Edge Colourings and the Depression of a Graph*

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Abstract

An *edge ordering* of a graph G is an injection $f : E(G) \to \mathbb{Z}$, where $\mathbb{Z}$ denotes the set of integers. A path in G for which the edge ordering f increases along its edge sequence is called an *f-ascent*; an f-ascent is *maximal* if it is not contained in a longer f-ascent. The *depression* of G is the smallest integer k such that any edge ordering f has a maximal f-ascent of length at most k. We apply the concept of ascents to edge colourings using possibly less than $|E(G)|$ colours and consider the problem of determining the minimum number of colours required such that there exists an edge colouring c for which the length of a shortest maximal c-ascent is equal to the depression of G.

Keywords: edge ordering of a graph, increasing path, monotone path, depression, edge colouring

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1 Introduction

Suppose the edges of a finite graph are totally ordered. The problem of determining, or bounding, the height – the length of a longest maximal path along which the edges increase with respect to the order – was first studied by Chvátal and Komlós [5]. They considered the problem of determining the minimum value of the height (over all total edge orderings) for complete graphs. This is a difficult

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problem and the result is known only for $1 \leq n \leq 8$ (see [2, 5]). The related problem of determining the maximum value of the flatness – the length of a shortest maximal path – was first introduced in [7].

Partial orderings have been used to determine an upper bound on the minimum value of the height. Proper edge colourings are an example of such a partial order. In this paper we study proper edge colourings in relation to the maximum value of the flatness. In particular, we consider the following question: What is the smallest integer r such that there exists a proper edge colouring $c : E(G) \to \{1, 2, \ldots, r\}$ for which a shortest maximal ascent has length k? We focus our attention on the case where k is the maximum value of the flatness, and also consider the case where it is smaller.

Formal definitions are provided in the next section. Further background and a summary of known results follow in Section 3. Our main results are established in Section 4. These include general bounds, classes of graphs which attain some of these bounds, an upper bound for a particular class of trees, and an upper bound for complete graphs. The paper concludes with a list of open problems, in Section 5.

2 Definitions and Background

We consider simple, finite graphs $G = (V(G), E(G))$. For basic graph theoretic definitions we refer the reader to the book [4] or any of its predecessors.

A *branch vertex* of a tree is a vertex of degree at least three. Let $\ell(T)$ and $B(T)$ respectively denote the sets of all leaves and branch vertices of the tree T. For $v \in V(T)$ and $l \in \ell(T)$, a (v, l)-*endpath*, or v-*endpath* if the leaf is unimportant, or *endpath* if neither v nor l is important, is a path P from v to l such that each internal vertex of P has degree two in T. A *spider* $S(a_1, a_2, \ldots, a_r)$ is a tree with exactly one branch vertex v and v-endpaths (also called *legs*) of lengths $1 \leq a_1 \leq a_2 \leq \cdots \leq a_r$.

An *edge ordering* of a graph G is an injection $f : E(G) \to \mathbb{Z}$, where $\mathbb{Z}$ denotes the set of integers. Denote the set of all edge orderings of G by $\mathcal{F}(G)$. A path G for which $f \in \mathcal{F}(G)$ increases along its edge sequence is called an f-*ascent*; an f-ascent is *maximal* if it is not contained in a longer f-ascent. The *height* of an edge ordering, denoted by $H(f)$, is the length of a longest maximal f-ascent, while the *flatness* of an edge ordering f, denoted by $h(f)$, is the length of a shortest maximal f-*ascent* of G.

In [2] the *altitude* of G was defined as $\alpha(G) = \min_{f \in \mathcal{F}(G)} \{H(f)\}$. The interpretation of the altitude of a graph G is that any edge ordering $f \in \mathcal{F}(G)$ has an f-ascent of length at least $\alpha(G)$, and $\alpha(G)$ is the largest integer for which this statement is true.

In [7] the *depression* of G was defined as $\varepsilon(G) = \max_{f\in\mathcal{F}(G)}\{h(f)\}$. The interpretation of the depression of a graph G is that any edge ordering f has a maximal f-ascent of length at most $\varepsilon(G)$, and $\varepsilon(G)$ is the smallest integer for which this statement is true.

For a proper edge colouring $c : E(G) \to \mathbb{Z}$ we define a *(maximal) c-ascent* and the *flatness* of c, denoted by $h(c)$, analogously to edge orderings.

Given a proper edge colouring $c : E(G) \to \mathbb{Z}$ of the graph G, a c-ascent λ is simply called an *ascent* if the colouring is clear, and if λ has length k, it is also called a (k,c)-*ascent*. If the path λ with vertex sequence $v_0, v_1, \ldots, v_k$ or edge sequence $e_1, e_2, \ldots, e_k$ forms a c-ascent, we denote this fact by writing λ as $v_0v_1\cdots v_k$ or $e_1e_2\cdots e_k$.

There is an elementary relationship between edge colourings and the altitude of a graph. Upper bounds on α are established using edge colourings in the following method, also used in [2, 3, 6, 8, 11, 16, 17]. We colour the edges of the graph (not necessarily obtaining a proper colouring) in $t \geq 2$ colours. We then obtain an edge ordering f by first labelling all the edges of one colour with consecutive integers, and then the edges of the next colour, etc. In any f-ascent, once we use edges of one colour, we cannot use edges of a previous colour since all such edges have smaller labels. In [17], this method, together with Vizing's Theorem on the chromatic index χ_1 (see Corollary 8.19 in [4]), is used to show that $\alpha(G) \leq \Delta(G) + 1$, where $\Delta(G)$ denotes the maximum degree of G. For other work on the altitude of graphs the reader is referred to e.g. [1, 12, 14].

3 Known Results

Let $\tau(G)$ denote the length of a longest path in G, called the *detour length* in G. If we assume that G is connected and of size at least two, then

$$2 \leq \varepsilon(G), \alpha(G) \leq \tau(G).$$

By taking the edge ordering f for the path P_n, $n \geq 3$, to increase along its edge sequence we see that $\varepsilon(P_n) = \tau(P_n) = n-1$. On the other hand, by taking the edge ordering for the path P_n, $n \geq 3$, as $1, n-1, 2, n-2, \ldots, \lceil\frac{n}{2}\rceil$ along its edge sequence, we see that $\alpha(P_n) = 2$.

If a connected graph G has a vertex v that is adjacent to u, w, where u, w are end-vertices or adjacent vertices of degree two, then in any edge ordering f of G, either uvw or wvu is a maximal $(2, f)$-ascent, hence $\varepsilon(G) = 2$. In [7] it was shown that the converse of this statement is also true, which gives the following characterization of graphs with depression two.

Theorem 1. [7] *If G is connected, then $\varepsilon(G) = 2$ if and only if G has a vertex* *adjacent to two end-vertices or to two adjacent vertices of degree two.*

The characterization of graphs with depression three remains an open problem, however, trees with depression three were characterized in [13], and graphs with depression three and no adjacent vertices of degree three or higher were characterized in [15]. A lower bound for the depression of trees was established in [9] and it was shown that this bound gives the exact value of $\varepsilon(T)$ if the tree T has no adjacent vertices of degree three or higher.

It is reasonable to expect a link between the depression of a graph G and the diameter of its line graph $L(G)$, and indeed the following result appeared in [7].

Theorem 2. [7] *If* $\operatorname{diam}(L(G)) = 2$, *then* $\varepsilon(G) \leq 3$.

The difference $\operatorname{diam}(L(G)) - \varepsilon(G)$ can be arbitrarily large, a result that easily follows from Theorem 1. Much harder to see is that the difference $\varepsilon(G) - \operatorname{diam}(L(G))$ can also be arbitrarily large as shown by Gaber-Rosenblum and Roddity in [10].

The depression of complete graphs is a direct result of Theorems 1 and 2.

Corollary 3. [7] $\varepsilon(K_n) = 3$ *for all* $n \geq 4$.

The depression of cycles and spiders is also given in [7].

Proposition 4. [7] $\varepsilon(C_n) = \lceil \frac{n+1}{2} \rceil$ *for all* $n \geq 3$.

Proposition 5. [7] $\varepsilon(S(a_1, a_2, \ldots, a_r)) = \min\{a_1 + a_2, a_3 + 1\}$.

4 Main Results

4.1 The ε-ascent chromatic index of a graph

For an edge colouring we assume the colours are integers, and for any positive integer r, when we consider an r-edge colouring of a graph we assume without stating it explicitly that we used the colours $1, 2, \ldots, r$. We define the *ε-ascent chromatic index* of a graph, denoted $\chi_\varepsilon(G)$, as the minimum number of colours of a proper edge colouring c with $h(c) = \varepsilon(G)$.

Therefore, to prove that $\chi_\varepsilon(G) = k$, we must show that

(i) there exists a proper k-edge colouring c such that $h(c) = \varepsilon(G)$, i.e. $\chi_\varepsilon(G) \leq k$, and

(ii) for all proper $(k-1)$-edge colourings c, $h(c) < \varepsilon(G)$, i.e. $\chi_\varepsilon(G) \geq k$.

For example, in the case of C_4 (for which $\varepsilon(C_4) = 3$), if we colour the edges in sequence 1, 2, 3, 2, then the resulting colouring has flatness three, and since any proper colouring in only two colours has flatness two, we conclude that $\chi_\varepsilon(C_4) = 3$.

Remark 6. *For any graph G,*

(a) $\varepsilon(G) \leq \chi_\varepsilon(G)$,

(b) $\chi_1(G) \leq \chi_\varepsilon(G)$, *and*

(c) $\chi_\varepsilon(G) \leq |E(G)|$.

As mentioned in Remark 6, $\chi_\varepsilon(G) \geq \chi_1(G)$. We next characterize graphs G for which $\chi_\varepsilon(G) = \chi_1(G)$.

Proposition 7. *For any graph G, $\chi_\varepsilon(G) = \chi_1(G)$ if and only if $\varepsilon(G) \leq 2$.*

Proof. If $\varepsilon(G) = 1$, then G has exactly one edge and thus $\chi_\varepsilon(G) = \chi_1(G) = 1$. If $\varepsilon(G) = 2$, then G has at least two adjacent edges. Any proper edge colouring of G has the property that its maximal ascents have length at least two, thus $\chi_\varepsilon(G) = \chi_1(G)$. Now suppose $\varepsilon(G) \geq 3$ and $\chi_\varepsilon(G) = k$, and let $c : E(G) \to \{1, 2, \ldots, k\}$ be a proper edge colouring of G in $\chi_\varepsilon(G)$ colours whose maximal ascents have lengths at least three. Since G has no maximal $(2, c)$-ascents, the colours 1 and k do not occur at the same vertex, for if $c(uv) = 1$ and $c(vw) = k$, then uvw is a maximal $(2, c)$-ascent. Therefore each edge e such that $c(e) = k$ can be recoloured with colour 1 to give a proper $(k-1)$-edge colouring of G. Thus $\chi_1(G) < \chi_\varepsilon(G)$. □

4.2 A lower bound

As mentioned in Remark 6, $\chi_\varepsilon(G) \geq \chi_1(G)$, and by Proposition 7, equality holds if and only if $\varepsilon(G) = 2$. We now improve this bound for $\varepsilon(G) \geq 3$.

Proposition 8. *If $\varepsilon(G) \geq 2$, then $\chi_\varepsilon(G) \geq \chi_1(G) + \varepsilon(G) - 2$.*

Proof. If $\varepsilon(G) = 2$, the result follows from Remark 6. Let G be a graph with $\varepsilon(G) = k \geq 3$. Suppose, to the contrary, that $\chi_\varepsilon(G) = m \leq \chi_1(G) + k - 3$ and consider a proper m-edge colouring c of G for which $h(c) = \varepsilon(G)$. Suppose that an edge e for which $c(e) = 1$ is adjacent to an edge e' with $c(e') = \chi_1(G)$. Then any maximal ascent containing ee' has length at most $k-1$, a contradiction. In general, if an edge e for which $c(e) = i$ is adjacent to an edge e' for which $c(e') = \chi_1(G)+i-1$, then $h(c) \leq i+1+(\chi_1(G)+k-3)-(\chi_1(G)+i-1) = k-1$. Thus, for $1 \leq i \leq k-2$, an edge assigned colour i is not adjacent to an edge

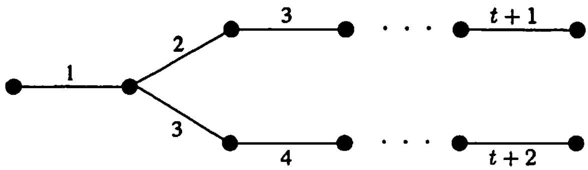

Figure 1: An edge colouring of $S(1,t,t)$ with flatness $t+1$.

assigned colour $\chi_1(G)+i-1$. Therefore we may reassign the edges coloured i with colour $\chi_1(G)+i-1$ for each i such that $1 \le i \le k-2$. Since $k \ge 3$, this gives us a proper edge colouring in $m-(k-2) \le \chi_1(G)-1$ colours, a contradiction. □

Corollary 9. *If* $\varepsilon(G) \ge 2$, *then* $\chi_\varepsilon(G) \ge \Delta(G)+\varepsilon(G)-2$.

For each $k \ge 3$, there exists a graph G with $\varepsilon(G)=k$ for which $\chi_\varepsilon(G)$ realizes the bound in Proposition 8. For example, consider the spider $S(1,t,t)$, where $t \ge 2$. By Proposition 5, $\varepsilon(S(1,t,t))=t+1$. Furthermore, $\chi_1(S(1,t,t))=3$, and by Proposition 8, $\chi_\varepsilon(S(1,t,t)) \ge t+2$. To establish $\chi_\varepsilon(S(1,t,t)) \le t+2$ we consider the proper $(t+2)$-edge colouring c of $S(1,t,t)$ shown in Figure 1, for which it is easy to verify that $h(c)=t+1$. Thus for all $t \ge 2$, $\varepsilon(S(1,t,t))=t+1$ and $\chi_\varepsilon(S(1,t,t))=t+2$.

The difference $\chi_\varepsilon(G)-[\chi_1(G)+\varepsilon(G)-2]$ can also be arbitrarily large. Consider the spider $S(t,t,t)$. By Proposition 5, $\varepsilon(S(t,t,t))=t+1$. Let v be the vertex of $S(t,t,t)$ with degree three, $e_{1,1}, e_{2,1}, e_{3,1}$ the edges incident with v, and $\lambda_i = e_{i,1}e_{i,2}\cdots e_{i,t}$ the v-endpath containing $e_{i,1}$. Let c be a proper r-edge colouring with $h(c)=t+1$. Without loss of generality we may assume that $c(e_{1,1}) < c(e_{2,1}) < c(e_{3,1})$. Necessarily, $c(e_{3,1}) < c(e_{3,2}) < \cdots < c(e_{3,t})$, otherwise λ_3 contains a maximal (k,c)-ascent, where $k \le t$, which is a contradiction. By a similar argument, $c(e_{1,1}) > c(e_{1,2}) > \cdots > c(e_{1,t})$. Hence $r \ge 2t+1$. If we let $c(e_{2,i}) = c(e_{3,i})$ for $2 \le i \le t$, then $r \le 2t+1$ and the resulting edge colouring has the required flatness. Thus $\chi_\varepsilon(S(t,t,t)) \le 2t+2$, and $\chi_\varepsilon(S(t,t,t)) - [\chi_1(S(t,t,t))+\varepsilon(S(t,t,t))-2] = t-1$.

4.3 Paths and cycles

In this section we determine the ε-ascent chromatic index for paths and cycles. We also show that the only graphs for which $\chi_\varepsilon(G)=\varepsilon(G)$ are paths and even cycles.

Proposition 10. $\chi_\varepsilon(P_n)=n-1$ *for all* $n \ge 2$.

Proof. Since $\varepsilon(P_n) = n - 1 = |E(P_n)|$, it follows from Remark 6 that $\chi_\varepsilon(T) = n - 1$. □

Proposition 11. $\chi_\varepsilon(C_n) = \lceil \frac{n}{2} \rceil + 1$ *for all* $n \geq 3$.

Proof. By Proposition 4 and Remark 6, $\chi_\varepsilon(C_n) \geq \varepsilon(C_n) = \lceil \frac{n+1}{2} \rceil$. Let $C_n = e_1 e_2 \cdots e_n$.

Case 1: n is even. Define the edge ordering c_e of C_n by

$$c_e(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq \frac{n}{2} + 1 \\ n - i + 2 & \text{if } \frac{n}{2} + 1 < i \leq n. \end{cases}$$

It is easy to verify that c_e is a proper $(\frac{n}{2} + 1)$-edge colouring with $h(c_e) = \varepsilon(C_n)$. Since n is even, $\frac{n}{2} + 1 = \lceil \frac{n+1}{2} \rceil = \varepsilon(C_n)$. Hence, by Remark 6, $\chi_\varepsilon(C_n) = \frac{n}{2} + 1$, and the result holds.

Case 2: n is odd. Define the edge colouring c_o of C_n by

$$c_o(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq \lceil \frac{n}{2} \rceil + 1 \\ n - i + 2 & \text{if } \lceil \frac{n}{2} \rceil + 1 < i \leq n. \end{cases}$$

It is easy to verify that c_o is a proper $(\lceil \frac{n}{2} \rceil + 1)$-edge colouring with $h(c_o) = \varepsilon(C_n)$. Since n is odd, $\lceil \frac{n}{2} \rceil + 1 = \lceil \frac{n+1}{2} \rceil + 1 = \varepsilon(C_n) + 1$. To prove the result we must show that any proper $(\frac{n+1}{2})$-edge colouring of C_n has flatness at most $\frac{n-1}{2}$. Suppose, to the contrary, that there exists a proper $(\frac{n+1}{2})$-edge colouring c'_o of C_n with flatness $\varepsilon(c_o) = \frac{n+1}{2}$. Let $c'_o(e_1) = 1$. Since any maximal c'_o-ascent in C_n has length at least $\frac{n+1}{2}$, it follows that

$$c'_o(e_i) = \begin{cases} i & \text{if } 2 \leq i \leq \frac{n+1}{2} \\ n - i + 2 & \text{if } \frac{n+1}{2} < i \leq n. \end{cases}$$

But then $c'_o(e_k) = c'_o(e_{k+1})$ for $k = \frac{n+1}{2}$, and c'_o is not a proper edge colouring, a contradiction. □

As mentioned in Remark 6, $\chi_\varepsilon(G) \geq \varepsilon(G)$. We next characterize graphs G for which $\chi_\varepsilon(G) = \varepsilon(G)$.

Proposition 12. *If G is connected and $\chi_\varepsilon(G) = \varepsilon(G)$, then $G = C_{2n}$ or $G = P_n$ for $n \geq 2$.*

Proof. Let G be a graph such that $\chi_\varepsilon(G) = \varepsilon(G)$. By Corollary 9, $\Delta(G) \leq 2$, which implies that G is either a path or a cycle. If $G = P_n$, then $\varepsilon(G) = n - 1$. Hence, from Proposition 10, $\chi_\varepsilon(G) = n - 1 = \varepsilon(G)$. From Proposition 4,

$\varepsilon(C_n) = \lceil \frac{n+1}{2} \rceil$. If $G = C_{2k+1}$, then $\varepsilon(G) = k+1$ and by Proposition 11, $\chi_\varepsilon(G) = k+2$. If $G = C_{2k}$, then $\varepsilon(G) = k+1$ and by Proposition 11, $\chi_\varepsilon(G) = k+1$. □

4.4 Trees

Remark 6 states that $\chi_\varepsilon(G) \leq |E(G)|$. In this section we characterize trees T for which $\chi_\varepsilon(T) = |E(T)|$. The problem of characterizing graphs in general for which $\chi_\varepsilon(G) = |E(G)|$ is included in the list of open problems in Section 5. In this section we also bound $\chi_\varepsilon(T)$ for trees with $\varepsilon(T) = 3$ and in the process introduce a variation on the parameter $\chi_\varepsilon(G)$.

Theorem 13. *Let T be a tree. Then $\chi_\varepsilon(T) = |E(T)|$ if and only if $T = P_n$ or $T = K_{1,n}$, $n \geq 2$.*

Proof. Suppose $T = P_n$, where $n \geq 2$. Then $\varepsilon(T) = n-1 = |E(T)|$, and by Remark 6, $\chi_\varepsilon(T) = |E(T)|$. Suppose $T = K_{1,n}$ where $n \geq 2$. Then $\chi_1(T) = n = |E(T)|$, and again by Remark 6, $\chi_\varepsilon(T) = |E(T)|$.

Conversely, suppose that $T \neq P_n$ and $T \neq K_{1,n}$ for $n \geq 2$. If T contains a vertex which is adjacent to two leaves, then $\varepsilon(T) = 2$ and by Proposition 7, $\chi_\varepsilon(T) = \chi_1(T)$. Furthermore, since T is not a star, $\text{diam}(L(T)) \geq 2$ which implies $\chi_1(T) < |E(T)|$.

Assume then that no vertex of T is adjacent to two leaves, that is, assume $\varepsilon(T) \geq 3$. Let f be an edge ordering of T with $h(f) = \varepsilon(T)$. Suppose also that T has at least three endpaths of length two or more. Let e_1, e_2 and e_3 be pendant edges on three such endpaths. Necessarily each e_i is either the initial or final edge of a maximal f-ascent in T. Without loss of generality we may assume that e_1 and e_2 are both initial edges of a maximal f-ascent in T. Let c be an edge colouring of T such that $c(e_1) = c(e_2) = \min_{e \in E(T)}\{f(e)\}$ and for all other edges $e \notin \{e_1, e_2\}$, $c(e) = f(e)$. Then $h(c) = h(f) = \varepsilon(T)$, which implies $\chi_\varepsilon(T) < |E(T)|$.

Suppose then that T contains at most two endpaths of length two or more. Consider the case where $B(T) = \{v\}$. Since v is not adjacent to two leaves and v has exactly two endpaths of length two or more, $T = S(1, k_1, k_2)$, where $2 \leq k_1 \leq k_2$. By Proposition 5, $\varepsilon(S(1, k_1, k_2)) = 1 + k_1$. Now we describe an edge colouring c using fewer than $|E(T)|$ labels such that $h(c) = 1 + k_1$. Assign the three edges incident with v the three smallest labels under f where the pendant edge receives the smallest of these labels. For the edges not incident with v, the values of f increase along each of the endpaths and the edges that are the same distance from v are assigned the same values. Thus at least two edges will be assigned the same value under the colouring c and it is easily verified that $h(c) = 1 + k_1 = \varepsilon(T)$.

Suppose then that $|B(T)| = k \geq 2$. Denote the number of leaves of T by l. Then $l \geq 2 + k$ and if any vertex in $B(T)$ has degree four or more, $l > 2 + k$ (see Theorem 3.7 in [4]). Necessarily, there exist at least two vertices in $B(T)$, say v_1 and v_2, such that each v_i is incident with exactly one edge that does not lie on a v_i-endpath. Then each v_i has at least two v_i-endpaths. Since no vertex in T is adjacent to two leaves and T has at most two endpaths of length two or more, each v_i has an endpath which is a pendant edge and another which has length two or more, and the degree of v_1 and v_2 is three. Let λ_i be the v_i-endpath of length two or more, y_i the pendant edge of λ_i, w_i the pendant edge incident with v_i, and f an edge ordering of T with $h(f) = \varepsilon(T)$. Then each y_i is either the initial or final edge in a maximal f-ascent in T. If both are initial or both are final edges of a maximal f-ascent in T, then, as before, there exists an edge colouring c of G using fewer than $|E(G)|$ labels such that $h(c) = h(f) = \varepsilon(T)$ and we are done. We assume then, without loss of generality, that y_1 is the initial edge of a maximal f-ascent of T and y_2 is the final edge of a maximal f-ascent of T.

We focus our attention on the vertex v_1. Let the endpath λ_1 be denoted by the edges $e_1, e_2, \ldots, e_t$ where $t \geq 2$ and $e_t = y_1$, and e' be the edge incident with v_1 not on a v_1-endpath. We may assume λ_1 is an f-ascent of T of which e_t is the initial edge; if not, then for some i, $2 \leq i \leq t-1$, $f(e_{i+1}) < f(e_i) > f(e_{i-1})$, and there exists an edge colouring c with $c(e_{i+1}) = c(e_{i-1})$ and $h(c) = h(f)$. We also may assume that $f(w_1) > f(e_1)$, otherwise $w_1 e_1$ is a maximal $(2, f)$-ascent and $\varepsilon(T) = 2$, a contradiction. If $f(e') < f(e_1)$, let j be the largest index such that $f(e_j) > f(e')$. If $j < t$, relabel e_{j+1} with $f(e')$, otherwise, relabel e_t with $f(e')$. In either case this new colouring has flatness $\varepsilon(T)$ using fewer than $|E(G)|$ labels. If $f(w_1) > f(e')$, then we define the edge colouring c of T by $c(w_1) = c(y_2) = \max_{e \in E(T)}\{f(e)\}$ and for all other edges e, $c(e) = f(e)$. Since $h(c) = h(f) = \varepsilon(T)$, $\chi_\varepsilon(T) < |E(T)|$.

We assume then that $f(w_1) < f(e')$. Let α be a shortest maximal f-ascent containing the path $w_1 e'$. Necessarily, the length of α is at least $\varepsilon(T)$. Define the edge colouring c as follows: $c(w_1) = f(e_1)$, $c(e_1) = f(w_1)$, for $2 \leq i \leq t$, $c(e_i) = f(e') + i - 2$, and for $e \notin \{e_1, e_2, \ldots, e_t, w_1, e'\}$, $c(e) = f(e)$. Note that the maximal f-ascent containing λ_1 and w_1 is now a maximal c-ascent with its direction reversed. Furthermore, the maximal f-ascent α is also a maximal c-ascent. Also, note that any shortest maximal c-ascent containing $e_1 e'$ has the same length as α, whose length is at least $\varepsilon(T)$. Thus $h(c) = h(f) = \varepsilon(T)$. Moreover, at least two edges receive the same label, hence $\chi_\varepsilon(T) < |E(T)|$. □

Next we discuss a bound of $\chi_\varepsilon(T)$ for trees T with $\varepsilon(T) = 3$. We begin with a slightly more general result.

Theorem 14. *Let T be a tree with $\varepsilon(T) \geq 3$. Then there exists a proper edge colouring c of T using at most $\chi_1(T) + 2$ colours such that $h(c) \geq 3$.*

Proof. Since $\varepsilon(T) \geq 3$ it follows that $|E(T)| \geq 3$, and by Theorem 1, no vertex of T is adjacent to two leaves. Note that for any tree T, $\chi_1(T) = \Delta(T)$. If $|E(T)| = 3$, then $T = P_4$, and since $\chi_1(P_4) = 2$ and $\varepsilon(P_4) = 3$, the result follows. Suppose then that $|E(T)| = 4$. The only tree T with four edges and $\varepsilon(T) \geq 3$ is $T = P_5$ and again, since $\chi_1(P_5) = 2$ and $\varepsilon(P_5) = 4$, the result follows.

Suppose the result is true for all trees T with $3 \leq |E(T)| < t$ for some $t \geq 5$ and consider a tree T' with $|E(T')| = t$ and $\varepsilon(T') \geq 3$.

Suppose T' is a path, say $T' = v_0v_1 \ldots v_t$. Let $T = T' - \{v_{t-1}, v_t\}$. By the induction hypothesis there exists a proper edge colouring c of T using at most 4 colours such that $h(c) \geq 3$. If $c(v_{t-3}v_{t-2}) \geq 3$, then let $c(v_tv_{t-1}) = 1$ and $c(v_{t-1}v_{t-2}) = 2$, otherwise let $c(v_tv_{t-1}) = 4$ and $c(v_{t-1}v_{t-2}) = 3$. In either case $v_tv_{t-1}v_{t-2}v_{t-3}$ is a $(3, c)$-ascent in T' which implies that we may extend the colouring c to T' so that it has flatness at least three using at most $\chi_1(T') + 2$ colours.

Suppose now that T' is not a path. Then there exist at least three endpaths in T'. Recall that $B(T')$ is the set of branch vertices of T'.

Case 1 There exists a v-endpath of length two for some $v \in B(T')$. Let vxy be a v-endpath of length two and $T = T' - \{x, y\}$. Since T' does not contain a vertex adjacent to two leaves, it follows that T also does not contain a vertex adjacent to two leaves. Hence $\varepsilon(T) \geq 3$ and by the induction hypothesis T has a proper edge colouring c using at most $\chi_1(T) + 2$ colours such that $h(c) \geq 3$. Let $\deg_T(v) = k$ and C_v be the set of colours assigned to edges incident with v. Note that $\chi_1(T) \geq k$, hence $\chi_1(T') \geq k + 1$, so there are at least $k + 3$ colours available to colour T'.

Suppose that either $\{1, 2\} \cap C_v = \varnothing$ or $\{k + 2, k + 3\} \cap C_v = \varnothing$. In the former case we extend the edge colouring c to T' by $c(vx) = 2$ and $c(xy) = 1$ and in the latter case by $c(vx) = k + 2$ and $c(xy) = k + 3$. In either case, in T', $h(c) \geq 3$, and since $\chi_1(T') \geq k + 1$, the result holds.

Suppose then that $\{1, 2\} \cap C_v \neq \varnothing$ and $\{k+2, k+3\} \cap C_v \neq \varnothing$. Then there exists $j \notin C_v$ such that $3 \leq j \leq k+1$. Suppose there exists $i \in C_v$ such that $i < j$ and for the edge uv assigned i, any edge e adjacent to uv such that $c(e) < c(uv)$ is incident with v. Then, since $h(c) \geq 3$, for any edge vw with $c(vw) > i$, there exists an edge e' incident with w but not with v such that $c(e') > c(vw)$. Thus, if we extend the edge colouring c to T' by $c(vx) = j$ and $c(xy) = j + 1$, then the resulting edge colouring of T' has flatness at least three. Therefore, we may assume that for any edge uv with $c(uv) < j$, there exists an edge uw such that

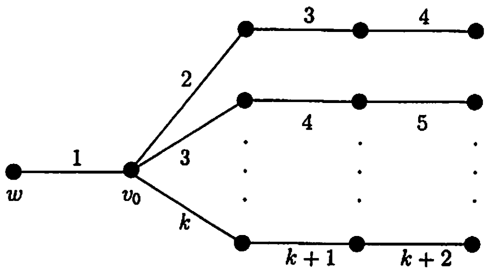

Figure 2: A $\chi_1(T') + 2$-edge colouring of $T' = S(1, 3, \ldots, 3)$ with flatness four.

$c(uw) < c(uv)$. Hence, if we extend the edge colouring c to T' by $c(vx) = j$ and $c(xy) = j - 1$, the resulting edge colouring of T' has flatness at least three.

Case 2 For all $v \in B(T')$, there does not exist a v-endpath of length two. Since $\varepsilon(T') \geq 3$, there exists an endpath of length three or more. Let v_0 be a branch vertex incident with at most one edge not on an endpath, $\deg_{T'}(v_0) = k$, and $\lambda = v_0 v_1 \cdots v_j$ be a longest v_0-endpath where necessarily $j \geq 3$. Let $T = T' - \{v_j, v_{j-1}\}$.

- Suppose $\varepsilon(T) \geq 3$. By the induction hypothesis there exists a proper edge colouring c of T in at most $\chi_1(T) + 2$ colours with flatness at least three. If $c(v_{j-3}v_{j-2}) \geq 3$, then let $c(v_j v_{j-1}) = 1$ and $c(v_{j-1}v_{j-2}) = 2$, otherwise let $c(v_j v_{j-1}) = 4$ and $c(v_{j-1}v_{j-2}) = 3$. In either case $v_j v_{j-1} v_{j-2} v_{j-3}$ is a $(3, c)$-ascent in T', which implies that we may extend the colouring c to T' so that it has flatness at least three.

- Suppose $\varepsilon(T) = 2$. Then in T', v_0 is incident with a pendant edge $e_0 = v_0 w$, and $j = 3$. If T' is a spider, then $T' \cong S(1, 3, \ldots, 3)$ and Figure 2 shows an edge colouring in $\chi_1(T') + 2$ colours with flatness four. Hence assume T' has at least two branch vertices and there are $k - 2 \geq 1$ v_0-endpaths of length three which we denote by $\lambda_1, \lambda_2, \ldots, \lambda_{k-2}$. Let e_i be the edge incident with v_0 which lies on endpath λ_i and e' the edge incident with v_0 that does not lie on an endpath. Let T_1 be the component of $T - \{e_1, e_2, \ldots, e_{k-2}\}$ which contains e_0, and note that $\deg_{T_1}(v_0) = 2$. Also, no vertex of T_1 is adjacent to two leaves, hence $\varepsilon(T_1) \geq 3$.

  By the induction hypothesis there exists a proper edge colouring c of T_1 in at most $\chi_1(T_1) + 2$ colours such that $h(c) \geq 3$, and without loss of generality we may assume that $c(e') > c(e_0)$. Then we may also assume without loss of generality that $c(e_0) = 1$.

  Let $c(e') = m$. We colour the edges of λ_i with $i + 1, i + 2, i + 3$, for

$1 \le i \le k-2$ and $i \neq m-1$, and the edges of λ_{m-1} (if it exists, in which case $m < k$) with k, $k+1$, $k+2$ to obtain a proper edge colouring c' of T'. Since $k \le \chi_1(T')$ and $\chi_1(T_1) \le \chi_1(T')$, it follows that c' uses at most $\chi_1(T')+2$ colours.

Clearly, the maximal ascents contained in the union of the v_0-endpaths, v_0w, $\lambda_1, \lambda_2, \ldots, \lambda_{k-2}$ all have length at least three (four in fact). If $c(e') < c(e_i)$, then e' followed by λ_i is a $(4, c')$-ascent. Suppose $c(e') > c(e_i)$. Then since $h(c) \ge 3$, there exists an edge e'' adjacent to e' such that $e_0e'e''$ is a $(3, c')$-ascent. Then $e_ie'e''$ is also a $(3, c')$-ascent. Hence $h(c') \ge 3$ and we are done. □

We combine the lower bound from Proposition 8 with the result from Theorem 14 in the following corollary.

Corollary 15. *For any tree T with $\varepsilon(T) = 3$, $\chi_1(T)+1 \le \chi_\varepsilon(T) \le \chi_1(T)+2$.*

Theorem 14 does not provide an upper bound for $\chi_\varepsilon(T)$ when $\varepsilon(T) \ge 4$ since we are only guaranteed an edge colouring with flatness at least three. This motivates a generalization of the parameter $\chi_\varepsilon(G)$.

We define the *k-ascent chromatic index* of a graph G, $\chi_{(k)}(G)$, as the minimum number of colours so that there exists a proper edge colouring with flatness k, where $2 \le k \le \varepsilon(G)$. Note that $\chi_{(2)}(G) = \chi_1(G)$.

We now restate Theorem 14 using the parameter $\chi_{(k)}(G)$.

Theorem 16. *Let T be a tree with $\varepsilon(T) \ge 3$. Then $\chi_{(3)}(T) \le \chi_1(T)+2$.*

The bound in Theorem 16 does not hold for all graphs in general. For example, consider the graph G shown in Figure 3. Since $c(e_1) = c(e_4) = 1$, $c(e_2) = c(e_6) = 2$, $c(e_3) = c(e_5) = 3$ is a proper 3-edge colouring of G, we conclude that $\chi_1(G) = 3$, and from Theorem 1 and Proposition 2, it follows that $\varepsilon(G) = 3$. Suppose $\chi_{(3)}(G) = \chi_\varepsilon(G) \le \chi_1(G)+2 = 5$. Then there exists a proper 5-edge colouring c of G with $h(c) = 3$. Necessarily, the edge coloured 1 is not adjacent to the edge coloured 5. Moreover, $\{c(e_2), c(e_3), c(e_4)\} \cap \{1, 5\} = \varnothing$, otherwise at least one of e_1e_2, e_1e_3, e_4e_5 or their reverse is a maximal $(2, c)$-ascent. Therefore, without loss of generality we may assume that $c(e_1) = 1$ and $c(e_5) = 5$. Since $|E(G)| = 6$, two edges are assigned the same colour, and since no other edge is assigned 1 or 5, $c(e_2) = c(e_6) = i$ where $i \in \{2, 3, 4\}$. If $i = 2$, then e_6e_5 is a maximal $(2, c)$-ascent, and if $i = 4$, then $c(e_4) \in \{2, 3\}$ and e_4e_6 is a maximal $(2, c)$-ascent. Suppose then that $i = 3$. If $c(e_4) = 2$, then e_4e_6 is a maximal $(2, c)$-ascent, and if $c(e_4) = 4$, then e_6e_4 is a maximal $(2, c)$-ascent. This completes all cases and we conclude that there does not exist a proper 5-edge colouring of G with flatness three. Hence, $\chi_{(3)}(G) = \chi_\varepsilon(G) = 6 > \chi_1(G)+2$.

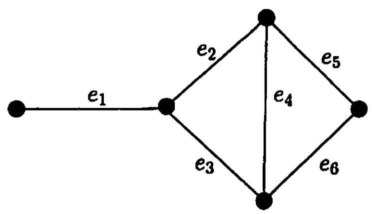

Figure 3: A graph G with $\chi_{(3)}(G) = \chi_\varepsilon(G) = 6 > \chi_1(G) + 2$.

4.5 Complete graphs

In this section we consider the problem of determining $\chi_\varepsilon(K_n)$. Trivially, we note that $\chi_\varepsilon(K_2) = 1$, and since $\chi_1(K_3) = |E(K_3)| = 3$, it follows that $\chi_\varepsilon(K_3) = 3$. From Proposition 3, $\varepsilon(K_n) = 3$ for all $n \geq 4$, hence, for $n \geq 4$ the problem involves determining the minimum number of colours required for there to exist a proper edge colouring of K_n with flatness three.

We first determine an upper bound for $\chi_\varepsilon(K_n)$.

Theorem 17. $\chi_\varepsilon(K_n) \leq 2n - 3$ *for all* $n \geq 2$.

Proof. As noted previously, $\chi_\varepsilon(K_2) = 1$ and $\chi_\varepsilon(K_3) = 3$, thus the result holds for $n = 2$ and $n = 3$. Consider K_n where $n \geq 4$. Let $V(K_n) = \{v_0, v_1, \dots, v_{n-1}\}$ and define the edge colouring c by $c(v_i v_j) = i + j$. Clearly c is a proper edge colouring of K_n, and furthermore, c uses $2n - 3$ colours. To complete the proof we need only show that every $(2, c)$-ascent is contained in a longer c-ascent.

Suppose $\lambda = v_a v_b v_c$ is a $(2, c)$-ascent of K_n. Note that $a \neq n - 1$ since $v_{n-1} v_b$ is assigned the largest value over all edges incident with v_b and hence $v_{n-1} v_b v_c$ is not a c-ascent. We now consider the following cases.

Case 1 $c = n - 1$. If $b \leq n - 3$ and $a \leq n - 3$, or $b \leq n - 4$ and $a = n - 2$, then there exists an edge $v_{n-1} v_k$ such that $k \neq a$ and $c(v_{n-1} v_k) > b + n - 1$, which implies that λ is not a maximal c-ascent. If $b = n - 3$ and $a = n - 2$, then there exists an edge $v_a v_k$ such that $c(v_a v_k) < c(v_a v_b)$, and λ is not a maximal c-ascent. If $b = n - 2$, then there exists an edge $v_a v_k$ such that $k < n - 2$, which implies $c(v_a v_k) < c(v_a v_b)$, and again, λ is not a maximal c-ascent.

Case 2 $b = n - 1$. By the definition of the edge colouring c, it follows that $v_a v_{n-1}$ is assigned the largest value over all edges incident with v_a. Thus, since $n \geq 4$, there exists an edge $v_a v_k$ such that $k \neq c$ and $c(v_a v_k) < c(v_a v_{n-1})$, which implies that λ is not a maximal c-ascent.

Case 3 $n - 1 \notin \{a, b, c\}$. By the definition of the edge colouring c, $c(v_b v_c) <$

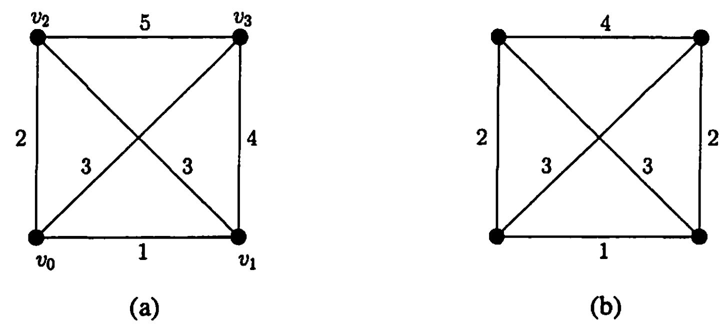

Figure 4: (a) A 5-edge colouring of K_4 with flatness 3. (b) A 4-edge colouring of K_4 with flatness 2.

$c(v_c v_{n-1})$, which implies that λ is not a maximal c-ascent.

We have now considered all possible cases for a $(2, c)$-ascent λ and shown that in each case λ is not a maximal c-ascent. □

Figures 4(a) and 5 depict the edge colouring c for K_4 and K_5 respectively as defined in the proof of Theorem 17. Next we determine $\chi_\varepsilon(K_4)$ and $\chi_\varepsilon(K_5)$.

Proposition 18. $\chi_\varepsilon(K_4) = 5$.

Proof. By Theorem 17, $\chi_\varepsilon(K_4) \leq 5$.

To complete the proof we now show that $\chi_\varepsilon(K_4) \geq 5$. Suppose, to the contrary, that there exists a proper 4-edge colouring of K_4 with flatness three. Necessarily, an edge assigned colour 1 is not incident with an edge coloured 4. This implies the colours 1 and 4 are used once each, and the colours 2 and 3 are each used twice. Any proper edge colouring with this configuration is equivalent to the one shown in Figure 4(b) and both of the ascents 2 3 are maximal ascents. Thus $\chi_\varepsilon(K_4) \geq 5$. □

Proposition 19. $\chi_\varepsilon(K_5) = 7$.

Proof. By Theorem 17, $\chi_\varepsilon(K_5) \leq 7$.

We now show that $\chi_\varepsilon(K_5) \geq 7$. Suppose, to the contrary, that there exists a proper 6-edge colouring c of K_5 with $h(c) = 3$. Let $V(K_5) = \{v_0, v_1, v_2, v_3, v_4\}$. Without loss of generality let $c(v_0v_1) = 1$. If $c(v_iv_j) = 6$, then $i, j \geq 2$, otherwise $h(c) = 2$. Without loss of generality let $c(v_2v_3) = 6$. To avoid a maximal $(2, c)$-ascent, $c(e) \in \{2, 3, 4, 5\}$ for each $e \in E(K_5) - \{v_0v_1, v_2v_3\}$. Then, since

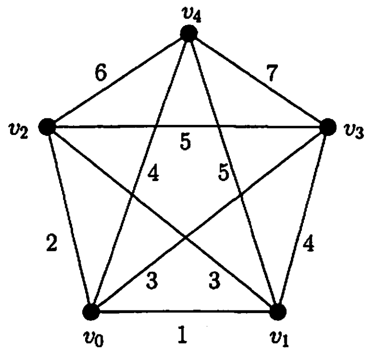

Figure 5: A 7-edge colouring c of K_5 with $h(c) = \varepsilon(K_5) = 3$.

the maximum size of an independent edge set of K_5 is two and there are 8 edges in $E(K_5) - \{v_0v_1, v_2v_3\}$, $|c^{-1}(i)| = 2$ for each $i \in \{2,3,4,5\}$. Additionally, for each colour $i \in \{2,3,4,5\}$, there exists $x \in \{0,1,2,3\}$ such that $c(v_xv_4) = i$. Note that $c(v_0v_4) \neq 5$ or else $v_1v_0v_4$ is a maximal $(2,c)$-ascent, a contradiction. By similar arguments, $c(v_1v_4) \neq 5$, $c(v_2v_4) \neq 2$, and $c(v_3v_4) \neq 2$. Thus, either $c(v_2v_4) = 5$ or $c(v_3v_4) = 5$, and we assume without loss of generality that $c(v_2v_4) = 5$. Since another edge must also be assigned the colour 5, either $c(v_0v_3) = 5$ or $c(v_1v_3) = 5$ and without loss of generality we may assume that $c(v_0v_3) = 5$. If $c(v_1v_3) = 2$, then $v_1v_3v_0$ is a maximal $(2,c)$-ascent. Hence $3 \leq c(v_1v_3) \leq 4$. Since $c(v_2v_3) = 6$ and $c(v_0v_3) = 5$, if $c(v_3v_4) = 3$, then $c(v_1v_3) = 4$ and $v_3v_4v_2$ is a maximal $(2,c)$-ascent. This implies $c(v_3v_4) = 4$ and $c(v_1v_3) = 3$. Now $c(v_0v_4), c(v_1v_4) \in \{2,3\}$ and since $c(v_1v_3) = 3$, $c(v_1v_4) = 2$ and $c(v_0v_4) = 3$. Moreover, since $c(v_1v_2), c(v_0v_2) \in \{2,4\}$ and $c(v_1v_4) = 2$, $c(v_1v_2) = 4$ and $c(v_0v_2) = 2$. Then $v_0v_2v_1$ is a maximal $(2,c)$-ascent and the result follows. □

For $2 \leq n \leq 5$, the bound provided in Theorem 17 is best possible. However, this is not always the case as we illustrate next with K_6.

Proposition 20. $7 \leq \chi_\varepsilon(K_6) \leq 8$.

Proof. Suppose $\chi_\varepsilon(K_6) \leq 6$. Then there exists a proper 6-edge colouring c of K_6 with $h(c) = 3$. Since the maximum size of an independent edge set of K_6 is three, any colour can be used at most three times. Moreover, an edge coloured 1 is not adjacent to an edge coloured 6, otherwise $h(c) = 2$. Suppose the colours 1 and 6 are each used exactly once. Since $|E(K_6)| = 15$, there are 13 edges to be coloured with colours from the set $\{2,3,4,5\}$, and by the pigeonhole principle

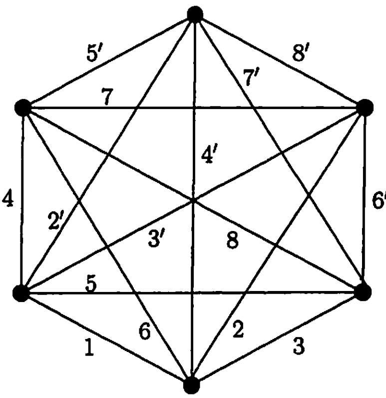

Figure 6: An 8-edge colouring c of K_6 with $h(c) = \varepsilon(K_6) = 3$.

one of these colours is used at least four times, a contradiction. Hence, three edges are coloured 1 or 6. Without loss of generality we assume two edges are coloured 1 and one is coloured 6; necessarily these edges form an independent edge set. Since there are 12 edges of K_6 to be coloured with colours from the set $\{2, 3, 4, 5\}$, each colour is used exactly three times. Consider the three edges coloured 5. Since only one edge is coloured 6, there exists at least one edge, say e, coloured 5 which is not adjacent to the edge coloured 6. Furthermore, the edge e is adjacent to an edge e' which is coloured 1. Thus $e'e$ is a maximal $(2, c)$-ascent, which is a contradiction. Therefore $\chi_\varepsilon(K_6) \geq 7$.

To show that $\chi_\varepsilon(K_6) \leq 8$, we show that the 8-edge colouring of K_6 in Figure 6 has flatness three. Note that in the figure the labels k and k' are assumed to be the same label for each $k \in \{2, 3, \ldots, 8\}$ and the notation is used to differentiate between edges which are assigned the same colour. We lexicographically consider all $(2, c)$-ascents which are the first two edges of a maximal c-ascent and include in brackets the colour of an edge which extends the ascent to a $(3, c)$-ascent. The $(2, c)$-ascents we need to consider are 1 2 (6′), 1 2′ (8′), 1 3 (6′), 1 3′ (6′), 1 4 (5′), 1 4′ (8′), 1 5 (6′), 1 6 (7), 2 3 (6′), 2 3′ (4), 2 4′ (5′), 2 6 (8), 2′ 3′ (6′), 2′ 4 (7), 2′ 4′ (6), 2′ 5 (6′), 3 4′ (8′), 3 6 (7), 4 5 (6′), 4′ 6 (7). Hence $h(c) = 3$. □

5 Open Problems

In Section 4.2 we proved that $\chi_\varepsilon(G) \geq \chi_1(G) + \varepsilon(G) - 2$ and we also showed that the difference $\chi_\varepsilon(G) - [\chi_1(G) + \varepsilon(G) - 2]$ can be arbitrary. Prove or disprove:

Problem 1. *The ratio* $\chi_\varepsilon(G)/\left[\chi_1(G)+\varepsilon(G)-2\right]$ *is bounded.*

Problem 2. *Characterize the class of graphs for which* $\chi_\varepsilon(G) = |E(G)|$.

Problem 3. *Characterize the class of graphs for which* $\chi_\varepsilon(G) = \chi_1(G) + 1$.

Problem 4. *Determine* $\chi_\varepsilon(K_n)$ *for* $n \geq 6$.

Problem 5. *Bound or determine explicitly* χ_ε *for other families of graphs e.g.* $K_{m,n}$.

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