Characterizing Homogeneous Digraphs

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Abstract

A digraph is called *homogeneous* if every connected induced subdigraph with two or more vertices is either strong or acyclic. The class of homogeneous digraphs contains acyclic digraphs, round digraphs, and symmetric digraphs. Tournaments which are homogeneous have been studied by Guido, Moon and others, and characterized by Moon. In this paper we give a characterization of homogeneous digraphs. Our characterization reveals a nice structural property of this class of digraphs and shows that all homogeneous digraphs can be obtained from acyclic digraphs, round digraphs, and symmetric digraphs by the operation of substitution.

1 Introduction and the main theorem

We consider digraphs which do not contain loops or multiple arcs but may contain two arcs of opposite directions between two vertices. Paths and cycles in a digraph are always assumed to be directed. Our notation and terminology generally follow that in [2].

A digraph D is *strong* if for every pair of distinct vertices x, y, there exists a path from x to y and a path from y to x. Clearly, every strong digraph with at least two vertices contains a cycle. A digraph which does not contain any cycle is called *acyclic*. Thus a digraph with only one vertex is both strong and acyclic.

We call a digraph D *homogeneous* if every connected induced subdigraph of D with two or more vertices is either strong or acyclic. Since every subdigraph of an acyclic digraph is acyclic, acyclic digraphs are all

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JCMCC **85** (2013), pp. 227-236

homogeneous. The class of homogeneous digraphs also contains every *symmetric digraph*, i.e., a digraph in which there is either no arc or a pair of arcs (of opposite directions) between any two distinct vertices.

Homogeneous tournaments have been studied in [3, 4, 7]. In [7], Moon proved every homogeneous tournament is obtained from a highly regular tournament by substituting the vertices with acyclic tournaments (see Theorem 1.3 below). Homogeneous tournaments are also called *Moon tournaments* in [4].

The class of homogeneous digraphs contains yet another class of digraphs studied in [6]. A digraph D is *round* if the vertices of D can be circularly ordered $v_1, v_2, \ldots, v_n$ such that, for each i, the out-neighbours of v_i appear consecutively following v_i and the in-neighbours of v_i appear consecutively preceding v_i in the circular ordering.

In this paper we prove that all homogeneous digraphs can be obtained from acyclic digraphs, round digraphs, and symmetric digraphs by the operation of substitution (see Theorem 1.6 below).

Let D be a digraph. An arc uv of D is *symmetric* if vu is also an arc of D; otherwise uv is *non-symmetric*. A path (resp. cycle) in D consists of only non-symmetric arcs is called an *oriented* path (resp. cycle). If uv is an arc of D (symmetric or not), then we say that u *dominates* v and denote it by $u \rightarrow v$; in this case, we also say that u is an *in-neighbour* of v and v is an *out-neighbour* of u. For each vertex v, the set of all in-neighbours of v is denoted by $I(v)$ and the set of all out-neighbours of v is denoted by $O(v)$. For convenience, we shall use $\tilde{I}(v), \tilde{O}(v), B(v)$ to denote $I(v) - O(v), O(v) - I(v), I(v) \cap O(v)$ respectively.

A digraph D is *semicomplete* if for any two distinct vertices u, v, there is at least one arc (symmetric or not) arc between u and v. Tournaments are semicomplete digraphs which contain only non-symmetric arcs. A digraph D is called *locally semicomplete* if for each vertex v, the subdigraphs of D induced by $I(x)$ and $O(x)$ are both semicomplete, cf. [1]. Locally semicomplete digraphs which contain only non-symmetric arcs are called *local tournaments*, cf. [5].

Locally semicomplete digraphs and round digraphs are intimately related. A characterization of round digraphs in terms of locally semicomplete digraphs is obtained in [6].

Theorem 1.1 *[6] A connected digraph D is round if and only if it is locally semicomplete and for every vertex v, the subdigraphs induced by $\tilde{I}(v)$ and $\tilde{O}(v)$ are both acyclic, and the subdigraph induced by $B(v)$ contains no oriented cycle.* ◇

Corollary 1.2 *Every round digraph is homogeneous.*

Proof: Since every induced subdigraph of a round digraph is round, it suffices to show that every connected round digraph with two or more vertices is either strong or acyclic. So let D be a connected round digraph with at least two vertices. By Theorem 1.1, D is locally semicomplete. Suppose that D is not strong. Then according to [1] the strong components of D can be ordered $S_1, S_2, \ldots, S_k$ in such a way that every vertex of S_i dominates every of S_{i+1} for each $i = 1, 2, \ldots, k-1$. It follows that for each $i = 1, 2, \ldots, k$, there is a vertex v such that S_i is contained in $\tilde{I}(v)$ or $\tilde{O}(v)$. Hence by Theorem 1.1 S_i is acyclic for each $i = 1, 2, \ldots, k$. Therefore D is acyclic. ◇

Let H be a digraph on vertices $v_1, v_2, \ldots, v_r$ and let $L_1, L_2, \ldots, L_r$ be a collection of digraphs. Then $H[L_1, L_2, \ldots, L_r]$ is the new digraph obtained from H by replacing v_i with L_i and adding an arc from every vertex of L_i to every vertex of L_j if and only if $v_i \to v_j$ in H ($1 \leq i \neq j \leq r$). We say that the new digraph is obtained from H by *substitution*. Note that if $D = H[L_1, L_2, \ldots, L_r]$, then $H, L_1, L_2, \ldots, L_r$ are induced subdigraphs of D.

According to Corollary 1.2 if a tournament is round then it is homogeneous. It turns out that round tournaments constitute the class of homogeneous tournaments and can be constructed from 'special' round tournaments by the operation of substitution. A tournament is *highly regular* if it has an odd number of vertices which can be labeled as $v_1, v_2, \ldots, v_{2k+1}$ so that $O(v_i) = \{v_{i+1}, v_{i+2}, \ldots, v_{i+k}\}$ (additions are modulo $2k+1$) for each $i = 1, 2, \ldots, 2k+1$.

Theorem 1.3 *[7] Let T be a homogeneous tournament. Then $T = T_0[T_1, T_2, \ldots, T_{2k+1}]$, where T_0 is highly regular and T_i is acyclic for each $i = 1, 2, \ldots, 2k+1$, that is, T is obtained from a highly regular tournament T_0 by substituting the vertices of T_0 with acyclic tournaments.* ◇

Proposition 1.4 *Suppose that H is a symmetric digraph on r vertices and $R_1, R_2, \ldots, R_r$ are homogeneous digraphs. Then $D = H[R_1, R_2, \ldots, R_r]$ is a homogeneous digraph.*

Proof: Since every induced subdigraph of a symmetric digraph is symmetric and every induced subdigraph of a homogeneous digraph is homogeneous, every induced subdigraph of D is obtained from a symmetric digraph by substituting the vertices with homogeneous digraphs. Thus it suffices to show that if D is connected, then D is strong or acyclic. So assume that D is connected. When $r \geq 2$, H is connected and hence strong, implying that $D = H[R_1, R_2, \ldots, R_r]$ is strong. When $r = 1$, $D = R_1$ is homogeneous by assumption and therefore is strong or acyclic. ◇

Proposition 1.5 *Suppose that H is a homogeneous digraph on k vertices and $S_1, S_2, \ldots, S_k$ are acyclic digraphs. Then $D = H[S_1, S_2, \ldots, S_k]$ is a homogeneous digraph.*

Proof: Again as above, it suffices to show that if D is connected, then it is either strong or acyclic. Thus assume that D is connected, which implies that H is connected. When $k = 1$, $D = S_1$ is acyclic. When $k \geq 2$. D is acyclic if H is acyclic, and is strong if H is strong. ◇

By Corollary 1.2 and Propositions 1.4 and 1.5, a homogeneous digraph can be constructed as follows: take a symmetric digraph, substitute the vertices of the symmetric digraph with round digraphs, and then substitute the vertices of the resulting digraph with acyclic digraphs. We show that every homogeneous digraph can be obtained in this way.

Theorem 1.6 *Let D be a homogeneous digraph. Then $D = H[L_1, L_2, \ldots, L_r$ where H is symmetric, $L_i = R_i[S_{i_1}, S_{i_2}, \ldots, S_{i_k}]$, R_i is round, and S_{i_j} is acyclic for each $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, k$.*

2 Proofs

Let D be a digraph and let S and S' be two induced subdigraphs of D. If every vertex in S dominates every vertex in S', then we say that S *completely dominates* S' and denote this by $S \Rightarrow S'$. In the case when $S = \{x\}$, we simply write $x \Rightarrow S'$; similarly when $S' = \{y\}$, we write $S \Rightarrow y$. We say that S is *coned by* x if $S \Rightarrow x$ whenever there is an arc from S to x, and $x \Rightarrow S$ whenever there is an arc from x to S. If S is coned by every vertex in $D - S$, then we say that S is *coned in* D. We say that S a *maximal coned* subdigraph in D if $S \neq D$, S is coned in D, and S is not properly contained in any coned subdigraph other than D.

Lemma 2.1 *Let D be a connected digraph. Suppose that S' and S' are both maximal coned subdigraphs in D and $V(S') \cap V(S'') \neq \emptyset$. Then $S'' - S'$ is coned in D. Moreoever D is obtained from a semicomplete digraph on two vertices by substituting the two vertices with S' and $S'' - S'$.*

Proof: Denote by S the subdigraph induced by $V(S')\cup V(S'')$. Since both S and S' are maximal coned subdigraphs, S properly contains S' and S''. We claim that $S = D$. Suppose not. Let x be any vertex not in S. If x dominates a vertex in S, then we must have $x \Rightarrow S'$ and $x \Rightarrow S''$ as both S' and S'' are coned in D and $V(S')\cap V(S'') \neq \emptyset$; similarly, if x is dominated by a vertex in S, then $S' \Rightarrow x$ and $S'' \Rightarrow x$. Thus S (which properly contains S') is coned in D. This contradicts the assumption that S' is a maximal coned subdigraph in D.

Since $D = S$ is connected, there is an arc between S' and $S'' - S'$. Suppose that the arc is from S' to a vertex y in $S'' - S'$. Then $S' \Rightarrow y$ because S' is coned in D. This implies that $x \rightarrow y$ for every vertex x in $S' - S''$. Since S'' is coned in D, $x \Rightarrow S''$ for every vertex $x \in S' - S''$. In particular, $S' - S'' \Rightarrow S'' - S'$. Again using the assumption that S' is coned in D, we have that $S' \Rightarrow S'' - S'$. A similar proof shows that if there is an arc from $S'' - S'$ to S' then $S'' - S' \Rightarrow S'$. Hence $S'' - S'$ is coned in D and D is obtained from a semicomplete digraph on two vertices by substituting the two vertices with S' and $S'' - S'$. ◇

We remark that Lemma 2.1 does not assume D is homogeneous.

Lemma 2.2 *Let D be a homogeneous digraph that is not obtained from a symmetric digraph with at least two vertices by substitution. Then the set of non-symmetric arcs of D induces a connected spanning subdigraph of D.*

Proof: Let D' be obtained from D by deleting all symmetric arcs. It suffices to show that D' is connected. Suppose not. Let $C_1, C_2, \ldots, C_k$ be the connected components of D' and let $S_1, S_2, \ldots, S_k$ be the subdigraphs of D induced by $V(C_1), V(C_2), \ldots, V(C_k)$ respectively. Then $k \geq 2$. We show that if there is an arc between S_i and S_j $(i \neq j)$, then $S_i \Rightarrow S_j$ and $S_j \Rightarrow S_i$. So suppose that there is an arc between $x \in V(S_i)$ and $y \in V(S_j)$. Note that the arc between x and y is symmetric as it is not in D'. For each vertex $z \in V(S_j)$ which is adjacent to y in C_i, there must be an arc between x and z, as otherwise $\{x, y, z\}$ induces a connected subdigraph of D which is neither strong nor acyclic. Since C_j is connected, it follows that there is a symmetric arc between x and every vertex of S_j. The same argument shows that there is a symmetric arc between y and every vertex of S_i. Since x and y are chosen arbitrarily, there is a symmetric arc between every vertex in S_i and every vertex in S_j, i.e., $S_i \Rightarrow S_j$ and $S_j \Rightarrow S_i$. Hence D is obtained a symmetric digraph on k vertices by substituting the k vertices with $S_1, S_2, \ldots, S_k$, contradicting the hypothesis. ◇

Lemma 2.3 *Let D be a homogeneous digraph that is not obtained from a symmetric digraph with at least two vertices by substitution. Then, for* *every $v \in V(D)$, the subdigraph of D induced by $B(v)$ contains no oriented cycle.*

Proof: Let x and y be any two vertices such that there is a non-symmetric arc between them. We show that if the subdigraph of D induced $B(x)$ contains an oriented cycle then the same cycle is contained in the subdigraph induced by $B(y)$. Assume that the arc between x and y is from x to y; the proof is similar if the arc is from y to x. Let $C : u_1 \to u_2 \to \cdots \to u_t \to u_1$ be an oriented cycle contained in the subdigraph induced by $B(x)$. Since D is a homogeneous digraph, the subdigraph induced by $\{x, y, u_i\}$ contains a cycle and hence is strong for each i. This implies that $y \to u_i$ for each $i = 1, 2, \ldots, t$. Again the assumption that D is a homogeneous digraph implies that the subdigraph induced by $V(C) \cup \{y\}$ is strong. Thus there exists some j such that $u_j \to y$. Hence we have both $y \to u_j$ and $u_j \to y$. In fact, this must be true for each $j = 1, 2, \ldots, t$; as otherwise there would be some l such that $y \to u_l$ and $u_l \to y$ but $u_{l+1} \not\to y$. Hence the subdigraph induced by $\{u_l, u_{l+1}, y\}$ is neither strong nor acyclic, contradicting the hypothesis.

By Lemma 2.2, the non-symmetric arcs of D induce a connected spanning subdigraph of D. Suppose that there is a vertex v such that the subdigraph induced by $B(v)$ contains an oriented cycle. It follows from the above that the same cycle is contained in $B(u)$ for every vertex u of D, which is not true when u is in the cycle. ◇

Lemma 2.4 *Let D be a homogeneous digraph that is not obtained from a symmetric digraph with at least two vertices by substitution. Suppose that $S_1, S_2, \ldots, S_k$ are the maximal coned subdigraphs of D. Then the following statements hold.*

(i) $V(S_i) \cap V(S_j) \neq \emptyset$ for all $i \neq j$;

(ii) S_i is acyclic for each i;

(iii) $D = R[S_1, S_2, \ldots, S_k]$ where R is a round digraph.

Proof: Statement (i) follows from Lemmas 2.1 and 2.2 and the assumption that D is strong and is not obtained from a symmetric digraph on two vertices by substitution. For Statement (ii), suppose that some S_p contains a cycle. Let Q be a strong component of S_p that contains a cycle. By Lemma 2.2, there is a non-symmetric arc between Q and a vertex x not in Q. The vertex x may or may not be in S_p. But in either case the connected subdigraph induced by $V(Q) \cup \{x\}$ is neither strong nor acyclic, since Q

is a strong component of S_p which is coned in D. This contradicts the assumption that D is homogeneous.

Clearly, $D = R[S_1, S_2, \ldots, S_k]$ for some digraph R. The assumption that $S_1, S_2, \ldots, S_k$ are maximal coned in D implies that R does not contain any coned proper subdigraph with two or more vertices. We show that R is round. Note that R is an induced subdigraph of D and hence is homogeneous. It follows that for each vertex v the subdigraphs of R induced by $\tilde{O}(v)$ and $\tilde{I}(v)$ are acyclic. Lemma 2.3 ensures that the subdigraph of R induced by $B(v)$ contains no oriented cycle for each vertex v. Thus in view of Theorem 1.1 we only need to show that R is locally semicomplete.

Suppose that R is not locally semicomplete. Then there is a vertex u such that $O(u)$ or $I(u)$ contains a pair of non-adjacent vertices. Assume that $x, y \in O(u)$ are two non-adjacent vertices; the proof for the other case is similar. We claim that there must exist a vertex z such that $\tilde{O}(z)$ or $\tilde{I}(z)$ contains a pair of non-adjacent vertices. Since R is homogeneous, the subdigraph induced by $\{u, x, y\}$ is either strong or acyclic in R. This implies that ux and uy are either both symmetric or both non-symmetric. If both are non-symmetric, then we are done. So assume both ux and uy are symmetric. By Lemma 2.2, there is a sequence of vertices $w_1, w_2, \ldots, w_q$ such that $w_1 = u$, $w_q \in \{x, y\}$, and w_i and w_{i+1} are connected by a non-symmetric arc. Suppose that $w_1 \rightarrow w_2$. (A similar discussion applies if $w_2 \rightarrow w_1$.) Then $w_2 \Rightarrow \{x, y\}$. The subdigraph induced by $\{w_2, x, y\}$ is either strong or acyclic in R, which implies that either both w_2x, w_2y are non-symmetric or both are symmetric. If both are non-symmetric, then we have found such a vertex z, i.e., $z = w_2$. Otherwise we proceed by considering the 'next' vertex in the sequence $w_3, \ldots, w_q$. Since the arc between w_{q-1} and $w_q \in \{x, y\}$ is non-symmetric, there exists some j such that either $w_j \Rightarrow \{x, y\}$ or $\{x, y\} \Rightarrow w_j$ and the arcs between w_j and $\{x, y\}$ are both non-symmetric. That is, x, y are non-adjacent vertices in $\tilde{O}(w_j)$ or in $\tilde{I}(w_j)$. So from now on we assume that z is such a vertex that either $\tilde{O}(z)$ or $\tilde{I}(z)$ contains a pair of non-adjacent vertices x, y. Since D is strong, so is R. If $x, y \in \tilde{O}(z)$, then R contains a path P from $\{x, y\}$ to z, and if $x, y \in \tilde{I}(z)$, then R contains a path P from z to $\{x, y\}$. We assume that z, x, y are chosen so that the indicated path P has the minimum length. We further assume that $x, y \in \tilde{O}(z)$; the proof is similar if $x, y \in \tilde{I}(z)$.

Let $P : x_0 \rightarrow x_1 \rightarrow \ldots \rightarrow x_r$ be the path (i.e., $x_0 \in \{x, y\}$ and $x_r = z$). Assume by symmetry that $x_0 = x$. The choice of z, x, y and P implies that there is no arc from x_i to x_j for all i, j with $j - i > 1$. Moreover, the subdigraph induced by $V(P) \cup \{y\}$ is strong as it is connected and contains a cycle, so there is an arc from y to $V(P)$. The choice of P implies that it must be yx_1. Note also that either both xx_1, yx_1 are symmetric or non-symmetric and when $r \geq 3$ they are both non-symmetric. Let $U \subset V(R)$ consist of all vertices $v \in \tilde{O}(z) \cap B(x_1)$ when xx_1, yx_1 are symmetric or of all vertices $v \in \tilde{O}(z) \cap \tilde{I}(x_1)$ when xx_1, yx_1 are non-symmetric. Then U contains both x and y. Let W be a minimal subset of U containing x, y such that the subdigraph induced by W is coned in the subdigraph induced by U. Since the subdigraph induced by W is not coned in R, there exists $c, c' \in W$ and $w \notin W$ such that either cw is an arc and $c'w$ is not or wc is an arc wc' is not. Assume that cw is an arc and $c'w$ is not; the other case can be treated analogously. We give a preference to such a w that cw is non-symmetric. Note that the definition of W implies that w is not in U.

Consider first the case when xx_1, yx_1 are symmetric arcs. Then $r = 2$. Suppose that x_1x_2 is non-symmetric. If cw is non-symmetric, then there is an arc from w to x_1 as otherwise the connected subdigraph induced by w, x_1, c is neither strong nor acyclic. In fact the arc wx_1 is symmetric as otherwise the connected subdigraph induced by w, x_1, c' is neither strong nor acyclic. By considering the connected subdigraphs induced respectively by x_1, x_2, w and x_2, c', w we see that x_2w is a non-symmetic arc. Thus w is in U, a contradiction. So cw is symmetric. Then wx_2 is an arc as otherwise the connected subdigraph induced by x_2, c, w is neither strong nor acyclic. And by considering the connected subdigraph induced by x_2, c', w we see that wx_2 is non-symmetric. Similarly, by considering the connected subdigraph induced by x_1, x_2, c', w we see that x_1w is an arc. In fact x_1w must be non-symmetric, as otherwise x_1, x_2, w induce a connected subdigraph which is neither strong nor acyclic. It follows that wv is an arc for all $v \in U$. Thus W can be partitioned into two sets W' and W'' where W' consists of all vertices v such that wv is symmetric. Note that $c \in W'$ and $c' \in W''$. By considering the subdigraph induced by w, v, v' for all possible $v \in W'$ and $v' \in W''$, we see that $W'' \Rightarrow W'$. Since the subdigraph induced by W is acyclic, all arcs from W'' to W' are non-symmetric. Clearly, x, y are either both in W' or both in W''. This is a contradiction to the definition of W. Suppose that x_1x_2 is symmetric. If cw is symmetric, then the same argument as above shows that wx_2 is a non-symmetric arc. This implies that x_1w is an arc. If x_1w is non-symmetric, then a similar argument as above leads to a contradiction to the definition of W. So x_1w is symmetric. On the other hand if cw is non-symmetric, then wx_1 is an arc as otherwise the connected subdigraph induced by x_1, c, w is neither strong nor acyclic. And wx_1 must be symmetric as otherwise the connected subdigraph induced by x_1, c', w is neither strong nor acyclic. So we have shown that for every such vertex w, wx_1 is a symmetric arc. Hence $B(x_1)$ contains not only U and x_2 but also all such vertices w. It is now easy to see that the subdigraph induced by a set of vertices in $B(x_1)$ containing U, x_2, and such vertices w is coned in R. This is a contradiction.

Consider now the case when xx_1, yx_1 are non-symmetric. Suppose that $r = 2$. Then x_1w and wx_2 are non-symmetric arcs. If cw is non-symmetric, then partition W into W' and W'' where W' consists of all vertices $v \in W$ such that vw is an arc. We have $W'' \Rightarrow W'$. This contradicts the definition of W. If cw is symmetric, then for each vertex $v \in W$, either vw is a symmetric arc or wv is a non-symmetric arc. Thus W can be partitioned into W', W'' such that wv is symmetric for each $v \in W'$ and wv is non-symmetric for each $v \in W''$. Furthermore, $W'' \Rightarrow W'$, which contradicts the definition of W. Suppose that $r \geq 3$. If cw is non-symmetric, then by considering the connected subdigraph induced by $V(P) \cup \{c, w\}$ we see that there is an arc from w to $V(P)$. Let α be the largest subscript such that wx_α is an arc. Similarly, by considering the connected subdigraph induced by $V(P) \cup \{c', w\}$ we see that there is an arc from $V(P)$ to w. Let β be the smallest subscript such that $x_\beta w$ is an arc. Assume $\alpha = 1$. Then $\beta > 1$. Since $w \notin U$, $\beta < r$. By considering the connected subdigraph induced by $\{c, w, x_1 \ldots x_\beta\}$ we see that there is an arc from $\{x_1, \ldots, x_\beta\}$ to c. Let W' consist of all vertices $v \in W$ such that there is an arc from $\{x_1, \ldots, x_\beta\}$ to v and let $W'' = W - W'$. Then W' contains c and W'' contains x, y. It is now easy to see that $W' \Rightarrow W''$, a contradiction to the definition of W. Assume $\alpha > 1$. If there is no arc from W to any vertex in $\{x_\alpha, \ldots, x_r\}$, then let W' consist of all vertices $v \in W$ such that vw is an arc and let $W'' = W - W'$. Then $W'' \Rightarrow W'$ as otherwise there are vertices $v \in W'$ and $v' \in W''$ such that the connected subdigraph induced by $\{v, v', w, x_\alpha, \ldots, x_r\}$ is neither strong nor acyclic. This is a contradiction to the definition of W. If there are arcs from W to $\{x_\alpha, \ldots, x_r\}$, then let γ be the largest subscript such that there is an arc from W to x_γ. Let W' consist of vertices $v \in W$ such that vx_γ is an arc and let $W'' = W - W'$. Then we have $W'' \Rightarrow W'$, a contradiction to the definition of W. If cw is symmetric, let W' consist of all vertices $v \in W$ such that wv is a symmetric arc and let $W'' = W - W'$. Then W', W'' is a partition of W and $W'' \Rightarrow W'$, which once again contradicts the definition of W. ◇

We are now ready to prove Theorem 1.6 which is restated below for clarity.

Theorem 1.6 Let D be a homogeneous digraph. Then $D = H[L_1, L_2, \ldots, L_r]$ where H is symmetric, $L_i = R_i[S_{i_1}, S_{i_2}, \ldots, S_{i_k}]$, R_i is round, and S_{i_j} is acyclic for each $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, k$.

Proof: Let D be a homogeneous digraph. It is always possible to express D in the form $H[L_1, L_2, \ldots, L_r]$ where H is a symmetric digraph (possibly on a single vertex) and L_i is not be obtained from a symmetric digraph with at least two vertices by substitution for each $i = 1, 2, \ldots, r$. By Lemma 2.4(iii), each L_i is obtained from a round digraph by substituting