Global Domination Stable Trees

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Abstract

A set of vertices in a graph G is a global dominating set of G if it dominates both G and its complement $\overline{G}$. The minimum cardinality of a global dominating set of G is the global domination number of G. We explore the effects of graph modifications (edge removal, vertex removal, and edge addition) on the global domination number. In particular, for each graph modification, we study the global domination stable trees, that is, the trees whose global domination number remains the same upon the modification. We characterize these stable trees having small global domination numbers.

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Dedicated to C.M. Mynhardt in honor of her 60th birthday.

1 Introduction

For notation and graph theory terminology in general, we follow [3]. For a graph $G = (V, E)$, a subset $S \subseteq V$ is a *dominating set*, denoted DS, of G if every vertex of $V \setminus S$ is adjacent to at least one vertex of S. The *domination number* $\gamma(G)$ is the cardinality of a minimum DS of G, and a DS of G with cardinality $\gamma(G)$ is called a $\gamma(G)$*-set*. Sampathkumar [5] introduced global domination as follows: a set $S \subseteq V$ is a *global dominating set*, denoted GDS, of G if S is a DS of both G and its complement $\overline{G}$. The *global domination number* $\gamma_g(G)$ of G (and of $\overline{G}$) is the minimum cardinality of a GDS of G, and a GDS of this size is a $\gamma_g(G)$*-set*. For more details on domination and global domination, see [1, 3, 4].

Brigham and Dutton [2] were the first to study the effects of graph modifications (edge removal, edge addition, vertex removal) on the global domination number. They observed that when an edge is removed from G, it is added to $\overline{G}$, and vice versa. We note that under any of the three graph modifications, the global domination number can decrease, stay the same, or increase. Brigham and Dutton [2] focused on the critical graphs, that is, those whose global domination number changes when the graph is modified.

In this paper, we consider the graphs whose global domination number remains the same when the graph is modified. Specifically, we define the following:

For any graph G, we denote the graph formed by removing an edge e from the edge set of G as $G - e$, by removing a vertex v and all of its incident edges as $G - v$; and by adding an edge e from the edge set of $\overline{G}$ to G as $G + e$.

Definition 1 *Let G be a graph with $\gamma_g(G) = k$.*

1. *If $\gamma_g(G) = \gamma_g(G - e)$ for an arbitrary edge $e \in E(G)$, then G is a k_g-edge minus stable graph, denoted k_g-EMS.*
2. *If $\gamma_g(G) = \gamma_g(G + e)$ for an arbitrary edge $e \in E(\overline{G})$, then G is a k_g-edge plus stable graph, denoted k_g-EPS.*
3. *If $\gamma_g(G) = \gamma_g(G - v)$ for an arbitrary vertex $v \in V(G)$, then G is a k_g-vertex stable graph, denoted k_g-VS.*

In Section 2, we present terminology and some preliminary results. For $k \in \{2, 3\}$, we characterize the k_g-EMS trees in Section 3, the k_g-EPS trees in Section 4, and the k_g-VS trees in Section 5.

2 Terminology and Preliminary Results

For a graph $G = (V, E)$, the *open neighborhood* of v is $N_G(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood of* v is $N_G[v] = \{v\} \cup N_G(v)$. If the graph G is clear from the context, we simply write $N(v)$ and $N[v]$ rather than $N_G(v)$ and $N_G[v]$, respectively. For a set $S \subseteq V$, its *open neighborhood* is the set $N(S) = \bigcup_{v \in S} N(v)$, and its *closed neighborhood* is the set $N[S] = N(S) \cup S$. For a set S of vertices and $u \in S$, a vertex v is a *private neighbor* of u (with respect to S) if $N[v] \cap S = \{u\}$. The *external private neighborhood of* u with respect to S is $\text{pn}[u, S] = \{v \in V \setminus S : N(v) \cap S = \{u\}\}$. We denote the subgraph induced in G by a subset S as

$G[S]$. A vertex of degree one is called a *leaf*, and its neighbor is a *support vertex*. If a vertex is adjacent to two or more leaves, it is called a *strong support vertex*.

Since a GDS of G dominates both G and $\overline{G}$, we make the following straightforward, but useful observation.

Observation 1 *For any graph G and $\gamma_g(G)$-set S, every vertex in $V(G)\setminus S$ is adjacent to some vertex in S and is nonadjacent to some vertex in S.*

We next give a property of vertex removal stable and edge removal stable graphs.

Lemma 2 *Let G be a k_g-EMS or a k_g-VS graph for $k \geq 2$. If every vertex in a $\gamma_g(G)$-set S is a support vertex, then S contains no strong support vertices.*

Proof. Let G be a k_g-EMS or a k_g-VS graph, and let S be a $\gamma_g(G)$-set such that every vertex of S is a support vertex in G. Assume to the contrary that $v \in S$ is a strong support vertex. Let L_v be the set of leaf neighbors of v and $v' \in L_v$. Then, since v' is an isolate in $G - vv'$, every $\gamma_g(G - vv')$-set S' contains v'. Also, since v is a strong support vertex in G, it is a support vertex in $G - vv'$, and hence either v or a leaf neighbor from $L_v \setminus \{v'\}$ is in S'. Moreover, since each vertex x in $S \setminus \{v\}$ is a support vertex in G, x is a support vertex in $G - vv'$, implying that x or a leaf neighbor of x is in S'. Note that no leaf neighbor of x is in $\{v, v'\}$. Thus, $\gamma_g(G - vv') = |S'| \geq |S \setminus \{v\}| + 2 = \gamma_g(G) + 1$, so G is not a k_g-EMS graph.

Hence, we may assume that G is a k_g-VS graph. Since v is a strong support vertex in T, $L_v \subset S'$, for every $\gamma_g(G - v)$-set. Since every vertex in $S \setminus \{v\}$ is a support vertex in $G - v$, we have that S' contains at least $|S \setminus \{v\}|$ additional vertices. Hence, $\gamma_g(G - v) = |S'| \geq |L_v| + |S \setminus \{v\}| \geq 2 + \gamma_g(G) - 1 = \gamma_g(G) + 1$, contradicting that G is a k_g-VS graph. □

Our focus is on global domination stable trees with small global domination number k, namely, $k \in \{2, 3\}$. We note that if G is a non-trivial graph, then $\gamma_g(G) \geq 2$. To aid in our discussions, we let $S = \{a, b\}$ or $S = \{a, b, c\}$ be a $\gamma_g(G)$-set for the graph G under consideration, and define the following sets depending upon S. Let $A = \text{epn}_G(a, S)$, $B = \text{epn}_G(b, S)$, and $C = \text{epn}_G(c, S)$. Further, let $AB = N_G(a) \cap N_G(b) \cap (V \setminus S)$, $AC = N_G(a) \cap N_G(c) \cap (V \setminus S)$, $BC = N_G(b) \cap N_G(c) \cap (V \setminus S)$, and $ABC = N_G(a) \cap N_G(b) \cap N_G(c) \cap (V \setminus S)$.

We next determine properties of the sets associated with a GDS of a tree. The lemmas follow directly from Observation 1 and the property that a tree has no cycles, so we state them without proof.

Lemma 3 *Let T be a tree with $\gamma_g(T)$-set S. If $\gamma_g(T) = 2$ and $S = \{a, b\}$, then*

1. $AB = \emptyset$,
2. *each of A and B is an independent set.*

Lemma 4 *Let T be a tree with $\gamma_g(T)$-set S. If $\gamma_g(T) = 3$ and $S = \{a, b, c\}$, then*

1. $ABC = \emptyset$,
2. $|AB| \leq 1$, $|AC| \leq 1$, *and* $|BC| \leq 1$,
3. *at least one of AB, AC, and BC is empty,*
4. *each of A, B, and C is an independent set.*

We conclude this section by defining a family of trees that we will use later in our characterizations. A *caterpillar* is a tree C for which the removal of all the leaves of T results in a path $(v_1, v_2, \cdots, v_k)$, which is called the *spine* of C. The code of caterpillar C is the ordered k-tuple $(x_1, x_2, \cdots, x_k)$, where x_i is the number of leaves adjacent to v_i, $1 \leq i \leq k$. We note that the reverse sequence also defines the caterpillar, that is, the code $(x_1, x_2, \cdots, x_k) = (x_k, x_{k-1}, \cdots, x_1)$. Figure 1 is an example of a caterpillar with code $(1, 0, 3, 2)$ or $(2, 3, 0, 1)$.

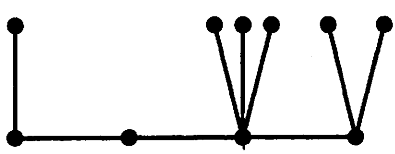

Figure 1: Caterpillar $(1, 0, 3, 2)$

3 Edge Removal

In this section, we consider trees whose global domination number does not change when any arbitrary edge is removed, that is, the k_g-EMS trees. We characterize such trees for $k \in \{2, 3\}$. Let P_n denote the path on on n vertices.

Theorem 5 *A tree T is a 2_g-EMS tree if and only if T is the path P_4 or a non-trivial star.*

Proof. If $T = P_4$ or T is a non-trivial star, then $\gamma_g(T) = \gamma_g(T - e) = 2$ for any edge $e \in E(T)$. Hence, T is a 2_g-EMS tree.

Let T be a 2_g-EMS tree with $\gamma_g(T)$-set $S = \{a, b\}$, and let the sets A, B, and AB be defined as in Section 2. By Lemma 3, we have $AB = \emptyset$ and each of A and B is an independent set. Since T is connected, either $ab \in E(T)$ or there is an edge between a vertex in A, say a', and a vertex in B, say b'. Note that since T is a tree, exactly one of the edges ab and $a'b'$ is in $E(T)$. If $ab \in E(T)$, then the vertices of $A \cup B$ are leaves in T, and if $a'b' \in E(T)$, then the vertices of $(A \cup B) \setminus \{a', b'\}$ are leaves in T.

First, assume that $ab \in E(T)$. If $A = \emptyset$ or $B = \emptyset$, then T is a non-trivial star and the result holds. Hence, we may assume that $|A| \geq 1$ and $|B| \geq 1$, that is, a and b are support vertices in T. It follows from Lemma 2 that $|A| = |B| = 1$, and so $T = P_4$ as desired.

Next, assume that $a'b' \in E(T)$. If $|A| = |B| = 1$, then again $T = P_4$ and the result holds. If $|A| \geq 2$ or $|B| \geq 2$, then removing a pendant edge increases the global domination number, contradicting that T a 2_g-EMS tree. □

Theorem 6 *A tree T is a 3_g-EMS tree if and only if T is the path P_7 or the caterpillar* $(1, 1, 1)$.

Proof. It is straightforward to check that the path P_7 and the caterpillar $(1, 1, 1)$ are 3_g-EMS trees.

Assume that T is a 3_g-EMS tree, and let $S = \{a, b, c\}$ be a $\gamma_g(T)$-set with the associated sets defined in Section 2. By Lemma 4, we have that $ABC = \emptyset$, and at least one of AB, AC, and BC is empty. Without loss of generality, assume that $AC = \emptyset$. Also, by Lemma 4, $|AB| \leq 1$, $|BC| \leq 1$, and each of A, B, and C is an independent set. Moreover, $T[S]$ has at most two edges. We consider three cases depending on the number of edges in $T[S]$.

Case 1. S is an independent set, that is, $T[S]$ has no edges.

Case 1(a). $|AB| = |BC| = 1$.

Then, $A \cup B \cup C$ is an independent set in T, that is, $A \cup B \cup C$ are leaves in T. We show that $|A| \leq 1$ and $|C| \leq 1$. By symmetry, it suffices to show that $|A| \leq 1$. Suppose to the contrary that $|A| \geq 2$, and let $a' \in A$. Then, a' is an isolated vertex in $T' = T - aa'$, and so $a' \in S'$ for every $\gamma_g(T')$-set S'. Moreover, at least one additional vertex from $\{a\} \cup A$ is in S'. Then exactly one vertex must dominate both b and c, implying that $x \in BC$ is in S'. It follows that $B = C = \emptyset$. But then $\{x, a\}$ is a GDS of T with cardinality less than $\gamma_g(T)$, a contradiction. Hence, we may assume that $|A| \leq 1$ and $|C| \leq 1$.

Let $B = \emptyset$. If A or C is empty, then $T \in \{P_5, P_6\}$ and $\gamma_g(T) = 2$, a contradiction. Hence, $|A| = |C| = 1$, and so $T = P_7$ as desired.

Thus, we may assume that $B \neq \emptyset$, and let $b' \in B$. If either A or C is not empty, say $A \neq \emptyset$, then a' is in every $\gamma_g(T - aa')$-set S', where $a' \in A$. Moreover b or b' is in S'. But since $|S'| = 3$, at least one of a and c is not dominated by S', a contradiction. Hence, $A = C = \emptyset$. Then $\{x, y, b\}$, where $AB = \{x\}$ and $BC = \{y\}$, is a $\gamma_g(T)$-set and every vertex in $\{x, y, b\}$ is a support vertex. Lemma 2 implies that T is the caterpillar $(1, 1, 1)$, and the result holds.

Case 1(b). $|AB| = 1$ and $BC = \emptyset$.

Since T is connected, there is an edge between a vertex in C and a vertex in $A \cup B$. Without loss of generality, say that $b' \in B$ is adjacent to $c' \in C$. Also note that since T is a tree, the only edge with both its endpoints in $A \cup B \cup C$ is the edge $b'c'$. Thus every vertex in $(A \cup B \cup C) \setminus \{b', c'\}$ is a leaf in T. If $|B| \geq 2$, then $\gamma_g(T - cc') > \gamma_g(T)$, a contradiction. Hence, we may assume that $B = \{b'\}$.

If $|C| \geq 3$ or $|A| \geq 2$, then removing a pendant edge incident to either a or c increases the global domination number, contradicting that T is a 3_g-EMS tree. Hence, $1 \leq |C| \leq 2$ and $|A| \leq 1$. If $|A| = 1$ and $|C| = 2$, then removing a pendant edge increases the global domination number, a contradiction. If $A = \emptyset$ and $|C| = 1$, then $T = P_6$ and $\gamma_g(P_6) = 2$, a contradiction. If $A = \emptyset$ and $|C| = 2$, or if $|A| = 1$ and $|C| = 1$, then $T = P_7$, as desired.

Case 1(c). $AB = BC = \emptyset$.

Since T is a tree, it is connected via edges between vertices in $A \cup B \cup C$. Recall that by Lemma 4, each of A, B, and C is an independent set. Without loss of generality, assume that $a'b' \in E(T)$ and $b''c' \in E(T)$, where $a' \in A$, $b', b'' \in B$ (note that b'' may equal b'), and $c' \in C$. Now all vertices of $(A \cup B \cup C) \setminus \{a', b', b'', c'\}$ are leaves of T, otherwise T has a cycle. If any vertex in S is adjacent to a leaf, then the removal of this pendant edge increases the global domination number, contradicting that T is a 3_g-EMS tree. Thus, we may assume that $A = \{a'\}$, $B = \{b', b''\}$, and $C = \{c'\}$. If $b' = b''$, then T is the caterpillar $(1, 1, 1)$, and if $b' \neq b''$, then $T = P_7$, as desired.

<u>Case 2</u>. $T[S]$ has exactly two edges.

Without loss of generality, let ab and bc be the edges of $T[S]$. Since T is a tree, $AB = BC = \emptyset$ and $A \cup B \cup C$ is an independent set, and so $A \cup B \cup C$ is a set of leaves in T. If none of A, B, and C is empty, then Lemma 2 implies that $|A| = |B| = |C| = 1$, and so T is the caterpillar $(1,1,1)$, and the result holds. If either of A or C is empty, then $\gamma_g(T) = 2$, a contradiction. Thus, we may assume $|A| \geq 1$, $|C| \geq 1$, and $B = \emptyset$. But then removing an edge incident to b decreases the global domination number from 3 to 2, contradicting that T is a 3_g-EMS tree.

<u>Case 3</u>. $T[S]$ has exactly one edge.

Relabelling the vertices if necessary, we may assume that ab is the edge in $T[S]$. Since T is a tree $AB = \emptyset$, and either $BC \neq \emptyset$ or there is an edge between a vertex in C and a vertex in $A \cup B$.

Assume first that $BC \neq \emptyset$. It follows that $A \cup B \cup C$ is an independent set, and the vertices of $A \cup B \cup C$ are leaves in T. If $B = \emptyset$, then $\{a, c\}$ is a GDS of T with cardinality less than $\gamma_g(T)$, a contradiction. Hence, $|B| \geq 1$. If $|A| \geq 2$, then removing a pendant edge incident to a increases the global domination number, a contradiction. Thus, $|A| \leq 1$. If $A = \emptyset$, then $\{b, c\}$ is a GDS for $G - bx$, where $BC = \{x\}$, contradicting that T is a 3_g-EMS tree. Hence, $|A| = 1$. If $|B| \geq 2$, then removing a pendant edge incident to b increases the global domination number; and if $C \neq \emptyset$, then removing a pendant edge incident to c increases the global domination number. Thus, $C = \emptyset$ and $|B| = 1$, that is, T is the caterpillar $(1,1,1)$, as desired.

Therefore, we may assume that $BC = \emptyset$. Since T is connected, there exists a vertex $c' \in C$ with a neighbor in $A \cup B$. Without loss of generality, let $b' \in B$ be a neighbor of c'. Since T is a tree, it follows that $b'c'$ is the only edge with both endvertices in $A \cup B \cup C$. Thus, $(A \cup B \cup C) \setminus \{b', c'\}$ is a set of leaves in T. If $A = \emptyset$, then $\{b, c\}$ is a GDS of T with cardinality less than $\gamma_g(T)$, a contradiction. Hence, $|A| \geq 1$. If $|A| \geq 2$ and ($|B| \geq 2$ or $|C| \geq 2$), then removing a pendant edge incident to a increases the global domination number. If $|A| \geq 2$ and $|B| = |C| = 1$, then $\{a, c'\}$ is a GDS of T, a contradiction. Thus, $|A| = 1$. If $|C| = 1$, then $S' = \{a, b, c'\}$ is a $\gamma_g(T)$-set with properties of a previous case, namely, one edge in $T[S']$ and $BC' \neq \emptyset$. Hence, we may assume that $C \setminus \{c'\} \neq \emptyset$, that is, $|C| \geq 2$. Now if b has a leaf neighbor, then removing a pendant edge incident to b increases the global domination number, a contradiction. Similarly, if $|C| \geq 3$, then removing a pendant edge incident to c increases the global domination, and we have another contradiction. Thus, $|B| = 1$ and $|C| = 2$, and so $T = P_7$.
□

4 Vertex Removal

In this subsection, we consider trees whose global domination number stays the same upon the removal of any arbitrary vertex. We characterize the 2_g-VS and 3_g-VS trees.

Theorem 7 *A tree T is a 2_g-VS tree if and only if T is one of the paths P_3, P_4, and P_5.*

Proof. It is straightforward to see that $\gamma_g(P_j) = \gamma_g(P_j - v) = 2$, where v is an arbitrary vertex of P_j and $j \in \{3, 4, 5\}$.

Assume that T is a 2_g-VS tree. Let $S = \{a, b\}$ be a $\gamma_g(T)$-set with the sets A, B, and AB as defined in Section 2. By Lemma 3, $AB = \emptyset$ and each of A and B is an independent set.

Since T is connected and $AB = \emptyset$, either $ab \in E(T)$ or there is an edge between a vertex in A, say a', and a vertex in B, say b'. If $ab \in E(T)$, then the vertices of $A \cup B$ are leaves in T, and if $a'b' \in E(T)$, then $ab \notin E(T)$ and the vertices of $(A \cup B) \setminus \{a', b'\}$ are leaves in T, for otherwise a cycle is formed.

First, assume that $ab \in E(T)$. If either of A and B is empty, then T is the non-trivial star. If $A = B = \emptyset$, then $T = P_2$ and removing a vertex decreases the global domination number from 2 to 1, contradicting that T is a 2_g-VS tree. Hence, T has at least three vertices. If $T = P_3$, we are finished, so assume that T has at least four vertices. Assume, without loss of generality, that $A = \emptyset$ and $B \neq \emptyset$. But then removing the center vertex of the star of order $n \geq 4$ increases the global domination number from 2 to $n - 1$, again a contradiction. Thus, we may assume that $|A| \geq 1$ and $|B| \geq 1$. Lemma 2 implies that $|A| = |B| = 1$, and so $T = P_4$ and the result holds.

Next, assume that $a'b' \in E(T)$. Note that $|A| \geq 1$ and $|B| \geq 1$ as $a' \in A$ and $b' \in B$. If $|A| = |B| = 1$, then $T = P_4$ and the result holds. Thus, without loss of generality, we may assume that $|A| \geq 2$. Lemma 2 implies that $|A| = |B| = 2$, or $|A| \geq 2$ and $|B| = 1$. If $|A| = |B| = 2$, or if $|A| \geq 3$ and $|B| = 1$, then $\gamma_g(T - a) > \gamma_g(T)$, contradicting that T is a 2_g-VS tree. Hence, $|A| = 2$ and $|B| = 1$, and so, $T = P_5$. □

Let S_3 be the graph formed from the star $K_{1,3}$ by subdividing each edge exactly once.

Theorem 8 *A tree T is a 3_g-VS tree if and only if T is the path P_8, the caterpillar $(1, 0, 1, 0, 1)$, or the subdivided star S_3.*

Proof. It is straightforward to check that the path P_8, the caterpillar $(1,0,1,0,1)$, and the subdivided star S_3 are 3_g-VS trees.

Let T be a 3_g-VS tree with $\gamma_g(T)$-set S, where the sets A, B, C, AB, AC, BC, and ABC are defined as in Section 2. Lemma 4 implies that $ABC = \emptyset$ and at least one of AB, AC, and BC is empty. Without loss of generality, assume that $AC = \emptyset$. Also, from Lemma 4, we have $|AB| \le 1$, $|BC| \le 1$, and each of A, B, and C is an independent set. We note that since T is a tree, $T[S]$ has at most two edges. We consider cases based on the number of edges in $T[S]$.

Case 1. $T[S]$ has no edges, that is, S is an independent set.

First, assume that $|AB| = |BC| = 1$. Since T is a tree, $A \cup B \cup C$ is an independent set, that is, each vertex in $A \cup B \cup C$ is a leaf in T. Let $x \in S$ and $X = \text{epn}_T(x, S)$. If $X = \emptyset$, then $\gamma_g(T-x) = 2$, contradicting that T is a 3_g-VS tree. Thus, none of A, B, and C is empty, that is, each of a, b, and c is a support vertex. Hence, Lemma 2 implies that $|A| = |B| = |C| = 1$ and T is the caterpillar $(1,0,1,0,1)$, as desired.

Next, consider the case where $|AB| = 1$ and $BC = \emptyset$. Since T is connected and $BC = \emptyset$, there is an edge between a vertex in C and a vertex in $A \cup B \cup AB$. Without loss of generality, say that $b' \in B \cup AB$ is adjacent to $c' \in C$. Also note that since T is a tree, $b'c'$ is the only edge with both its endpoints in $V(T) \setminus S$. Thus, every vertex in $(A \cup B \cup C) \setminus \{b', c'\}$ is a leaf in T. For a vertex $x \in S$, the leaves of X are isolated in $T-x$ and are in every $\gamma_g(T-x)$-set S'. Hence, if x is a strong support vertex of T, then at least two isolated vertices are in S' and at least two additional vertices are in S' to dominate the remaining vertices of $T-x$, a contradiction. Hence none of a, b, and c is a strong support vertex. First assume that $b' \in AB$, that is, $AB = \{b'\}$. Then, $|A| \le 1$, $|B| \le 1$, and $|C| \le 1$. If $A = \emptyset$ (respectively, $B = \emptyset$), then $\gamma_g(T-a) = 2$ (respectively, $\gamma_g(T-b) = 2$), a contradiction. Thus, $|A| = |B| = 1$. If $|C| = 1$, then $\gamma_g(T-c) = 4$, again a contradiction. Hence, $C = \emptyset$, and so T is the subdivided star S_3. Thus we may assume that $b' \in B$. It follows that $|A| \le 1$, $1 \le |B| \le 2$, and $1 \le |C| \le 2$. If $A = \emptyset$, then $\gamma_g(T-a) = 2$, a contradiction. Thus, $|A| = 1$. If $|B| = |C| = 2$, then $\gamma_g(T-b) > \gamma_g(T)$, a contradiction. Hence, at least one of B and C has cardinality one. If $|B| = |C| = 1$, then $T = P_7$ and removing a leaf from T results in the path P_6 with $\gamma_g(P_6) = 2$, a contradiction. If $|B| = 1$ and $|C| = 2$, then $T = P_8$; while if $|B| = 2$ and $|C| = 1$, then T is the caterpillar $(1,0,1,0,1)$, as desired.

Finally, assume that $AB = BC = \emptyset$. Since T is a tree, it is connected via edges between vertices in $A \cup B \cup C$. Without loss of generality, assume that $a'b' \in E(T)$, where $a' \in A$ and $b' \in B$. Furthermore, we may assume that $b''c' \in E(T)$, where $b'' \in B$ (note b'' can be b') and $c' \in C$. Then, the vertices of $(A \cup B \cup C) \setminus \{a', b', b'', c'\}$ are leaves of T, for otherwise a cycle is formed. For a vertex $x \in S$, the leaves of X are isolated in $T - x$ and are in every $\gamma_g(T - x)$-set S'. Hence, if x is a strong support vertex of T, then at least two isolated vertices are in S' and at least two additional vertices are in S' to dominate the remaining vertices of $T - x$, a contradiction. It follows that $1 \leq |A| \leq 2$ and $1 \leq |C| \leq 2$. Moreover, $1 \leq |B| \leq 2$ if $b' = b''$, and $2 \leq |B| \leq 3$ if $b' \neq b''$.

Assume that $b' = b''$. If $|A| = |B| = 1$ or $|B| = |C| = 1$, then $\gamma_g(T - b) = 2$, a contradiction. Hence, if $|B| = 1$, then $|A| = |C| = 2$, and so T is the caterpillar $(1, 0, 1, 0, 1)$, and the result holds. If $|A| = |B| = 2$ (respectively, $|B| = |C| = 2$), then $\gamma_g(T - a) > \gamma_g(T) = 3$ (respectively, $\gamma_g(T - c) > \gamma_g(T) = 3$), a contradiction. If $|B| = 2$ and $|A| = |C| = 1$, then T is the subdivided star S_3.

Next, assume that $b' \neq b''$. Recall that $1 \leq |A| \leq 2$, $2 \leq |B| \leq 3$, and $1 \leq |C| \leq 2$. If $|A| = |C| = 2$, then $\gamma_g(T - a) > \gamma_g(T) = 3$, a contradiction. Hence, at least one of A and C has cardinality one. Without loss of generality, let $|A| = 1$. If $|B| = 2$ and $|C| = 1$, then $T = P_7$, which we have seen is not a 3_g-VS tree. If $|B| = |C| = 2$, then $T = P_8$ and the result holds. Thus, we may assume that $|B| = 3$. If $|C| = 2$, then $\gamma_g(T - b) > \gamma_g(T) = 3$, a contradiction. If $|C| = 1$, then T is the caterpillar $(1, 0, 1, 0, 1)$, as desired.

Case 2. $T[S]$ has exactly two edges.

Without loss of generality, let ab and bc be the edges of $T[S]$. Then, $A \cup B \cup C$ is an independent set, and so $A \cup B \cup C$ is a set of leaves in T. If either of A or C is empty, then $\{b, c\}$ (respectively, $\{a, b\}$) is a GDS of T with cardinality less than $\gamma_g(T)$, a contradiction. Thus, we may assume $|A| \geq 1$ and $|C| \geq 1$. Moreover, if $B = \emptyset$, then $\{a, c\}$ is a GDS of $T - b$ with cardinality less than $\gamma_g(T)$, contradicting that T is a 3_g-VS tree. Thus, $|B| \geq 1$. But then Lemma 2 implies that $|A| = |B| = |C| = 1$. Hence, T is the caterpillar $(1, 1, 1)$, and $\{b, c\}$ is a GDS of $T - a'$, where $A = \{a'\}$, with cardinality less than $\gamma_g(T)$, contradicting that T is a 3_g-VS tree.

Case 3. $T[S]$ has exactly one edge.

Let $ab \in E(T)$. Since T is a tree $AB = \emptyset$, and either $BC \neq \emptyset$ or there is an edge between a vertex in C and a vertex in $A \cup B$.

Assume first that $BC \neq \emptyset$. Now the vertices of $A \cup B \cup C$ are leaves in T. If $B = \emptyset$, then $\{a, c\}$ is a GDS of T with cardinality less than $\gamma_g(T)$, a contradiction. Hence, $|B| \geq 1$. If $A = \emptyset$, then $\{b, c\}$ is a GDS of $T - x$ for $x \in BC$, again a contradiction. Hence, $|A| \geq 1$ and $|B| \geq 1$. Lemma 2 implies that $C = \emptyset$ or $|A| = |B| = |C| = 1$. If $C = \emptyset$, then $\{a, b\}$ is a GDS of $T - c$ with cardinality less than $\gamma_g(T)$, a contradiction.

If $|A| = |B| = |C| = 1$, then T is the caterpillar $(1, 1, 0, 1)$. But then $\{a, c\}$ is a GDS of $T - b'$, where $B = \{b'\}$, with cardinality less than $\gamma_g(T)$, a contradiction.

Thus, we may assume that $BC = \emptyset$. Since T is connected, there is an edge between a vertex in C and a vertex in $A \cup B$. Without loss of generality, say that $b' \in B$ is adjacent to $c' \in C$. Also note that since T is a tree, the only edge with both its endpoints in $A \cup B \cup C$ is the edge $b'c'$. Thus every vertex in $(A \cup B \cup C) \setminus \{b', c'\}$ is a leaf in T. If $A = \emptyset$, or if $|B| = |C| = 1$, then $\gamma_g(T) = 2$, a contradiction. Hence, $A \neq \emptyset$, and at least one of b and c has a leaf neighbor in T. If $A = \{a'\}$, then $\{b, c\}$ is a GDS of $T - a'$ with cardinality less than $\gamma_g(T)$, a contradiction. Hence, $|A| \geq 2$. But then $\gamma_g(T - a) \geq \gamma_g(T)$, a contradiction. □

5 Edge Addition

In this subsection, we consider trees whose global domination number stays the same upon the addition of an arbitrary edge. We characterize the 2_g-EPS and 3_g-EPS trees.

Theorem 9 *A tree T is a 2_g-EPS tree if and only if T is one of the paths P_2 and P_4 or a star of order $n \geq 4$.*

Proof. Vacuously, $T = P_2$ is a 2_g-EPS tree. Henceforth, we assume that $n \geq 3$. It is straightforward to check that if T is the path P_4 or a star on $n \geq 4$ vertices, then T is a 2_g-EPS tree.

Assume that T is a 2_g-EPS tree with $\gamma_g(T)$-set S where the sets A, B, and AB are defined in Section 2. By Lemma 3, $AB = \emptyset$ and each of A and B is an independent set. Since $n \geq 3$, at least one of A and B, say A, is not empty. If $|A| = 1$ and B is empty, then $T = P_3$, which is not 2_g-EPS. If $|A| \geq 2$ and $B = \emptyset$, then T is a star of order at least four and the result holds. Hence, we may assume that $A \neq \emptyset$ and $B \neq \emptyset$.

Let $ab \in E(T)$. Then, $A \cup B$ is an independent set, that is, $A \cup B$ is a set of leaves in T. If $|A| = |B| = 1$, then $T = P_4$, and the result holds. Assume, without loss of generality, that $|A| \geq 2$. Then, adding an edge between b and a leaf in A increases the global domination number, contradicting that T is a 2_g-EPS tree.

Thus, we may assume that $ab \notin E(T)$. Since T is connected and $AB = \emptyset$, there is an edge between a vertex $a' \in A$ and a vertex $b' \in B$. Since T is a tree, the only edge with both its endpoints in $A \cup B$ is the edge $a'b'$. Thus every vertex in $(A \cup B) \setminus \{a', b'\}$ is a leaf in T.

Note that $A \neq \emptyset$ and $B \neq \emptyset$ because $a' \in A$ and $b' \in B$. Assume that $|A| \geq 2$. Let $e = ba''$, where $a'' \in A \setminus \{a'\}$. Then, $\gamma_g(T + e) \geq 3$, a contradiction. Hence, $|A| \leq 1$, and analogously, $|B| \leq 1$, which implies $|A| = |B| = 1$, and so $T = P_4$. □

For the purpose of characterizing the 3_g-EPS trees, we define families $\mathcal{T}_p$, $1 \leq p \leq 7$, of caterpillars with codes for positive integers i, j, and k, as follows:

$\mathcal{T}_1$: $(i, 0, j)$, for $i \geq 3$ and $j \geq 3$,

$\mathcal{T}_2$: (i, j, k), for $i \geq 2$, $j \geq 1$, and $k \geq 2$,

$\mathcal{T}_3$: $(i, j, 0, k)$, for $j \geq 2$ and where $i \geq 2$ or $k \geq 2$,

$\mathcal{T}_4$: $(i, 0, j, 0, k)$, for $i \geq 1$, $j \geq 1$, and $k \geq 1$,

$\mathcal{T}_5$: $(i, j, 0, 0, k)$, for $i \geq 2$, $j \geq 1$, and $k \geq 1$,

$\mathcal{T}_6$: $(i, 0, j, 0, 0, k)$, for $i \geq 1$, $j \geq 0$, and $k \geq 1$,

$\mathcal{T}_7$: $(i, 0, 0, j, 0, 0, k)$, for $i \geq 1$, $j \geq 0$, and $k \geq 1$.

Define $\mathcal{H}$ as the family of graphs obtained from the caterpillar $(1, 1, 1)$ with spine x, y, z, adjacent to leaves x', y', z', respectively, by adding $i \geq 0$ new vertices adjacent to x', $j \geq 1$ new vertices adjacent to y', and $k \geq 0$ new vertices adjacent to z'. Figure 2 is an example of a tree $H \in \mathcal{H}$ with $(i, j, k) = (0, 3, 2)$.

Let $\mathcal{F} = \bigcup_{i=1}^{7} \mathcal{T}_i \cup \mathcal{H}$.

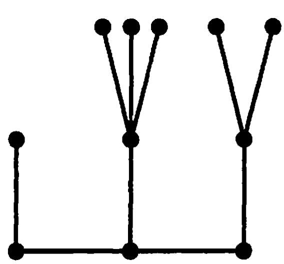

Figure 2: H, $(0, 3, 2)$

Theorem 10 *A tree T is a 3_g-EPS tree if and only if $T \in \mathcal{F}$.*

Proof. We leave it to the reader to check that if T is a tree in $\mathcal{F}$, then T is indeed a 3_g-EPS tree.

Assume that T is a 3_g-EPS tree with $\gamma_g(T)$-set $S = \{a, b, c\}$ and the associated sets as defined in Section 2. By Lemma 4, we have that $ABC = \emptyset$, and at least one of AB, AC, and BC is empty. Without loss of generality, assume that $AC = \emptyset$. Then, by Lemma 4, $|AB| \leq 1$ and $|BC| \leq 1$, and also each of A, B, and C is an independent set.

We note that since T is a tree, $T[S]$ has at most two edges. We consider the cases based on the number of edges in $T[S]$.

Case 1. $T[S]$ has no edges, that is, S is an independent set.

Case 1(a). $|AB| = |BC| = 1$.

Since T is a tree, $A \cup B \cup C$ is an independent set, that is, each vertex in $A \cup B \cup C$ is a leaf in T. If $B = \emptyset$, then $\{a, c\}$ is a GDS of $T + ab$, contradicting that T is a 3_g-EPS tree. Hence, $|B| \geq 1$. If $|A| \geq 1$ and $|C| \geq 1$, then T is the caterpillar $(i, 0, j, 0, k)$, where $i \geq 1$, $j \geq 1$, and $k \geq 1$, and so $T \in \mathcal{T}_4 \subseteq \mathcal{F}$.

Therefore, we may assume that at least one of A and C is empty. Without loss of generality, let $A = \emptyset$. If $B = \{b'\}$, then $\{x, c\}$ is a GDS of $G + xb'$ where $AB = \{x\}$, contradicting that T is a 3_g-EPS tree. Thus, $|B| \geq 2$. If $C = \emptyset$, then $\{x, b\}$ is a GDS of $T + bc$, a contradiction. If $C = \{c'\}$, then $\{b, c'\}$ is a GDS of $T + ba$, again a contradiction. Thus, $|C| \geq 2$, and so T is the caterpillar $(1, j, 0, k)$, where $j = |B| \geq 2$ and $k = |C| \geq 2$. Hence, $T \in \mathcal{T}_3 \subseteq \mathcal{F}$.

Case 1(b). $|AB| = 1$ and $BC = \emptyset$.

Since T is connected and $BC = \emptyset$, there is an edge between a vertex in C and a vertex in $A \cup B$. Without loss of generality, say that $b' \in B$ is adjacent to $c' \in C$. Also note that since T is a tree, the only edge with both its endpoints in $A \cup B \cup C$ is the edge $b'c'$. Thus every vertex in $(A \cup B \cup C) \setminus \{b', c'\}$ is a leaf in T. If $A = \emptyset$, then $\{b, c\}$ is a GDS of $T + ab$, contradicting that T is a 3_g-EPS tree. Hence, $|A| \geq 1$. If $|C| \geq 2$, then T is the caterpillar $(i, 0, j, 0, 0, k)$, where $i = |A| \geq 1$, $j = |B| - 1 \geq 0$, and $k = |C| - 1 \geq 1$. Thus, $T \in \mathcal{T}_6 \subseteq \mathcal{F}$.

Thus, we may assume that $|C| = 1$, that is, $C = \{c'\}$. If $B = \{b'\}$, then $\{a, b'\}$ is a GDS of $G + b'c$, contradicting that T is a 3_g-EPS tree. Thus, $|B| \geq 2$. But then, T is the caterpillar $(i, 0, j, 0, k)$, where $i = |A| \geq 1$, $j = |B| - 1 \geq 1$, and $k = 1$, and so $T \in \mathcal{T}_4 \subseteq \mathcal{F}$.

Case 1(c). $AB = BC = \emptyset$.

Since T is a tree, it is connected via edges between vertices in $A \cup B \cup C$. Without loss of generality, assume that $a'b' \in E(T)$, where $a' \in A$ and $b' \in B$. Furthermore, we may assume that $b''c' \in E(T)$, where $b'' \in B$ (note that b'' can be b') and $c' \in C$. Now all vertices of $(A \cup B \cup C) \setminus \{a', b', b'', c'\}$ are leaves of T, otherwise a cycle is formed. Note that none of A, B, and C is empty because $a' \in A$, $b' \in B$, and $c' \in C$.

First, consider when $b' = b''$. If $|B| \geq 2$, then $T \in \mathcal{H} \subseteq \mathcal{F}$, and the result holds. Thus, we may assume that $|B| = 1$. If $|A| = 1$ (respectively, $|C| = 1$), then $\{a', c\}$ (respectively $\{c', a\}$) is a GDS of $T + a'b$ (respectively, $T + c'b$), a contradiction. Thus, $|A| \geq 2$ and $|C| \geq 2$, and so, T is the caterpillar $(i, 0, 1, 0, k)$, where $i = |A| - 1 \geq 1$ and $k = |C| - 1 \geq 1$. Hence, $T \in \mathcal{T}_4 \subseteq \mathcal{F}$.

Next, assume $b' \neq b''$. Then, $|B| \geq 2$. Assume that $|B| = 2$, that is, $B = \{b', b''\}$. If $|A| = |C| = 1$, then $T = P_7$, which is not a 3_g-EPS tree. Thus, $|A| \geq 2$ or $|C| \geq 2$. Assume, without loss of generality, that $|A| \geq 2$. Then, depending on $|C|$, T is either the caterpillar $(i, 0, 0, 0, 0, 1)$ or the caterpillar $(i, 0, 0, 0, 0, 0, k)$, where $i = |A| - 1 \geq 1$ and $k = |C| - 1 \geq 1$. Therefore, $T \in \mathcal{T}_6 \cup \mathcal{T}_7 \subseteq \mathcal{F}$.

Hence, we may assume that $|B| \geq 3$. Depending on $|A|$ and $|C|$, we have that T is one of the following caterpillars: $(1, 0, j, 0, 1)$, $(1, 0, j, 0, 0, k)$, $(i, 0, 0, j, 0, 1)$, or $(i, 0, 0, j, 0, 0, k)$, where $i = |A| - 1 \geq 1$, $j = |B| - 2 \geq 1$, and $k = |C| - 1 \geq 1$. Hence, $T \in \mathcal{T}_4 \cup \mathcal{T}_6 \cup \mathcal{T}_7 \subseteq \mathcal{F}$.

Case 2. $T[S]$ has exactly two edges.

Without loss of generality, let ab and bc be the edges of $T[S]$. Since T is a tree, $A \cup B \cup C$ is an independent set, and so $A \cup B \cup C$ is a set of leaves in T. If either of A or C is empty, then $\{b, c\}$ (respectively $\{a, b\}$) is a GDS of T with cardinality less than $\gamma_g(T)$, a contradiction. Thus, we may assume $|A| \geq 1$ and $|C| \geq 1$. If $|A| = 1$, say $A = \{a'\}$, then $\{b, c\}$ is a GDS of $T + ba'$, contradicting that T is a 3_g-EPS tree. Thus, we may assume that $|A| \geq 2$, and analogously, $|C| \geq 2$. If $|B| \geq 1$, then T is the caterpillar (i, j, k), where $i \geq 2$, $j \geq 1$, and $k \geq 2$, and so $T \in \mathcal{T}_2 \subseteq \mathcal{F}$.

We may assume that $B = \emptyset$. If $A = \{a', a''\}$, then $\{a', c\}$ is a GDS of $T + a'a''$, a contradiction. Therefore, $|A| \geq 3$, and analogously, $|C| \geq 3$. Then, T is the caterpillar $(i, 0, j)$, where $i \geq 3$ and $j \geq 3$, and so $T \in \mathcal{T}_1 \subseteq \mathcal{F}$.

Case 2. $T[S]$ has exactly one edge.

Without loss of generality, assume that $ab \in E(T)$. Since T is a tree $AB = \emptyset$, and either $BC \neq \emptyset$ or there is an edge between a vertex in C and a vertex in $A \cup B$. Assume first that $BC \neq \emptyset$. Now the vertices of $A \cup B \cup C$

are leaves in T. If $B = \emptyset$, then $\{a, c\}$ is a GDS of T with cardinality less than $\gamma_g(T)$, a contradiction. Hence, $|B| \geq 1$. If $B = \{b'\}$, then $\{a, c\}$ is a GDS of $T + ab'$, contradicting that T is a 3_g-EPS tree. Hence, $|B| \geq 2$.

If $C = \emptyset$, then $\{a, b\}$ is a GDS of $G + bc$, contradicting that T is a 3_g-EPS tree. Hence, $C \neq \emptyset$. Assume that $A = \emptyset$. If $|C| \geq 3$, then T is the caterpillar $(i, 0, j)$, where $i \geq 3$ and $j \geq 3$. Thus, $T \in \mathcal{T}_1 \subseteq \mathcal{F}$. Then, we may assume that $1 \leq |C| \leq 2$. If $C = \{c'\}$, then $\{b, x\}$, where $x \in BC$, is a GDS of $T + bc'$, a contradiction. If $C = \{c', c''\}$, then $\{b, c'\}$ is a GDS of $T + c'c''$, again contradicting that T is a 3_g-EPS tree.

Hence, we may assume that $a' \in A$ and $c' \in C$. If $|A| = |C| = 1$, then $\{b, c'\}$ is a GDS of $T + ba'$, a contradiction. We conclude that $|A| \geq 2$ or $|C| \geq 2$. Therefore, T is the caterpillar $(i, j, 0, k)$, where i, j, and k are positive integers satisfying $j \geq 2$ and at least one of i and k is at least two. Thus, $T \in \mathcal{T}_3 \subseteq \mathcal{F}$.

Finally, assume that $BC = \emptyset$. Since T is connected, $c' \in C$ has a neighbor in $A \cup B$. Without loss of generality, let $b' \in B$ be a neighbor of c'. Since T is a tree, it follows that $b'c'$ is the only edge in $T[V \setminus S]$. If $A = \emptyset$, then $\{b, c\}$ is a GDS of T with cardinality less than $\gamma_g(T)$, a contradiction. Hence, $|A| \geq 1$. If $A = \{a'\}$, then $\{b, c\}$ is a GDS of $T + a'b$, contradicting that T is a 3_g-EPS tree. Hence, $|A| \geq 2$. If $|B| = 1$, then $\{a, c\}$ is a GDS of $T + b'c$, again a contradiction. Thus, $|B| \geq 2$. If $|B| = 2$ and $|C| = 1$, then $\{a, c'\}$ is a GDS of $T + b''c'$, where $b'' \in B \setminus \{b'\}$, a contradiction. Thus, $|B| \geq 3$ or $|C| \geq 2$. If $|C| \geq 2$, then T is the caterpillar $(i, j, 0, 0, k)$, where $i = |A| \geq 2$, $j = |B| - 1 \geq 1$, and $k = |C| - 1 \geq 1$, and so $T \in \mathcal{T}_5 \subseteq \mathcal{F}$. If $|B| \geq 3$, then depending on the $|C|$, T is either the caterpillar $(i, j, 0, 1)$ or the caterpillar $(i, j, 0, 0, k)$, where $i = |A| \geq 2$, $j = |B| - 1 \geq 2$, and $k = |C| - 1 \geq 1$. Hence, $T \in \mathcal{T}_3 \cup \mathcal{T}_5 \subseteq \mathcal{F}$. □

6 Concluding Remarks

We have characterized the k_g-EMS, k_g-VS, and k_g-EPS trees with $k \in \{2, 3\}$. In doing so, we have also characterized the k_g-EMS, k_g-VS, and k_g-EPS graphs G with $k \in \{2, 3\}$ where the complement of G is a tree. It remains open to characterize the stable trees having global domination number at least 4. We conclude with a list of open problems suggested by this work.

1. Characterize the k_g-EMS trees for $k \geq 4$.
2. Characterize the k_g-VS trees for $k \geq 4$.

3. Characterize the k_g-EPS trees for $k \geq 4$.
4. Characterize the k_g-EMS, k_g-VS, and k_g-EPS graphs.

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