# The Game of timber!

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**Abstract**

We analyse TIMBER, a game played on a graphs. We find the $\mathcal{P}$ positions for both normal and misère play on paths and show how to win the game. In passing, we also show a correspondence with Dyck paths, the Catalan and Fine numbers. We present an algorithm for winning the Normal Play game on trees.

## 1 Introduction

The game of TIMBER arises out of the coming together (loosely) of the work of C. Mynhardt's work on "altitude" and "depression" on graphs, see [8, 9] for examples, and an offshoot of the game TOPPLING DOMINOES. The game TIMBER is played on a directed graph, with a domino on each edge. The

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edges will be denoted by $(x, y)$ with the arc starting at $x$ and ending at $y$. There are two players, herein called Alice and Bob, moving alternately. On a move, the player chooses a domino on some edge, say, $(x, y)$ and topples it in the direction of $y$. (This is the only time the direction of the edge is important.) The domino then topples the dominoes on the edges incident with $y$, independent of whether the edge is directed into or away from $y$, and the process of toppling the dominoes continues until no more dominoes topple. The toppled dominoes and corresponding edges are removed from the graph. More formally, let $G$ be a graph, a TIMBER position based on $G$ will be denoted by $G_t$ which will be $G$ with its edges directed. Note that for a given $G$ there will be many possible $G_t$. Given a graph $G$, a corresponding $G_t$ and an arc $(x, y)$, let $Y$ be the set of all vertices that can be reached by a path in $G$ starting at $y$ and not having $x$ as the second vertex. Choosing to topple the domino on the arc $(x, y)$ removes $Y$ and all edges incident with $Y$ from $G$ and $G_t$. For example, see Figure 1. We invite the reader to find the unique winning move in this position.

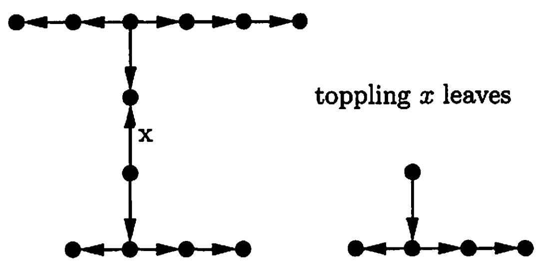


Figure 1: Example and Problem

We will attempt to make the paper self-contained but the reader can consult [1, 2] for more details on game theory. Under *Normal* play, the first player who cannot move is the loser; under *misère* play this player would be the winner. TIMBER is an impartial game, since both players have the same moves available, i.e., Alice and Bob can start with any domino. In impartial games there are just two *outcome classes:* $\mathcal{N}$-positions in which the Next player to move can force a win; and $\mathcal{P}$-positions in which the next player cannot force a win but the Previous player can. In Normal play, the end position is a $\mathcal{P}$-position (since the next player has no move at all and therefore has no way to force a win) and it would be an $\mathcal{N}$-position in misère play. Note that the outcome classes can be determined recursively starting with the end of the game. From a position $G_t$, if a player can move to a $\mathcal{P}$-position then $G_t$ is an $\mathcal{N}$-position, otherwise $G_t$ is a $\mathcal{P}$-position. We write, $o^+(G_t)$ for the outcome class of the position $G_t$ under Normal play and $o^-(G_t)$ under misère play.

Since TIMBER could be played on a set of disconnected components we should be using the Sprague-Grundy theory for impartial games, however, we found the game was too complicated for us to solve at this level of detail, even on a path. We do completely characterize the outcome class for all $G_t$, where $G$ is any connected graph, under Normal play and find the misère outcome classes where $G$ is a path or a 2-connected graph.

Our first result shows that, for most graphs, the outcome class is very easy to calculate.

**Theorem 1.1** *If $G$ is a connected graph with a 2-connected subgraph then for any $G_t$, $o^+(G_t) = \mathcal{N}$. If $G$ is a 2-connected graph then for any $G_t$, $o^-(G_t) = \mathcal{P}$.*

**Proof.** Let $G$ be a connected graph and $G_t$ be some TIMBER position of $G$. Let $xy$ be an edge of $G$ in the 2-connected subgraph and $(x, y)$ the corresponding arc in $G_t$. Toppling the domino on $(x, y)$ causes all the dominoes to be toppled. In Normal play, this is a winning move. In misère play it is a losing move and if $G$ is 2-connected the next player only has losing moves. □

Thus, in Normal Play, only graphs which are trees need be considered. The paper is divided into two main sections. In Section 2, TIMBER played on a path is analyzed. Although the algorithm for trees could be used to analyze the game on paths, the path analysis uncovers relationships to Dyck paths, Catalan and Fine numbers and is worthy of independent investigation. The Normal play version on paths was the original game analyzed by Lamoureux, Mellon, and Miller as a first year class project at Dalhousie University, supervised by the first author. The game was called TOPPLING PEAKS. See Section 2 and Figure 2 to see how the game got its name and why a possible relationship to 'altitude' and 'depressions' of graphs could be inferred. Nowakowski and Renault then found the misère analysis and generalized the game to graphs in general. In Section 3, we give a reduction algorithm to determine the outcome class of a tree. Surprisingly, *nimbers* are required even though we are only investigating outcome classes. In the last section we raise some questions.

## 2 TIMBER on paths.

TIMBER on paths, also called TOPPLING PEAKS, arose as an impartial variant of the partizan game TOPPLING DOMINOES [5] which is played with the same set-up: a line of dominoes, each domino marked $L$ or $R$ where in TOPPLING PEAKS an $L$ can only be toppled to the left and an $R$ to the right. See Figure 2.

After some preliminary analysis of TIMBER, using CGSuite [3], we quickly determined the number of $\mathcal{P}$-positions of length $n = 1, 2, \ldots, 10$ is 0, 1, 0, 2, 0, 5, 0, 14, 0, 42, respectively. Ignoring the 0s, the On-line Encyclopedia of Integer Sequences [11] suggests that the resulting sequence is the Catalan numbers, $\frac{1}{n+1}\binom{2n}{n}$. The first two references are [4, 6]. The second—nesting and roosting habits of the laddered parenthesis—suggests a pairing approach that is very useful in actual play. The first reference—A bijection of Dyck paths and its consequences—gives rise to the name of the game and hints at the solutions.

A position can be represented visually on a 2-dimensional graph: start at $(0,0)$ and let an $L$ be a line joining the lattice points $(x, y)$ and $(x+1, y+1)$ and $R$ be the line joining $(x, y)$ and $(x + 1, y - 1)$. We'll sometime refer to this as the *peak* representation. For example, all the representations of $LLRL$ are given in Figure 2.

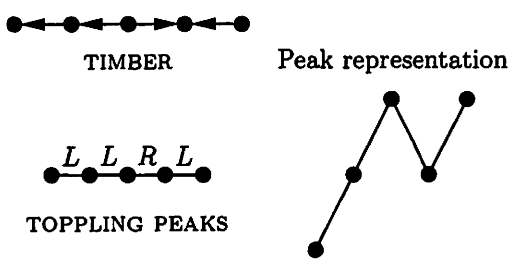


Figure 2: Three representations of one position.

A *Dyck path of length* $2n$ is one of these paths that also ends at $(2n, 0)$ and which never goes below the $x$-axis. See Theorems 2.3 and 2.4.

For normal play, Theorem 2.3 shows that $\mathcal{P}$-positions are exactly those positions with Dyck path representations. For misère play, there are two types of $\mathcal{P}$-positions. Theorem 2.4 shows that one kind, a subset of the Normal play $\mathcal{P}$-positions, correspond directly to the Dyck paths with no 'peaks' of height 1, i.e., a Dyck path with no sub-path $(j,0),(j+1,1),(j+2,0)$ for any $j$. The misère $\mathcal{P}$-positions of length $2n+1$ are formed from these $\mathcal{P}$-positions of length $2n$ by the addition of an $R$ at the start or an $L$ at the end. The extra $\mathcal{P}$-positions of length $2n+2$ are obtained by adding both the $L$ and the $R$ dominoes. In general, the number of misère $\mathcal{P}$-positions of length $2n$ is the number of Dyck paths of length $2n$ with no peaks of height 2 before the path returns to 0, but this is a counting argument since the representation of a length $2n$ $\mathcal{P}$-position does not have to be a Dyck path and can have a peak of height 2.

## 2.1 The Analysis.

We will refer to the two players as 'Alice' and 'Bob' usually with Alice to go first in a given position.

The main tool for characterizing outcome classes are the following theorems.

**Theorem 2.1** *[Partition Theorem][1][page 41] Suppose the positions of a finite, impartial, normal-play game can be partitioned into mutually exclusive sets A and B with the properties:*

*every option of a position in A is in B; and*

*every position in B has at least one option in A.*

*Then A is the set of $\mathcal{P}$-positions and B is the set of $\mathcal{N}$-positions.*

**Theorem 2.2** *[Misère Partition Theorem] Suppose the positions of a finite, impartial, misère-play game can be partitioned into mutually exclusive sets A and B with the properties:*

*every option of a position in A is in B; and*

*every position in B that has an option has at least one option in A.*

*every position with no option is in B.*

*Then A is the set of $\mathcal{P}$-positions and B is the set of $\mathcal{N}$-positions.*

We now translate the intuitive 'peak' approach into one where it is easier to give proofs and more useful when we consider trees.

Given the alphabet $\{L,R\}$, for a word $w$, let $|w|_L$ be the number of $L$s in $w$, $|w|_R$ the number of $R$s in $w$ and $w_{[i,j]}$ the subword $w_iw_{i+1}\cdots w_j$. Let $WP$ be the set of words $w$ such that for any $i$, $|w_{[0,i]}|_L \geqslant |w_{[0,i]}|_R$ and $|w|_L = |w|_R$.

**Theorem 2.3** *In Normal play, the $\mathcal{P}$-positions of* TOPPLING PEAKS *are exactly those which correspond to word of $WP$.*

**Proof.** Let $w \in WP$ be a position and $n$ its length. First let's assume that Alice plays on an $R$ domino, say $w_i$. Then Alice has moved to the position $w_{[0,i-1]}$. As $w \in WP$, $|w_{[0,i]}|_L \geqslant |w_{[0,i]}|_R$. Hence $|w_{[0,i-1]}|_L = |w_{[0,i]}|_L \geqslant |w_{[0,i]}|_R = |w_{[0,i-1]}|_R + 1$, and $|w_{[0,i-1]}|_L \neq |w_{[0,i-1]}|_R$. So $w_{[0,i-1]} \notin WP$.

If Alice plays on an $L$ domino, instead, say $w_i$. Then Alice has moved to the position $w_{[i+1,n-1]}$. As $w \in WP$, $|w_{[0,i-1]}|_L \geqslant |w_{[0,i-1]}|_R$. Hence $|w_{[i+1,n-1]}|_L = \frac{n}{2} - |w_{[0,i]}|_L = \frac{n}{2} - |w_{[0,i-1]}|_L - 1 \leqslant \frac{n}{2} - |w_{[0,i-1]}|_R - 1 = |w_{[i+1,n-1]}|_R + 1$, and $|w_{[i+1,n-1]}|_L \neq |w_{[i+1,n-1]}|_R$. So $w_{[0,i-1]} \notin WP$.

Let $w \notin WP$ be a position. Assume $|w|_L = |w|_R$. Let $i = min\{0 \leqslant k \leqslant n-1 \mid |w_{[0,k]}|_L < |w_{[0,k]}|_R\}$. $w_k$ is an $R$ domino, and $|w_{[0,k]}|_L = |w_{[0,k]}|_R - 1$. By toppling $w_k$, Alice moves to $w_{[0,k-1]} \in WP$. Assume now $|w|_L \neq |w|_R$. Without loss of generality, we can assume $|w|_L < |w|_R$. Let $i = min\{0 \leqslant k \leqslant n-1 \mid |w_{[0,k]}|_L < |w_{[0,k]}|_R\}$. $w_k$ is an $R$ domino, and $|w_{[0,k]}|_L = |w_{[0,k]}|_R - 1$. By toppling $w_k$, Alice moves to $w_{[0,k-1]} \in WP$. □

Note that $w \in WP$ iff the corresponding position produces a Dyck path.

We now consider Misère play. Let $SWP$ be the set of words $w$ such that $w \in WP$ and $\forall w_1, w_2 \in WP$, $w \neq w_1 LR w_2$. We define $X = (SWP \backslash \{\emptyset\}) \cup \{Rw \mid w \in SWP\} \cup \{wL \mid w \in SWP\} \cup \{RwL \mid w \in SWP\}$. We note $\widetilde{w}$ the word obtained from $w$ after removing the first character if it is an $R$ and the last one if it is an $L$.

**Theorem 2.4** *In misère play, the $\mathcal{P}$-positions of* TOPPLING PEAKS *are exactly those which correspond to word of $X$.*

**Proof.** Let $w \in X$ be a position. Assume $w \in (SWP \backslash \{\emptyset\})$. From the normal play analysis, we know that Alice cannot move to a position in $SWP \subset WP$. Assume Alice can move to a position $Rw_0$ with $w_0 \in SWP$. Then it follows that $w = w_1 LR w_0$ for some $w_1$. As $w, w_0 \in WP$ then $w_1 \in WP$, which is not possible since $w \in SWP$. Similarly, we can prove Alice has no move to a position of the form $w_0 L$ or $Rw_0 L$ with $w_0 \in SWP$. Similarly, we can prove Alice has no move to a position in $X$ from a position in $X$.

Let $w \notin X \cup \{\emptyset\}$. Assume $w \in WP$. Then there exists $w_1, w_2 \in WP$ such that $w = w_1 LR w_2$, and we can choose them such that $w_2 \in SWP$. From $w$, Alice can move to $Rw_2 \in X$. Similarly, we can prove Alice has a move to a position in $X$ from a position in $(\{Rw \mid w \in WP\} \cup \{wL \mid w \in WP\} \cup \{RwL \mid w \in WP\}) \backslash X$.

Now assume $w_{[0,1]} = RR$. Alice can move to $R \in X$.

Now assume $w$ is none of the above forms. $\widetilde{w}$ starts with an $L$ and ends with a $R$, and is not in $WP$, so Alice has a move from $\widetilde{w}$ to a position $w_0 \in WP\backslash\{\emptyset\}$. Without loss of generality, we can assume it is by toppling a domino leftward. If $w_0 \in SWP$, the same move from $w$ leaves the position $w_0 \in X$ or $w_0L \in X$. Otherwise, there exists $w_1, w_2 \in WP$ such that $w_0 = w_1LRw_2$ and we can choose $w_2 \in SWP$. Alice can then move from $w$ to $Rw_2 \in X$ or $Rw_2L \in X$. □

## 2.2 Counting positions

Surprisingly, there are few games for which the number of $\mathcal{P}$-positions are known. In fact, the authors only know of [7] where the game was developed to relate the number of $\mathcal{P}$-positions to Bernoulli numbers of the second kind. Even for the basic game of NIM with $n$ pieces, it is not presently known how many $\mathcal{P}$-positions there are (see [11] sequence A048833).

Let $WP(n)$, $SWP(n)$ and $X(n)$ be the number of $\mathcal{P}$-positions of length $n$ respectively: (1) in normal play; (2) in misère play where the position is a Dyck path; and (3) in misère play in general.

A peak at height $k$ on a Dyck path is a point $(j, k)$ of the path that is preceded by the point $(j-1, k-1)$ and followed by the point $(j+1, k-1)$. The *Fine* number, $F_n$, is the number of Dyck paths of length $2n$ without peaks at height 1.

**Theorem 2.5** *Let $n$ be a nonnegative integer, then*

1. *$WP(n)$ is the number of Dyck paths of length $n$, specifically,*

$$WP(2n) = \frac{(2n)!}{n!(n+1)!}; \quad WP(2n+1) = 0.$$

2. *$SWP(n)$ is the number of Dyck paths without a peak of height 1; specifically*

$$SWP(2n) = \frac{1}{2}\sum_{i=0}^{n-2}(-1)^i c_{n-i}(\frac{1}{2})^i; \quad SWP(2n+1) = 0.$$

*3. $X(2n)$ is the number of Dyck paths of length $2n$ without peaks at height 2 before the first return to height 0; specifically $X(0) = 0$ and*

$$\begin{aligned} X(2n) &= F_{n-1} + F_n \text{ for } n > 0; \\ X(2n+1) &= 2SWP(2n) = \sum_{i=0}^{n-2} (-1)^i c_{n-i} (\frac{1}{2})^i. \end{aligned}$$

**Proof.** (1): From the definition, it is clear that $WP$ is the set of TOPPLING PEAKS positions that has a representation that is a Dyck path. Dyck paths only have even length. It is well known that the number of Dyck path of length $2n$ is the $n^{th}$ Catalan number $c_n = \frac{(2n)!}{n!(n+1)!}$, for example see [4] p.194. Thus

$$WP(2n) = \frac{(2n)!}{n!(n+1)!}; \quad WP(2n+1) = 0.$$

(2): Again, from the definition, it is also clear $SWP$ is the set of TOPPLING PEAKS position that has a representation that is a Dyck path without peaks at height 1. That is $SWP(2n) = F_n$. From [10], Theorem 2, we have the recursive formulas $F_n = \sum_{i=1}^{n-1} c_i * F_{n-i-1}$ and

$$F_n = \frac{1}{2} \sum_{i=0}^{n-2} c_{n-i} (\frac{-1}{2})^i.$$

(3): The number of TOPPLING PEAKS $\mathcal{P}$-position of length $2n$ in the misère version is $F_n + F_{n-1}$, where the second term counts the number of positions in $SWP(2n-2)$ which have had an $R$ appended at the beginning and an $L$ at the end. The number of TOPPLING PEAKS $\mathcal{P}$-position of length $2n+1$ in the misère version is $2 * F_n$.

Let $E_n$ be the number of Dyck paths of length $2n$ without peaks at height 2 before the first return to height 0. These can be counted by starting the sequence with an $L$ then a member of $SWP(2i)$ (a sequence without a peak of height 1) then an $R$ then any member of $WP(2n-2i-2)$ (that is a Dyck path of length $2n-2i-2$). This gives the first line in:

$$\begin{aligned} E_n &= \sum_{i=0}^{n-1} F_i * c_{n-i-1} \\ &= F_{n-1} + \sum_{i=1}^{n-1} c_i * F_{n-i-1} \\ &= F_{n-1} + F_n \\ &= X(2n) \end{aligned}$$

## 3 TIMBER on trees.

When TIMBER is played on a tree, a move is to choose an arc $(x, y)$ and remove from the tree the connected component containing $y$ but not $x$. There may be an immediate move that finishes the game. If there isn't we present a reduction algorithm each part of which preserves the outcome class of the tree. One step removes 'peaks'. The second merges two 'peak'-free paths incident with a common vertex.

The one move to finish the game is characterized first.

**Lemma 3.1** *Let $T$ be a tree, $v$ a leaf of $T$ and $x$ the vertex adjacent to $v$. Let $T_t$ be a* TIMBER *position which contains the arc $(v, x)$, then $o^+(T_t) = \mathcal{N}$, that is $T_t$ is a next-player win position.*

**Proof.** Alice wins by toppling the domino on the arc $(v, x)$. □

The next lemma shows that the two edges forming a 'peak' can be eliminated.

**Lemma 3.2** *Let $T_t^1, T_t^2$ be two* TIMBER *positions. Choose $y \in V(T_t^1)$, $z \in V(T_t^2)$ and let $x$ be a vertex disjoint from $T^1$ and $T^2$. Let $T_t$ be the position with vertex set $V(T) = V(T^1) \cup \{x\} \cup V(T^2)$ and arcs $E(T_t) = E(T_t^1) \cup \{(x, y), (x, z)\} \cup E(T_t^1)$. Let $T_t'$ be the position with vertex set $V(T') = V(T^1) \cup V(T^2)$ where $y$ and $z$ are identified, and the arcs $E(T') = E(T_t^1) \cup E(T_t^2)$. Then $o^+(T_t) = o^+(T_t')$.*

**Proof.** We show it by induction on the number of vertices of $T'$. If $V(T') = \{y\}$, then there is no move in $T_t'$ and $T_t$ is the TOPPLING PEAKS position $LR$. Hence $o^+(T) = \mathcal{P} = o^+(T')$. Assume now $|V(T')| > 1$. Assume Alice has a winning move in $T_t$. It cannot be by choosing $(x, y)$ or $(x, z)$ because Bob would choose the other and win. If the chosen arc removes $x$ from the game, choosing the same arc in $T'$ leaves the same position. Otherwise, choosing the same arc in $T_t'$ leaves a position which has the same outcome by induction. Hence Alice has a winning move in $T_t'$. The proof that Alice has a winning move in $T_t$ if she has one in $T_t'$ is similar. □

If the tree has been reduced to two paths directed away from a common vertex $x$, then no move in either path affects the other. In either path any number from 1 to all the edges can be removed. This is the game of

NIM played with 2 heaps. Suppose the size of the paths/heaps are $m$ and $n$. The Sprague-Grundy theory of Impartial games (see [1, 2] or numerous other books and papers) shows that the two heaps are equivalent to playing with one heap of size $m \oplus n$ where $\oplus$ is the XOR of integers. (Or write the numbers in binary and add without carrying. For example, $14 \oplus 5 = 1110_2 \oplus 101_2 = 1011_2 = 11$.) The next result shows that even if there is more of the tree affixed at $x$, we can still make the replacement. Every impartial game is equivalent to a playing NIM with single heap. The proof of the next result actually shows that when this merging of paths is done, the size of the equivalent NIM heap doesn't change.

**Lemma 3.3** *Let $T_0$ be a tree, $w \in V(T_0)$ a vertex, and $n, m \in \mathbb{N}$ two integers. Let $T_t$ be the position with vertex set $V(T_t) = V(T) \cup \{y_i\}_{1 \leqslant i \leqslant n} \cup \{z_i\}_{1 \leqslant i \leqslant m}$ and arcs*

$$\begin{aligned} E(T_t) &= E(T) \cup \{(y_i, y_{i+1})\}_{1 \leqslant i \leqslant n-1} \\ &\quad \cup \{(z_i, z_{i+1})\}_{1 \leqslant i \leqslant m-1} \cup \{(w, y_1), (w, z_1)\}. \end{aligned}$$

*Let $T_t'$ be the position with vertex set*

$$V(T') = V(T) \cup \{x_i\}_{1 \leqslant i \leqslant n \oplus m}$$

*and the arcs*

$$E(T') = E(T) \cup \{(x_i, x_{i+1})\}_{1 \leqslant i \leqslant (n \oplus m)-1} \cup \{(w, x_1)\}.$$

*Then $o^+(T_t) = o^+(T_t')$.*

**Proof.** We prove it by induction on $|V(T_0)| + n + m$ and show that $o^+(T_t + T_t') = \mathcal{P}$ which shows that $o^+(T_t) = o^+(T_t')$. If $n + m = 0$, $T = T_0 = T'$.

Assume now $|V(T_0)| + n + m > 0$. Any arc of $T_0$ is in both $T_t$ and $T_t'$, thus if Alice chooses such an edge in one of $T_t$ or $T_t'$ then Bob can choose the corresponding arc in $T_t'$ or $T_t$, which leaves a $\mathcal{P}$ position (either by induction or because the two remaining positions are the same). Assume Alice chooses the arc $(y_i, y_{i+1})$ (or $(w, y_1) = (y_0, y_1)$). If $(i \oplus m) < (n \oplus m)$, Bob can choose the arc $(x_{i \oplus m}, x_{(i \oplus m)+1})$ (or $(w, x_1)$ if $i \oplus m = 0$) which leaves a $\mathcal{P}$ position by induction. Otherwise, there exists $j < m$ such that $(i \oplus j = n \oplus m)$, and Bob can choose the arc $(z_j, z_{j+1})$ which leaves a $\mathcal{P}$ position by induction. Similarly, we can prove that Bob has a winning answer to any move of the type $(x_i, x_{i+1})$ or $(z_i, z_{i+1})$. □

A position which is neither of the form $T$ for Lemma 3.2 or Lemma 3.3 is called *minimal*. A *leaf-path* is a path from some vertex $x$ to a leaf $y$, $x \neq y$, consisting only of vertices, other than $y$ and possibly $x$, of degree 2.

**Lemma 3.4** *$\emptyset$ is the only minimal position which is $\mathcal{P}$.*

**Proof.** Let $T_t$ be a minimal position with at least one arc. If it has exactly one arc, it is obviously in $\mathcal{N}$, so we can assume $T_t$ has at least two arcs. Then there exist a vertex $w$ at which there are two leaf paths $\{x_i\}_{1\leqslant i\leqslant n}$ and $\{y_i\}_{1\leqslant i\leqslant m}$. If $(x_n, x_{n-1})$ or $(y_m, y_{m-1})$ is an arc, Alice can choose it and win. Now assume both $(x_{n-1}, x_n)$ and $(y_{m-1}, y_m)$ are arcs. As $T$ is minimal, it cannot be of the form from Lemma 3.2, so all $(x_i, x_{i+1})$, $(y_i, y_{i+1})$, $(w, x_1)$ and $(w, y_1)$ are arcs. But then it is of the form of Lemma 3.3, which is a contradiction. □

The next algorithm shows how to apply the previous results to find the winning move, if there is one.

**Algorithm 3.5** *Let $T^0$ be a tree and $T_t^0$ a* TIMBER *position.*

1. *Check each leaf $x \in V(T_0)$; if $(x, y) \in E(T^0)$ then toppling the domino on $(x, y)$ wins, by Lemma 3.1, and $o^+(T_t^0) = \mathcal{N}$ and the algorithm stops.*

   *If no such edge exists then let $n = 0$.*

2. *Flatten: Choose a vertex $x \in V(T^n)$ at which there is a leaf path $P$ to a leaf $y$ and with at least one edge directed toward $x$. If there is $u, v, w \in V(P)$ such that $(v, u), (v, w) \in E(T_t^n)$ then $T^{n+1}$ and $T_t^{n+1}$ are the tree and position obtained by applying Lemma 3.2 by removing $v$ and identifying $u$ and $w$. Let $n := n + 1$.*

   *If there is a new edge, say $(u, v)$ at the 'leaf' end of the path and it is directed away from the leaf vertex, then by Lemma 3.1, and $o^+(T_t^n) = o^(T_t^0) = \mathcal{N}$.*

   *If no 'inward' arc is produced, eventually all the leaf-paths from $x$ only have edges which are directed away from $x$.*

3. *Merge: If all vertices which have leaf-paths have all edges directed toward the respective leaves then there is a vertex $x$ which has two leaf-paths say, $P_1, P_2$. Form $T_t^{n+1}$ by applying Lemma 3.3. Let $n := n+1$ and repeat step 2.*

The merging of paths in Step 3 makes it awkward to describe how to find the winning move but with a little practice it becomes an easy matter for a player to discover it. The algorithm runs in $O(n^2)$ time. Each edge is visited most once before at least one edge is deleted or a merging (Step 3) takes place. An edge on a merged path needs only be looked once, that in Step 2 when it will be immediately deleted.

As an example, we solve the position given in the Introduction. That the position is an $\mathcal{N}$-position is shown by toppling the edge labelled '!' in the reduced position. This corresponds to toppling the edge labelled $y$. (After toppling $y$ the merge of the two paths results in the empty path.) Note that the Flatten step can be applied at any appropriate degree 2 vertex at anytime.

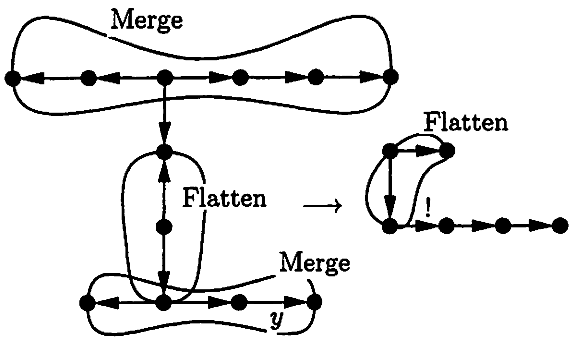


Figure 3: Example and Problem

# 4 Further Work

Lamoureaux, Mellon and Miller found the values of some families of positions that fit between the lines $y = 0$, $y = 1$, and $y = 0$, $y = 2$. The latter families required an induction with 9 cases.

**Problem 4.1** *Is there an efficient algorithm to find the value of a* TIMBER *position on a path?*

In the reduction (merge) from Lemma 3.3, both trees actually have the same value but this is not true in the reduction (flatten) of Lemma 3.2.

**Problem 4.2** *Is there any other reductions on trees (or paths) that preserve values?*

Given $G$ and $G_t$, suppose we regard the dominoes as physical. If a domino on $(x, y)$ is toppled and there is another domino on $(u, v)$ where the distance (in $G$) from $y$ to $u$ is the same as distance from $y$ to $v$ then this domino would have dominoes trying to topple it from both sides and so would remain standing! Call this game PHYMINOES. Then PHYMINOES is the same as TIMBER on bipartite graphs.

**Problem 4.3** *Is there an efficient algorithm to find the outcome class of a* PHYMINOES *on a non-bipartite graph?*

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