in a 2-stratified graph are considered to be colored red and those in the other subset are colored blue. A *red-blue coloring* of a graph G is an assignment of colors to the vertices of G, where each vertex is colored either red or blue. In a red-blue coloring, all vertices of G may be colored the same. A red-blue coloring in which at least one vertex is colored red and at least one vertex is colored blue thereby produces a 2-stratification of G.

We next describe how domination was defined in [3] with the aid of stratification. Let F be a 2-stratified graph in which some blue vertex ρ is designated as the root of F. The graph F is then said to *be rooted at* ρ. Since F is 2-stratified, F contains at least two vertices, at least one of each color. There may be blue vertices in F in addition to the root. By an *F-coloring* of a graph G, we mean a red-blue coloring of G such that for every blue vertex u of G, there is a copy of F in G with ρ at u. Therefore, every blue vertex u of G belongs to a copy F' of F rooted at u. A red vertex v in G is said to *F-dominate* a vertex u if $u = v$ or there exists a copy F' of F rooted at u and containing the red vertex v. The set S of red vertices in a red-blue coloring of G is an *F-dominating set* of G if every vertex of G is F-dominated by some vertex of S, that is, this red-blue coloring of G is an F-coloring. The minimum number of red vertices in an F-dominating set is called the *F-domination number* $\gamma_F(G)$ of G. An F-dominating set with $\gamma_F(G)$ vertices is a *minimum F-dominating set.* The F-domination number of every graph G is defined since $V(G)$ is an F-dominating set. This concept provides a generalization of domination and has been studied in many articles (see [7, 8] and [11] - [15] for example).

An edge version of this concept was introduced and studied in [16, 17]. Here a red-blue coloring of a graph G refers to an edge coloring of G in which every edge is colored red or blue. Let F be a connected graph of size 2 or more with a red-blue coloring, at least one edge of each color. One of the blue edges of F is designated as the *root edge* of F. The *underlying graph* of F is the graph H, obtained by removing the colors assigned to the edges of F. In this case, F is called a *color frame* of H. The simplest example of this is the unique color frame F_0 of the path P_3 in which one edge is red, the other is blue and the blue edge is its root edge shown in Figure 1, where a red edge is labeled r and a blue edge is labeled b. The five (distinct) color frames $F_1, F_2, \ldots, F_5$ of the path P_4 of size 3 are also shown in Figure 1, where each root edge is indicated by a double-line edge.

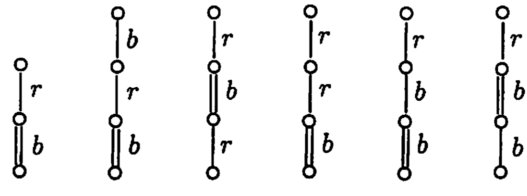

Figure 1: Color frames of P_3 and P_4

For a color frame F, an F*-coloring* of a graph G is a red-blue coloring of G in which every blue edge of G is the root edge of a copy of F in G. If G contains no subgraph isomorphic to F, then the only F-coloring of G is that in which every edge of G is red. The F*-chromatic index* $\chi'_F(G)$ of G is the minimum number of red edges in an F-coloring of G. Since the edge coloring of G that assigns red to every edge is an F-coloring of G, the number $\chi'_F(G)$ exists for every color frame F and every graph G. An F-coloring of G having exactly $\chi'_F(G)$ red edges is called a *minimum* F*-coloring* of G. Although these concepts are related to the vertex concepts discussed earlier through the line graph of a graph, this fact, as with proper colorings, has provided no benefit in the study of F-colorings. It turns out that F-colorings of graphs and the F-chromatic index provide a new framework for studying both edge independence (or matchings) and edge domination in graphs, as we describe next.

An edge e in a graph G is said to *dominate* itself as well as all edges adjacent to e. A set S of edges of G is an *edge dominating set* of G if every edge of G is dominated by some edge in S. The minimum size of an edge dominating set of G is the *edge domination number* of G and is denoted by $\gamma'(G)$. Moreover, $\gamma'(G)$ is the domination number of the line graph of G. An edge dominating set of size $\gamma'(G)$ is called a *minimum edge dominating set* of G. The edge domination number of a graph can be looked at in terms of F-colorings for a specific color frame F, namely, the color frame F_0 of P_3.

Theorem 1.1 [17] *Let G be a connected graph of size 2 or more. If F_0 is the color frame of P_3, then $\gamma'(G) = \chi'_{F_0}(G)$.*

Among the concepts that are fundamental in graph theory is that of matchings. Lovász and Plummer have written a book [19] devoted to the theory of matchings. A set of edges in a graph G is *independent* if no two edges in the set are adjacent in G. The edges in an independent set of edges of G form a *matching* in G. A matching of maximum size in G is a *maximum matching*. The *edge independence number* $\alpha'(G)$ of G is the number of edges in a maximum matching of G. The number $\alpha'(G)$ is also referred to as the *matching number* of G. A matching M in a graph G is a *maximal matching* of G if M is not a proper subset of any other matching in G. While every maximum matching is maximal, a maximal matching need not be a maximum matching. The minimum number of edges in a maximal matching of G is called the *lower edge independence number* or *lower matching number* $\alpha''(G)$ of G. Necessarily, $\alpha''(G) \leq \alpha'(G)$. It was shown in [18] that if G is a graph and k is an integer with $\alpha''(G) \leq k \leq \alpha'(G)$, then G contains a maximal matching of size k. Furthermore, it is known that if G is a connected graph of size 2 or more, then $\gamma'(G) = \alpha''(G)$. It then follows by Theorem 1.1 that the lower matching number of a graph can be expressed in terms of F_0-colorings where F_0 is the color frame of P_3.

Theorem 1.2 *Let G be a connected graph of size 2 or more. If F_0 is the color frame of P_3, then $\chi'_{F_0}(G) = \alpha''(G)$.*

For the five color frames $F_1, F_2, \ldots, F_5$ of the path P_4 of size 3 in Figure 1, many of the F_i-chromatic indexes are closely related to several well-studied edge domination parameters, as we describe next.

Theorem 1.3 [17] *Let G be a connected graph of size at least 3 and F_1 the blue-red-blue color frame of P_4. If G is either (i) triangle-free and $\delta(G) \geq 2$ or (ii) $\delta(G) \geq 3$, then $\chi'_{F_1}(G) = \gamma'(G)$.*

For each positive integer k, a set S of edges of G is a *k-edge dominating set* of G if every edge in $E(G) - S$ is dominated by at least k edges in S. Since $E(G)$ is vacuously such a set S, every graph G has a k-edge dominating set. The minimum size of a k-edge dominating set of G is the *k-edge domination number* of G and is denoted by $\gamma'_k(G)$. The number $\chi'_{F_2}(G)$ is related to $\gamma'_2(G)$.

Theorem 1.4 [17] *If G is a connected graph of size at least 3, then $\gamma'_2(G) \leq \chi'_{F_2}(G)$. Furthermore, for each pair a, b of integers with $2 \leq a \leq b$, there is a connected graph G such that $\gamma'_2(G) = a$ and $\chi'_{F_2}(G) = b$.*

A *total edge dominating set* in a connected graph G is a subset S of $E(G)$ such that every edge of G is adjacent to an edge of S. Thus the subgraph $G[S]$ of G induced a total edge dominating set S of G contains no component which is K_2. Moreover, if G is a nonempty graph that does not contain K_2 as a component, then $E(G)$ is a total edge dominating set. In particular, every connected graph of order at least 3 has a total edge dominating set. The *total edge domination number* $\gamma_t'(G)$ is the minimum size of a total edge dominating set of G. A total edge dominating set of size $\gamma_t'(G)$ is a *minimum total edge dominating set* of G. The number $\chi'_{F_3}(G)$ is related to $\gamma_t'(G)$.

Theorem 1.5 [17] *If G is a connected graph of size at least 3, then $\gamma_t'(G) \leq \chi'_{F_3}(G)$. Furthermore, for each positive integer k, there is a connected graph G_k such that $\chi'_{F_3}(G_k) - \gamma_t'(G_k) = k$.*

Although it appears that $\chi'_{F_4}(G)$ is not related to any known edge domination parameters, the parameter $\chi'_{F_5}(G)$ is related to an edge domination number. A set $S \subseteq E(G)$ is a *restrained edge dominating set* if every edge not in S is adjacent to an edge in S and to an edge in $E(G) - S$. Every graph has a restrained edge dominating set since $E(G)$ is such a set. The *restrained edge domination number* $\gamma_r'(G)$ is the minimum size of a restrained edge dominating set of G. The number $\chi'_{F_5}(G)$ is related to $\gamma_r'(G)$.

Theorem 1.6 [17] *If G is a connected graph G of size at least 3, then $\gamma_r'(G) \leq \chi'_{F_5}(G)$. Furthermore, for each pair a, b of positive integers with $a \leq b$, there is a connected graph G of size at least 3 such that $\gamma_r'(G) = a$ and $\chi'_{F_5}(G) = b$ if and only if $(a, b) \neq (1, 1)$.*

2 Color Frames of a Claw

The graph $K_{1,3}$ is often referred to as a *claw*. There are two color frames of a claw, both shown in Figure 2. The color frame Y_1 of a claw has exactly one red edge while Y_2 has exactly two red edges. In Y_1, there are therefore two blue edges and in Y_2 only one blue edge. By symmetry, we can choose either of the two blue edges in Y_1 as the root edge, while in Y_2, the only blue edge is the root edge of Y_2.

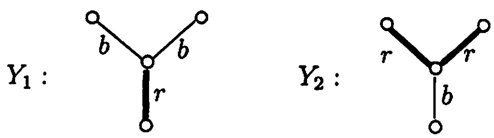

Figure 2: The two color frames of the claw $K_{1,3}$

In the vertex version, where F is a 2-stratified graph of a claw, F-colorings were studied by Chartrand, Haynes, Henning and Zhang in the paper [2], which was dedicated to Ernie Cockayne on the occasion of his

60th birthday. Hence, in the case of edge colorings, we study F-colorings of graphs for color frames F of a claw. Before beginning this study, it is useful to establish some additional definitions and notation.

For an F-coloring c of a graph G, let $E_{c,r}$ denote the set of red edges of G and $E_{c,b}$ the set of blue edges of G. (We also use E_r and E_b for $E_{c,r}$ and $E_{c,b}$, respectively, when the coloring c under consideration is clear.) Thus $\{E_r, E_b\}$ is a partition of the edge set $E(G)$ of G when $E_b \neq \emptyset$. Furthermore, let $G_r = G[E_r]$ denote the *red subgraph* induced by E_r and $G_b = G[E_b]$ the *blue subgraph* induced by E_b. Thus $\{G_r, G_b\}$ is a decomposition of G. If G is a disconnected graph with components G_1, G_2, ..., G_k where $k \geq 2$, then

$$\chi'_F(G) = \chi'_F(G_1) + \chi'_F(G_2) + \cdots + \chi'_F(G_k). \tag{1}$$

Thus, it suffices to consider only connected graphs. We refer to the books [4, 5] for graph theory notation and terminology not described in this paper.

If G is a connected graph with maximum degree $\Delta(G) \leq 2$ (and so G is a path or cycle), then the only Y_i-coloring ($i = 1, 2$) of G is the one that assigns the color red to every edge of G and so $\chi'_{Y_i}(G)$ is the size of G. On the other hand, if G contains a vertex u with $\deg u \geq 3$ and $uv, uw \in E(G)$, then the red-blue coloring that assigns the color blue to the edges uv and uw and the color red to all other edges is a Y_1-coloring of G; while the red-blue coloring that assigns the color blue to to uv and the color red to all other edges is a Y_2-coloring of G. This leads to the following observation.

Observation 2.1 *If G is a graph of size $m \geq 1$, then*

$$\chi'_{Y_1}(G) = \chi'_{Y_2}(G) = m \text{ if and only if } \Delta(G) \leq 2.$$

It suffices therefore to consider only those connected graphs with maximum degree at least 3. For example, if G is a star of size at least 3, then $\chi'_{Y_1}(G) = 1$ and $\chi'_{Y_2}(G) = 2$. In the case where G is a double star (a tree having diameter 3) of size at least 4, we have the following.

Observation 2.2 *If G is a double star of size at least 4, then*

$$\chi'_{Y_1}(G) = \begin{cases} 2 & \text{if } G \text{ contains a vertex of degree } 2 \\ 1 & \text{otherwise} \end{cases}$$

and $\chi'_{Y_2}(G) = 3$.

As another example, consider the Petersen graph P. A Y_i-coloring c_i of P is shown in Figure 3 for $i = 1, 2$, where a red edge is indicated by a bold edge. Since c_1 produces three red edges while c_2 produces eight red edges, $\chi'_{Y_1}(P) \leq 3$ and $\chi'_{Y_2}(P) \leq 8$. In fact, we have equality in each case.

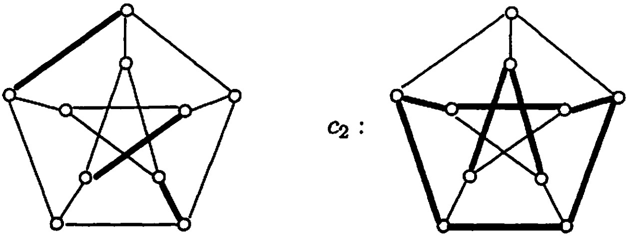

Figure 3: Y_i-colorings of the Petersen graph for $i = 1, 2$

First, we show that $\chi'_{Y_1}(P) = 3$. Let $S_{3,3}$ be the double star whose two central vertices have degree 3. The graph P is $S_{3,3}$-decomposable, that is, P contains three edge-disjoint copies of $S_{3,3}$ [4, p. 437]. Since P is 3-regular, in any Y_1-coloring of P, there cannot be a blue copy of $S_{3,3}$ and so $|E_r| \geq 3$. That is, $\chi'_{Y_1}(P) \geq 3$ and so $\chi'_{Y_1}(P) = 3$.

Next, we show that $\chi'_{Y_2}(P) = 8$. Assume, to the contrary, that there is a Y_2-coloring c of P with $|E_{c,r}| = k \leq 7$. Let $(u_1, u_2, u_3, u_4, u_5, u_1)$ and $(v_1, v_3, v_5, v_2, v_4, v_1)$ be the outside and inside 5-cycles, respectively, in a dawning of P such as in Figure 3, where $u_iv_i \in E(P)$ for $1 \leq i \leq 5$ and let G_r be the red subgraph of order n_r and size $m_r = k$ induced by c. Suppose, first, that $n_r < 10$ and let $u \in V(P) - V(G_r)$, say $u = u_1$. Since $u_1u_2, u_1v_1, u_1u_5 \in E_{c,b}$, it follows that

$$X = \{u_2v_2, u_2u_3, v_1v_3, v_1v_4, u_5v_5, u_5u_4\} \subseteq E_{c,r}.$$

If $X = E_{c,r}$, then the blue edge u_3u_4, for example, does not belong to a copy of Y_2. Thus $|E_{c,r}| = 7$ and $E_{c,r} = X \cup \{e\}$ for some edge $e \in E(P) - X$. If $e \in \{v_2v_5, v_2v_4, v_5v_3\}$, then the blue edge u_3u_4 does not belong to a copy of Y_2; while if $e \in \{u_3u_4, u_3v_3, u_4v_4\}$, then the blue edge v_2v_5 does not belong to a copy of Y_2. In either case, a contradiction is produced, which implies that $n_r = 10$.

Since $m_r = k \leq 7$, there are at least six vertices of degree 1 in G_r, say u, v, w, x, y, z have degree 1 in G_r. Because the independence number of P is 4, there are two vertices in $\{u, v, w, x, y, z\}$ that are adjacent in P, say $uv \in E(P)$. If uv is blue, then uv does not belong to a copy of Y_2. Thus uv must be red. We may assume that $uv = u_1v_1$. Since $\deg_{G_r} u_1 = \deg_{G_r} v_1 = 1$, it follows that for the four edges u_1u_2, u_1u_5, v_1v_3 and v_1v_4 are blue. Since each of these four blue edges must belong to a copy of Y_2, it follows that all of $u_2v_2, u_2u_3, u_5v_5, u_5u_4, v_3v_5, v_3u_3, v_4u_4, v_4v_2$ are red. However then, $|E_{c,r}| \geq 8$, which is a contradiction. Therefore, $\chi'_{Y_2}(P) = 8$.

If $G = C_{3k}$, $k \geq 1$, then $\chi'_{Y_1}(G) = 3k$ by Observation 2.1. Also $\alpha''(G) = k$. Consequently, $\chi'_{Y_1}(G) - \alpha''(G) = 2k$ can be arbitrarily large. On the other hand, if G contains no vertices of degree 2, then this cannot occur.

Proposition 2.3 *If G is a connected graph of order at least 4 having no vertex of degree 2, then $\chi'_{Y_1}(G) \leq \alpha''(G)$.*

Proof. Let M be a maximal matching of G of size $\alpha''(G)$ and consider a red-blue coloring of G such that $E_r = M$. Suppose that $e = uv$ is a blue edge. Since $M \cup \{e\}$ is not a matching in G, it follows that at least one of u and v is incident with exactly one red edge, say u is incident with exactly one red edge. Since $\deg u \geq 3$, it follows that u is incident with at least two blue edges. Hence e belongs to a copy of Y_1. Thus, this is a Y_1-coloring of G and so $\chi'_{Y_1}(G) \leq |M| = \alpha''(G)$. ■

The Y_1- and Y_2-chromatic indexes can now be determined for graphs belonging to another familiar class.

Proposition 2.4 *For each integer $n \geq 4$,*

$$\chi'_{Y_1}(K_n) = \lfloor n/2 \rfloor \text{ and } \chi'_{Y_2}(K_n) = n-1.$$

Proof. Since every maximum matching in K_n consists of $\lfloor n/2 \rfloor$ edges, it follows by Proposition 2.3 that $\chi'_{Y_1}(G) \leq \lfloor n/2 \rfloor$. Any red-blue coloring of K_n having fewer than $\lfloor n/2 \rfloor$ red edges has a blue edge incident only with blue edges and so the coloring of K_n is not a Y_1-coloring. Thus $\chi'_{Y_1}(G) \geq \lfloor n/2 \rfloor$ and so $\chi'_{Y_1}(K_n) = \lfloor n/2 \rfloor$.

Next, we show that $\chi'_{Y_2}(K_n) = n-1$. Let there be given a minimum Y_2-coloring of $G = K_n$. Thus, if uv is a blue edge, then at least one of u and v is incident with two or more red edges. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ with $0 \leq \deg_{G_r}(v_1) \leq \deg_{G_r}(v_2) \leq \cdots \leq \deg_{G_r}(v_n)$. If $\deg_{G_r}(v_2) \leq 1$, then v_1v_2 must be a red edge (for otherwise, the blue edge v_1v_2 does not belong to any copy of Y_2) and so $\deg_{G_r}(v_1) = \deg G_r(v_2) = 1$ while $\deg_{G_r}(v_i) \geq 2$ for $3 \leq i \leq n$. Otherwise, $\deg_{G_r}(v_i) \geq 2$ for $2 \leq i \leq n$. Hence, $|E_r| \geq n-1$ in either case, that is, $\chi'_{Y_2}(K_n) = |E_r| \geq n-1$. A red-blue coloring of K_n inducing a red C_{n-1} shows that $\chi'_{Y_2}(K_n) \leq n-1$ and so $\chi'_{Y_2}(K_n) = n-1$. ■

3 Relationships Between the Two Color Frames of a Claw

An edge $e = uv$ in a graph G is referred to as a *non-claw edge* if e belongs to no claw in G. Thus if e is a non-claw edge, then $\max\{\deg u, \deg v\} \leq 2$. Necessarily, every non-claw edge must be colored red in every Y_i-coloring c of G for $i = 1, 2$. Let $NC(G)$ be the set of all non-claw edges in a graph G. If G is a path or a cycle, then $NC(G) = E(G)$ and so $\chi'_{Y_1}(G) = \chi'_{Y_2}(G) = |NC(G)|$. We saw this earlier when considering the cycle C_{3k}. On the other hand, if G is a connected graph with maximum degree $\Delta(G) \geq 3$, then $NC(G) \neq E(G)$.

For a connected graph G, we now describe an upper bound for $\chi'_{Y_2}(G)$ in terms of $\chi'_{Y_1}(G)$ and the number of non-claw edges in G.

Theorem 3.1 *If G is a nontrivial connected graph containing ℓ non-claw edges, then*

$$\chi'_{Y_2}(G) \leq 3\chi'_{Y_1}(G) - 2\ell. \tag{2}$$

Proof. Suppose that $\chi'_{Y_1}(G) = k$ and that c_1 is a minimum Y_1-coloring of G resulting in the set $E_{c_1,r}$ of red edges in G. Let $E_{c_1,r} = X_0 \cup X_1 \cup X_2 \cup X_3$, where $X_0 = NC(G)$,

$$\begin{aligned} X_1 &= \{e = uv \in E_{c_1,r} : \deg u = 1 \text{ and } \deg v \geq 3\}, \\ X_2 &= \{e = uv \in E_{c_1,r} : \deg u = 2 \text{ and } \deg v \geq 3\}, \\ X_3 &= \{e = uv \in E_{c_1,r} : \deg u \geq 3 \text{ and } \deg v \geq 3\} \end{aligned}$$

such that $|X_i| = k_i \geq 0$ for $i = 1, 2, 3$. Since $|X_0| = \ell$, it follows that $k = \ell + k_1 + k_2 + k_3$. If any of the sets X_1, X_2 and X_3 are nonempty, let $X_1 = \{e_1, e_2, \ldots, e_{k_1}\}$, where $e_i = u_iv_i$, $\deg u_i = 1$ and $\deg v_i \geq 3$ for $1 \leq i \leq k_1$, $X_2 = \{f_1, f_2, \ldots, f_{k_2}\}$, where $f_j = w_jx_j$, $\deg w_j = 2$ and $\deg x_j \geq 3$ for $1 \leq j \leq k_2$ and $X_3 = \{h_1, h_2, \ldots, h_{k_3}\}$, where $h_t = y_tz_t$, $\deg y_t \geq 3$ and $\deg z_t \geq 3$ for $1 \leq t \leq k_3$. For each i with $1 \leq i \leq k_1$, let $v'_i \neq u_i$ be a vertex adjacent to v_i; for each j where $1 \leq j \leq k_2$, let $x'_j \neq w_j$ be a vertex adjacent to x_j; and for each t where $1 \leq t \leq k_3$, let $y'_t \neq z_t$ be a vertex adjacent to y_t and $z'_t \neq y_t$ be a vertex adjacent to z_t such that $y'_t \neq z'_t$. We now define a new red-blue coloring of G, denoted by c_2, such that

$$\begin{aligned} E_{c_2,r} &= E_{c_1,r} \cup \{v_iv'_i : 1 \leq i \leq k_1\} \cup \{x_jx'_j : 1 \leq j \leq k_2\} \\ &\quad \cup \{y_ty'_t, z_tz'_t : 1 \leq t \leq k_3\}. \end{aligned}$$

[Note that the four sets on the right-hand side may not be pairwise disjoint.] Let e be a blue edge in the coloring c_2. Then e is also a blue edge in the coloring c_1. Since c_1 is a Y_1-coloring of G, it follows that e belongs to a copy of Y_1 in the coloring c_1 of G and so is adjacent to a red edge $e_r \in E_{c_1,r} \subseteq E_{c_2,r}$. Suppose that $e = uv$ and $e_r = vw$ where $\deg v \geq 3$. It then follows by the definition of c_2 that there is a vertex $v' \neq w$ such that v' is adjacent to v and vv' is a red edge in the coloring c_2. Thus e belongs to a copy of Y_2 consisting of $\{e, e_r, vv'\}$. Hence c_2 is a Y_2-coloring with at most $\ell + 2k_1 + 2k_2 + 3k_2$ red edges. Since $k = \ell + k_1 + k_2 + k_3$, it follows that

$$\begin{aligned} \chi'_{Y_2}(G) &\leq \ell + 2k_1 + 2k_2 + 3k_2 = 3k - (2\ell + k_1 + k_2) \\ &\leq 3k - 2\ell = 3\chi'_{Y_1}(G) - 2\ell \end{aligned}$$

and so (2) holds. ∎

We noted that if G is a path or cycle of size $m \geq 1$, then $\chi'_{Y_1}(G) = \chi'_{Y_2}(G) = m$. Since $\ell = m$ in this case, equality holds in (2). Equality in (2) can also hold for connected graphs with claws. For example, the graph G_1 of Figure 4 has one non-claw edge and $\chi'_{Y_1}(G_1) = 2$ while $\chi'_{Y_2}(G_1) = 4$. Therefore, $\chi'_{Y_2}(G_1) = 3\chi'_{Y_1}(G_1) - 2\ell$. For each integer $p \geq 2$, there is a connected graph G_p having p non-claw edges such that $\chi'_{Y_1}(G_p) = 2p$ and $\chi'_{Y_2}(G_p) = 4p$. For each i with $1 \leq i \leq p$, let $S_i = S_{3,3}$ be the double star with central vertices u_i and v_i where u_i is adjacent to the two end-vertices w_i and x_i and v_i is adjacent to the two end-vertices y_i and z_i. Then G_p be the graph obtained from the graphs S_i $(1 \leq i \leq p)$ by (1) joining z_i and y_{i+1} for $1 \leq i \leq p-1$ and (2) adding a new vertex y_{p+1} and joining y_{p+1} to z_p. The graph G_3 is shown in Figure 4. Then G_p has the desired properties.

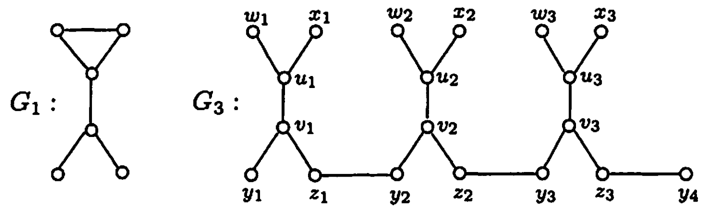

Figure 4: Graphs G for which $\chi'_{Y_2}(G) = 3\chi'_{Y_1}(G) - 2\ell$

For each connected graph G with $\Delta(G) \geq 3$ that we have encountered thus far, we have seen that $\chi'_{Y_1}(G) < \chi'_{Y_2}(G)$. This is, in fact, true in general. In order to show this, we first establish some preliminary results. An edge-induced subgraph F of a graph G is called a *Δ_k-subgraph* of G if $\Delta(F) = k$. In particular, each component of a Δ_2-subgraph is either a nontrivial path or a cycle. A Δ_k-subgraph F is *maximal* if F is not a proper subgraph of a Δ_k-subgraph.

Theorem 3.2 *Let G be a connected graph of size 3 or more. If F is a maximal Δ_2-subgraph of minimum size in G, then $\chi'_{Y_2}(G) = |E(F)|$.*

Proof. Since the red-blue coloring of G that assigns red to each edge in F and blue to the remaining edges of G is a Y_2-coloring, it follows that $\chi'_{Y_2}(G) \leq |E(F)|$.

It therefore remains to show that $\chi'_{Y_2}(G) \geq |E(F)|$. Among all minimum Y_2-colorings of G, let c be one such that the sum of the degrees of the vertices of degree 3 or more in the resulting red subgraph is minimum. Let $F_{c,r} = G[E_{c,r}]$ be the red subgraph of G induced by $E_{c,r}$. Then $\Delta(F_{c,r}) \geq 2$.

We show that $\Delta(F_{c,r}) = 2$. Assume, to the contrary, that $F_{c,r}$ contains a vertex v such that $\deg_{F_{c,r}} v = k \geq 3$, where $vv_1, vv_2, \ldots, vv_k$ are the edges in $F_{c,r}$ incident with v. Since c is a minimum Y_2-coloring, the red-blue coloring of G in which the color of vv_k is changed to blue is not a Y_2-coloring of G. This implies that there are edges uv_k and wv_k, where $v \neq u, w$, such that (1) uv_k is blue and u is incident with at most one red edge in $F_{c,r}$ and (2) wv_k is red and $\deg_{F_{c,r}} v_k = 2$. The red-blue coloring c' obtained from c by interchanging the colors of vv_k and $v_k u$ is also a minimum Y_2-coloring of G (see Figure 5 where each red edge is indicated by a bold line).

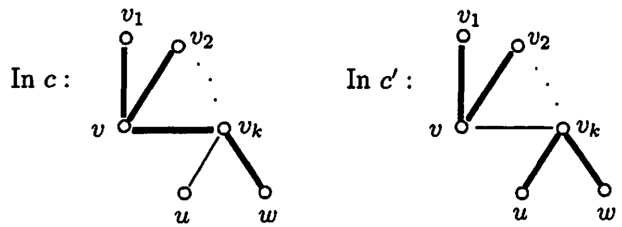

Figure 5: A step in the proof of Theorem 3.2

In the red subgraph $F_{c',r}$ of G induced by $E_{c',r}$, the degree of v_k is 2, the degree of u is at most 2 and the degree of v is $k-1$. Thus the number of vertices of degree 3 or more in $F_{c',r}$ is at most that of $F_{c,r}$, while the sum of degrees of the vertices of degree 3 or more in $F_{c',r}$ is less than that of $F_{c,r}$, which contradicts the defining property of c. Thus, $\Delta(F_{c,r}) = 2$. Therefore, $\chi'_{Y_2}(G) = |E(F_{c,r})| \geq |E(F)|$. ■

From the proof of Theorem 3.2, we have the following corollary.

Corollary 3.3 *If G is a connected graph of order 4 or more, then there is a minimum Y_2-coloring c of G such that the red subgraph of G induced by c is a maximal Δ_2-subgraph of G.*

We now show that with few exceptions, $\chi'_{Y_1}(G) < \chi'_{Y_2}(G)$ for a graph G.

Theorem 3.4 *If G is a connected graph of order at least 4 that is neither a path nor a cycle, then*

$$\chi'_{Y_1}(G) < \chi'_{Y_2}(G).$$

Proof. By Corollary 3.3, there exists a minimum Y_2-coloring c of G such that the red subgraph $F = G[E_{c,r}]$ induced by the set $E_{c,r}$ of red edges is a maximal Δ_2-subgraph of G. Consequently, every component of F is a path or a cycle. We claim that there is a set E' of edges of G with $|E'| < |E_{c,r}|$

such that the red-blue coloring c' of G with $E_{c',r} = E'$ is a Y_1-coloring of G. We construct the set E' as follows.

First, suppose that F has a component P that is a path. The set E' contains all edges belonging to components of size 1 in F. If P is a component of size 2 in F, say $P = (u, v, w)$, where $\deg_G v = 2$, then E' contains both edges of P. If, on the other hand, $\deg_G v \geq 3$, then let E' contain exactly one of uv and vw. Next, consider a component P of size $k \geq 3$, say $P = (v_1, v_2, \ldots, v_{k+1})$. If $\deg_G v_i = 2$ for all i with $2 \leq i \leq k$, then let $E(P) \subseteq E'$. If some vertex v_i $(2 \leq i \leq k)$ of P has degree 3 or more in G, then let v_{i_1} $(i_1 \geq 2)$ be the first vertex (after v_1) of P that has degree 3 or more in G. Let $v_1v_2, \ldots, v_{i_1-1}v_{i_1} \in E'$ and $v_{i_1}v_{i_1+1} \notin E'$. If $i_1 + 2 \leq k$, then repeat this procedure for $P^* = (v_{i_1+1}, v_{i_1+2}, \ldots, v_{k+1})$; that is, if $\deg_G v_i = 2$ for $i_1 + 2 \leq i \leq k$, then let $E(P^*) \subseteq E'$; otherwise, let v_{i_2} $(i_2 \geq i_1 + 2)$ be the first vertex (after v_{i_1+1}) on P that has degree 3 or more in G. Let $v_{i_1+1}v_{i_1+2}, \ldots, v_{i_2-1}v_{i_2} \in E'$ and $v_{i_2}v_{i_2+1} \notin E'$. We repeat this procedure until we arrive at a vertex v_{i_ℓ} for some $\ell \geq 2$ such that either $\deg_G v_i = 2$ for $i_\ell + 2 \leq i \leq k$ or $i_\ell + 2 \geq k + 1$. This procedure is illustrated in Figure 6 for a possible component $P = (v_1, v_2, \ldots, v_{k+1})$ of size $k = 17$ in F. In Figure 6, if an edge e of P is drawn in a bold line, then $e \in E'$; otherwise, $e \notin E'$. If an edge f is drawn in a dashed line, then f is not an edge of P but f is incident with an interior vertex of P that has degree at least 3 in G.

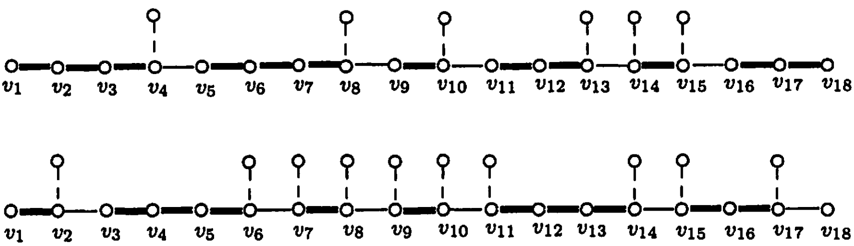

Figure 6: Selecting edges in E' for components that are paths

Second, suppose that F has a component C that is a cycle, say

$$C = (v_1, v_2, \ldots, v_k, v_{k+1} = v_1).$$

We consider three cases.

Case 1. $\deg_G v_i = 2$ *for all* i *with* $1 \leq i \leq k$. Let $E(C) \subseteq E'$.

Case 2. $\deg_G v_i \geq 3$ *for all* i *with* $1 \leq i \leq k$. If k is even, then let $v_iv_{i+1} \in E'$ for each even integer i and $v_iv_{i+1} \notin E'$ for each odd integer i. Now suppose that k is odd. Suppose that vv_k is an edge of G that is not on C, where v may or may not be on C. Then $vv_k \notin E(F)$. Let $vv_k \in E'$ and

$v_iv_{i+1} \in E'$ for each odd integer i with $1 \leq i \leq k-2$ and the remaining edges of C do not belong to E'. Thus for each cycle of size k in F, there are $\lceil k/2 \rceil$ edges added to the set E'. This is illustrated in Figure 7 for $C = C_3, C_4, C_5$. In Figure 7, if an edge e is drawn in a bold line, then $e \in E'$; otherwise, $e \notin E'$. If an edge f is drawn in a dashed line, then f is not an edge of C but f is incident with a vertex of C that has degree at least 3 in G.

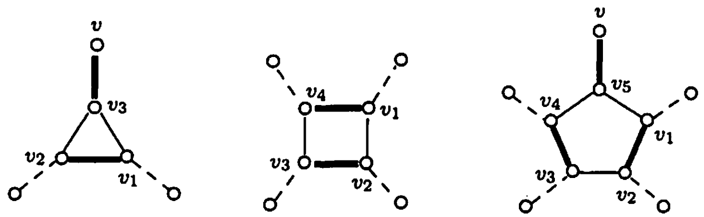

Figure 7: Selecting edges in E' for a cycle component in Case 2

Case 3. *There is at least one vertex of C having degree* 2 *and at least one vertex of G having degree* 3 *or more.* We may assume that $\deg_G v_1 = 2$. Let v_{i_1} ($i_1 \geq 2$) be the first vertex of C that has degree 3 or more in G. Let $v_1v_2, \ldots, v_{i_1-1}v_{i_1} \in E'$ and $v_{i_1}v_{i_1+1} \notin E'$. If $\deg_G v_i = 2$ for $i_1 + 2 \leq i \leq k$, then let $v_iv_{i+1} \in E'$ for $i_1 + 2 \leq i \leq k$; otherwise, let v_{i_2} ($i_2 \geq i_1 + 2$) be the first vertex of C that has degree 3 or more in G. Let $v_{i_1+1}v_{i_1+2}, \ldots, v_{i_2-1}v_{i_2} \in E'$ and $v_{i_2}v_{i_2+1} \notin E'$. We repeat this procedure until we arrive at a vertex v_{i_ℓ} for some $\ell \geq 2$ such that either $\deg_G v_i = 2$ for $i_\ell + 2 \leq i \leq k$ (and let $v_iv_{i+1} \in E'$ for $i_\ell + 2 \leq i \leq k$) or $i_\ell + 2 \geq k + 1$. This procedure is illustrated in Figure 8 for a possible component $C = (v_1, v_2, \ldots, v_k, v_{k+1} = v_1)$ of size $k = 17$ in F. In Figure 8, if an edge e of C is drawn in a bold line, then $e \in E'$; , otherwise, $e \notin E'$. If an edge f is drawn as a dashed line, then f is not an edge of C but f is incident with a vertex of C that has degree at least 3 in G.

Since $\Delta(G) \geq 3$, there is either a component P in F that is a path one of whose interior vertices has degree at least 3 in G or a component C in F that is a cycle in which some vertex has degree at least 3 in G.

- If z is an interior vertex of the path P such that $\deg_G z \geq 3$, then exactly one of the two edges incident with z on P belongs to E'.

- If z is a vertex on the cycle C such that $\deg_G z \geq 3$, then either (1) exactly one of the two edges on C incident with z belongs to E' or (2) neither of these two edges belong to E' but there is an edge zz' not on C that belongs to E'.

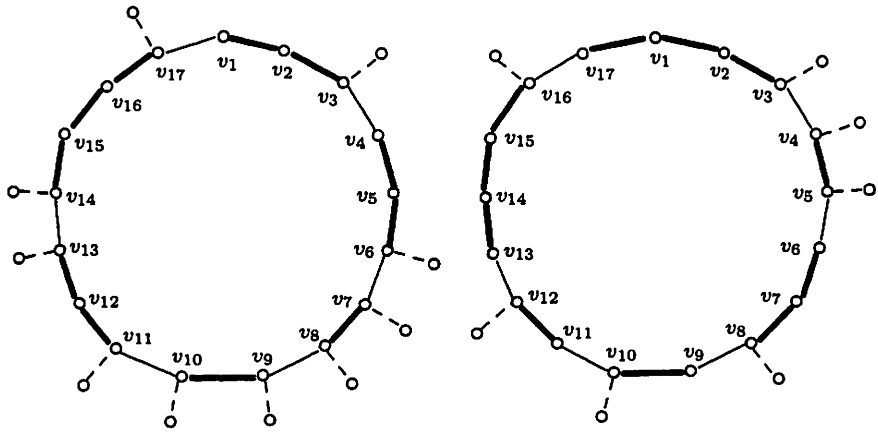

Figure 8: Selecting edges in E' for a cycle component in Case 3

Therefore, $|E'| < |E_{c,r}|$.

We now show that the red-blue coloring c' of G with $E_{c',r} = E'$ is a Y_1-coloring of G. Let $e = uv$ be a blue edge in the coloring c'. First, suppose that e is a red edge in the coloring c and so either u or v has degree 3 or more in G, say $\deg_G v \geq 3$, and v is incident with a red edge different from e and at least one blue edge in the coloring c. Thus e belongs to a copy of Y_1 in the coloring c'. Next, suppose that e is a blue edge in the coloring c. Thus e belongs to a copy Y of Y_2 in c, say $E(Y) = \{e = uv, ux, uy\}$, where (x, u, y) is a path in a component of F. By the definition of E', there are two possibilities.

(1) Exactly one of ux and uy belongs to E' and so one of ux and uy is red and the other is blue in the coloring c'. So e belongs to a copy of Y_1.

(2) Neither ux nor uy belongs to E' but there is an edge uu' not on C that belongs to E'. Thus e is adjacent to at least one blue edge and the red edge uu' in the coloring c'. Hence e belongs to a copy of Y_1.

Therefore, c' is a Y_1-coloring of G and so

$$\chi'_{Y_1}(G) = |E'| < |E(F)| = \chi'_{Y_2}(G),$$

as desired. ■

By Observation 2.1, Theorems 3.1 and 3.4, we have the following.

Corollary 3.5 *If G is a connected graph of order at least 4, then*

$$\chi'_{Y_1}(G) \leq \chi'_{Y_2}(G) \leq 3\chi'_{Y_1}(G).$$

By Corollary 3.5, if G is a connected graph of order at least 4 with $\chi'_{Y_1}(G) = a$ and $\chi'_{Y_2}(G) = b$, then $a \leq b \leq 3a$ and $b \geq 2$. We next show that every pair a, b of positive integers with $a \leq b \leq 3a$ and $b \geq 2$ can be realized as $\chi'_{Y_1}(G)$ and $\chi'_{Y_2}(G)$, respectively, for some connected graph G of order at least 4. To show this, we first present two lemmas.

Lemma 3.6 *If G is the corona of an n-cycle where $n \geq 3$, then*

$$\chi'_{Y_1}(G) = \lceil n/2 \rceil \text{ and } \chi'_{Y_2}(G) = n.$$

Proof. Let $G = cor(C_n)$ where $C_n = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$ for some integer $n \geq 3$. Suppose that $u_i v_i$ is the pendant edge of G at v_i for $1 \leq i \leq n$. First, we show that $\chi'_{Y_1}(G) = \lceil n/2 \rceil$. For each even integer $n \geq 4$, define a red-blue coloring c of G with

$$E_{c,r} = \{v_i v_{i+1} : i \text{ is odd}, 1 \leq i \leq n-1\}; \tag{3}$$

while for each odd integer $n \geq 3$, define a red-blue coloring c of G with

$$E_{c,r} = \{v_i v_{i+1} : i \text{ is odd}, 1 \leq i \leq n-2\} \cup \{u_n v_n\}. \tag{4}$$

Since c is a Y_1-coloring of G in each case, $\chi'_{Y_1}(G) \leq |E_{c,r}| = \lceil n/2 \rceil$.

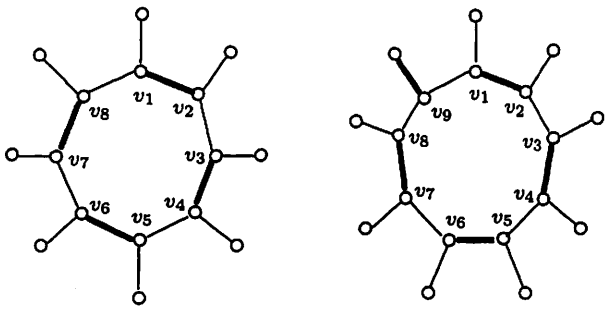

Figure 9: Illustrate Y_1-colorings for $cor(C_8)$ and $cor(C_9)$

To show that $\chi'_{Y_1}(G) \geq \lceil n/2 \rceil$, suppose that G has a Y_1-coloring c^* using at most $\lceil n/2 \rceil - 1$ red edges. Thus the size of the red subgraph G^*_r induced by c^* is at most $\lceil n/2 \rceil - 1$. Since G^*_r contains no isolated vertices and $n/2 > \lceil n/2 \rceil - 1$, the order of G^*_r is at most $n-1$ and so there is at least one vertex v of C_n such that $v \notin V(G^*_r)$, say $v = v_1$. However then, the blue edge $u_1 v_1$ does not belong to any copy of Y_1, which is impossible. Thus $\chi'_{Y_1}(G) \geq \lceil n/2 \rceil$ and so $\chi'_{Y_1}(G) = \lceil n/2 \rceil$.

Next, we show that $\chi'_{Y_2}(G) = n$. The red-blue coloring that assigns the color red to each edge of C_n in G and the color blue to the remaining edges of G is a Y_2-coloring of G with exactly n red edges. Hence $\chi'_{Y_2}(G) \leq n$. To show that $\chi'_{Y_2}(G) \geq n$, let c be a minimum Y_2-coloring of G. For each integer i with $1 \leq i \leq n$, at least one edge in $\{u_iv_i, v_iv_{i+1}\}$ must be red; for otherwise, the blue edge u_iv_i does not belong to any copy of Y_2 in G, which is a contradiction. Thus $\chi'_{Y_2}(G) = |E_{c,r}| \geq n$. Therefore, $\chi'_{Y_2}(G) = n$. ■

Lemma 3.7 *Let G_1 and G_2 be two vertex-disjoint graphs, where v_i is an end-vertex in G_i for $i = 1, 2$. If G is the graph obtained from G_1 and G_2 by identifying the vertex v_1 in G_1 with the vertex v_2 in G, then $\chi'_{Y_i}(G) = \chi'_{Y_i}(G_1) + \chi'_{Y_i}(G_2)$ for $i = 1, 2$.*

Proof. For a given $Y \in \{Y_1, Y_2\}$, let $\chi'_Y(G_i) = k_i$ for $i = 1, 2$. Suppose that v is the vertex of G obtained by identifying v_1 and v_2 in G_1 and G_2, respectively. Then $V(G) = (V(G_1) - \{v_1\}) \cup (V(G_2) - \{v_2\}) \cup \{v\}$ and $E(G) = E(G_1) \cup E(G_2)$ where $E(G_1) \cap E(G_2) = \emptyset$. Let c_i be a minimum Y-coloring of G_i for $i = 1, 2$. Since the red-blue coloring c of G defined by $c(e) = c_i(e)$ if $e \in E(G_i)$ is a Y-coloring of G having exactly $k_1 + k_2$ red edges, it follows that $\chi'_Y(G) \leq |E_{c,r}| = k_1 + k_2$. Suppose that $\chi'_Y(G) < k_1 + k_2$. Let c^* be a minimum Y-coloring of G and let $E_{c^*,r} = E_1 \cup E_2$, where E_i is the set of red edges in G_i for $i = 1, 2$. Hence, either $|E_1| < k_1$ or $|E_2| < k_2$, say the former. Then the red-blue coloring c_1^* with $E_{c_1^*,r} = E_1$ is not a Y-coloring of G_1 and so there is a blue edge $e \in E(G_1)$ that does not belong to any copy of Y in G_1. Since $\deg_G v = 2$, it follows that G contains no $K_{1,3}$ having edges in both G_1 and G_2. Thus e does not belong to any copy of Y in G, which is a contradiction. Thus $\chi'_Y(G) = \chi'_Y(G_1) + \chi'_Y(G_2)$. ■

Theorem 3.8 *For a given pair a, b of positive integers, there exists a connected graph G of order at least 4 such that $\chi'_{Y_1}(G) = a$ and $\chi'_{Y_2}(G) = b$ if and only if $a \leq b \leq 3a$ and $b \geq 2$.*

Proof. We have seen that if G is a connected graph of order at least 4 with $\chi'_{Y_1}(G) = a$ and $\chi'_{Y_2}(G) = b$, then $a \leq b \leq 3a$ and $b \geq 2$. It remains to verify the converse. If $b = a \geq 2$, then let $G = P_{b+1}$ and so $\chi'_{Y_1}(G) = \chi'_{Y_2}(G) = a$. If $b = a + 1$, let G be the graph obtained from $P_{b+1} = (v_1, v_2, \ldots, v_{b+1})$ by adding a new vertex v and joining v to the vertex v_2. Then $\chi'_{Y_1}(G) = b - 1 = a$ and $\chi'_{Y_2}(G) = b$. Thus, we may assume that $b = a + k$ where $k \geq 2$. We consider two cases, according to whether $k \leq a$ or $a + 1 \leq k \leq 2a$.

Case 1. $k \leq a$. Let $a = k + \ell$ where $\ell \geq 0$. Then $b = a + k = (k + \ell) + k = 2k + \ell$. If $\ell = 0$, then let $G = cor(C_{2k})$. By Lemma 3.6, $\chi'_{Y_1}(G) = k = a$ and $\chi'_{Y_2}(G) = 2k = b$. If $\ell \geq 1$, then let G be the graph obtained from the corona $cor(C_{2k})$ and the path $P_{\ell+1} = (v_1, v_2, \ldots, v_{\ell+1})$ by identifying the vertex v_1 of $P_{\ell+1}$ with an end-vertex of $cor(C_{2k})$. Then $\chi'_{Y_1}(G) = k+\ell = a$ and $\chi'_{Y_2}(G) = 2k+\ell = b$ by Lemmas 3.6 and 3.7.

Case 2. $a+1 \le k \le 2a$. Then $k = a+p$ for some p with $1 \le p \le a$. Then $b = a+(a+p) = 2a+p$. We consider three subcases.

Subcase 2.1. $p = a$. Then $b = 3a$. Let H_a be the tree obtained from the path $P_{3a+1} = (v_1, v_2, \ldots, v_{3a+1})$ by adding the pendant edge u_iv_i at each vertex v_i if $i \equiv 0, 2 \pmod 3$ and $2 \le i \le 3a$. The graph H_3 is shown in Figure 10 for $a = 3$. By Observation 2.2 and successive applications of Lemma 3.7, it follows that $\chi'_{Y_1}(H_a) = a$ and $\chi'_{Y_2}(H_a) = 3a = b$.

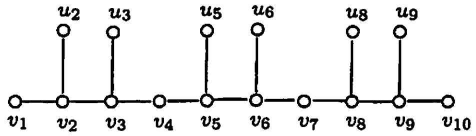

Figure 10: The graph H_3 with $\chi'_{Y_1}(H_3) = 3$ and $\chi'_{Y_2}(H_3) = 9$ in Subcase 2.1

Subcase 2.2. $p = a-1 \ge 1$. Then $b = 3a-1$. Let H_{a-1} be the tree constructed in Subcase 2.1 where a is replaced by $a-1$. Then $\chi'_{Y_1}(H_{a-1}) = a-1$ and $\chi'_{Y_2}(H_{a-1}) = 3(a-1)$. Let x be a peripheral vertex of H_{a-1} and let $H = K_{1,3}$. The graph G is obtained from H_{a-1} and H by identifying the vertex x in H_{a-1} and an end-vertex of H. (The graph G is shown in Figure 11 for $a = 3$.) Then $\chi'_{Y_1}(G) = (a-1)+1 = a$ and $\chi'_{Y_2}(G) = 3(a-1)+2 = 3a-1$ by Lemma 3.7.

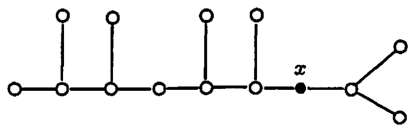

Figure 11: The graph G with $\chi'_{Y_1}(G) = 3$ and $\chi'_{Y_2}(G) = 8$ in Subcase 2.2

Subcase 2.3. $1 \le p \le a-2$. Then $a-p \ge 2$. Let H_p be the tree in Subcase 2.1 (where a is replaced by p) with $\chi'_{Y_1}(H_p) = p$ and $\chi'_{Y_2}(H_p) = 3p$. Let $H = cor(C_{2(a-p)})$ and so $\chi'_{Y_1}(H) = a-p$ and $\chi'_{Y_2}(H) = 2(a-p)$ by Lemma 3.6. Let x be a peripheral vertex of H_p. The graph G is obtained from H_p and H by identifying the vertex x in H_p and an end-vertex of H. The graph G is shown in Figure 12 for $p = 2$ and $a = 5$ where $H = cor(C_6)$. Then $\chi'_{Y_1}(G) = p+(a-p) = a$ and $\chi'_{Y_2}(G) = 3p+2(a-p) = 2a+p = b$ by Lemmas 3.6 and 3.7. ∎

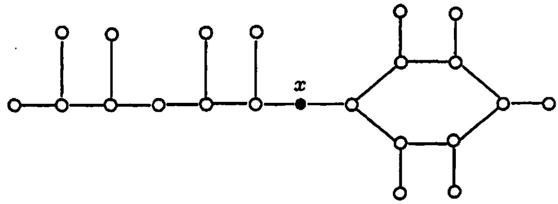

Figure 12: The graph G with $\chi'_{Y_1}(G) = 5$ and $\chi'_{Y_2}(G) = 12$ in Subcase 2.3

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