On the Null Space Structure Associated with Trees and Cycles

Shaun M. Fallat[a,1,*], Shahla Nasserasr[a,2]

[a] *Department of Mathematics and Statistics, University of Regina, Regina, Sask. Canada, S4S 0A2*

Abstract

In this work, we study the structure of the null spaces of matrices associated with graphs. Our primary tool is utilizing Schur complements based on certain collections of independent vertices. This idea is applied in the case of trees, and seems to represent a unifying theory within the context of the support of the null space. We extend this idea and apply it to describe the null vectors and corresponding nullities of certain symmetric matrices associated with cycles.

Keywords: Cycle, eigenvector, multiplicity, null space, Schur complement, independent vertices, tree, positive semidefinite matrix.
2000 MSC: 05C50, 15A18.

1. Introduction

Studying eigenvalues and eigenvectors associated with graphs has long been a topic of significant interest to both theorists and applied mathematicians (see, for example, the books [3, 4] or the survey paper [5]).

Our primary objective is to focus on the eigenvectors associated with graphs. Under various conditions, such as no zero coordinates, we verify interesting properties on the multiplicities of the corresponding eigenvalues. This idea is not novel and has been used, to a certain degree, in other works such as [1, 7, 14, 15, 17].

* Corresponding Author.
Email addresses: Shaun.Fallat@.uregina.ca (Shaun M. Fallat), Shahla.Nasserasr@uregina.ca (Shahla Nasserasr)

[1] Research supported in part by an NSERC Discovery Research Grant.
[2] Research supported in part by the Pacific Institute for the Mathematical Sciences and NSERC.

Here, we consider a unifying approach by utilizing Schur complements, which seeks to encompass a number of known results along with some new and interesting properties for two families of graphs. It is our hope that these ideas presented here will continue to produce results on the spectra of graphs. The two families we concentrate on here are *trees* and *cycles*. Both are natural starting points for investigating spectra of graph type problems, and both have a rich history in this area, see for instance [2, 8, 12, 14, 15, 17].

We will incorporate the notion of the support of a subspace as a measure of the "structure" for a subspace (see also [14]), which is formally defined in Section 2. This concept of support along with the incorporation of Schur complements (discussed in Section 2) leads to some interesting properties. In Section 3, we apply these techniques to trees and extend them further to the case of cycles in Section 4.

For integers $m, n \geq 1$, the set of real matrices of order $m \times n$ is denoted by $M_{m,n}$, and $M_{n,n}$ is abbreviated to M_n. For a given simple (no loops or multiple edges) graph G on n vertices, $S(G)$ denotes the set of all real symmetric matrices $A = [a_{ij}] \in M_n$ such that for $i \neq j$, $a_{ij} \neq 0$ if and only if $\{i, j\}$ is an edge in G; for each $i = 1, 2, \ldots, n$, a_{ii} is free to be chosen. Using the fact that the main diagonal of $A \in S(G)$ is free, we need only consider homogeneous linear systems instead of the conventional eigen-equations ($Ax = \lambda x$) as A and $A - \lambda I$ both lie in $S(G)$. For a vertex v in G, $N(v)$ denotes the set of vertices of G that are adjacent to v, and $N_p(v)$ denotes the set of pendent vertices of G that are adjacent to the vertex v. Thus, $N_p(v) \subseteq N(v)$, for any vertex v. If i is a pendent vertex (i.e., $|N(i)| = 1$), the unique neighbor of the vertex i, is denoted by i'. If L is a subset of the set of vertices of a graph G, then the graph obtained by deleting all of the vertices of L and their incident edges, is denoted by $G \backslash L$. For a positive integer n, K_n denotes the complete graph (all possible edges) on n vertices.

For $\alpha, \beta \subseteq \{1, \ldots, n\}$, we let $A[\alpha, \beta]$ and $A(\alpha, \beta)$ denote the submatrices of A obtained by keeping and deleting rows indexed by α and columns indexed by β, respectively, where both $A[\alpha, \alpha]$ and $A(\alpha, \alpha)$ are abbreviated to $A[\alpha]$ and $A(\alpha)$, respectively. For $A \in M_n$, the null space of A is denoted by $\mathrm{Nul}(A)$, and the nullity of A is denoted by $\dim\mathrm{Nul}(A)$. For a vector $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \\ x_{r+1} \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$, and $r \in \{1, 2, \ldots, n-1\}$, let $x(r) = \begin{bmatrix} x_{r+1} \\ x_{r+2} \\ \vdots \\ x_n \end{bmatrix}$, and for any index set α, we let $x[\alpha]$ denote the subvector of x with indices from α. If $L \subseteq \{1, 2, \ldots, n\}$, then the complement of L in $\{1, 2, \ldots, n\}$, denoted by L^c, is the set $\{1, 2, \ldots, n\} \backslash L$.

Definition 1. Consider a matrix $A \in M_n$ and let $L \subseteq \{1, 2, \ldots, n\}$. If $A[L]$ is invertible, then the *Schur complement* of $A[L]$ in A, denoted by $A/A[L]$, is defined to be

$$A/A[L] = A(L) - A[L^c, L]A[L]^{-1}A[L, L^c].$$

In particular, if A is symmetric, $L = \{1, 2, \ldots, \ell\}$ for $1 \leq \ell < n$, and A is partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix},$$

where A_{11} is of order $\ell \times \ell$ and invertible, then

$$A/A[L] = A/A_{11} = A_{22} - A_{12}^T A_{11}^{-1} A_{12}.$$

Throughout this work, the rows and columns of the matrix obtained by the Schur complement operation inherit the indexing of the original matrix. The same applies to the labeling of the vertices of the corresponding graphs.

2. Null space of graphs

For a given graph G and $A \in S(G)$, we intend to provide an upper bound on dimNul(A), by using Schur complements of A based on subsets of independent vertices in G. Let $V(G)$ denote the set of vertices of G. For $L \subset V(G)$, we let $N_L(j) = \{s \in V(G) : s \in N(j) \cap L\}$. A subset of the vertices, L, of a graph G is called an *independent set*, if there is no edge between any of the vertices in L.

In this setting, it is conceivable that such bounds could be applied to other related problems, including determination of, or useful bounds on, the maximum nullity over all A in $S(G)$. Our idea is based on the adjacency in G, and while A is fixed, it is meant to be arbitrary (up to certain conditions on the main diagonal). In addition, investigating the nullities of symmetric matrices associated with a graph and with additional constraints on the main diagonal (such as the adjacency matrix), are of current research interest.

Our study begins, naturally, by investigating combinatorial constraints on the nullity for a fixed matrix in $S(G)$ upon consideration of an independent set of vertices. The next result, while natural from a purely matrix theoretic standpoint, does, in turn, provide interesting insight into the influence that the adjacencies in G have on the nullity of a matrix in $S(G)$.

Theorem 2.1. Consider a graph G on n vertices and let $A \in S(G)$. Suppose $L = \{1, \ldots, \ell\}$ is an independent set of vertices in G such that $A[L]$ is invertible, then

i. $\dim\mathrm{Nul}(A) \leq n - \ell$,

ii. $Ax = 0$ if and only if

$$x_i = -\frac{1}{a_{ii}} \sum_{j \in N(i)} a_{ij} x_j, \text{ for } i \in L \tag{1a}$$

and

$$A/A[L]x(\ell) = 0, \tag{1b}$$

iii. $\dim\mathrm{Nul}(A) = \dim\mathrm{Nul}(A/A[L])$.

Proof. Observe that (i) follows as a simple consequence of the rank-nullity theorem in basic linear algebra, and the fact that $\mathrm{rank}A \geq \mathrm{rank}A[L] = \ell$. Furthermore, since $\mathrm{rank}(A/A[L]) = \mathrm{rank}A - \ell$, whenever $A[L]$ is invertible, it follows that

$$\dim\mathrm{Nul}(A/A[L]) = n - \ell - (\mathrm{rank}A - \ell) = n - \mathrm{rank}(A) = \dim\mathrm{Nul}(A),$$

which establishes (iii).

Upon closer inspection of the linear system $Ax = 0$, we see that if $A \in S(G)$ is partitioned in the following form, where $A_{11} = A[L] \in M_\ell$ is diagonal

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix},$$

then, for $i \in L$, the ith row of the linear system $Ax = 0$ is of the form $a_{ii}x_i + \sum_{j \in N(i)} a_{ij}x_j = 0$. Since A_{11} is invertible, $a_{ii} \neq 0$ for all $i \in L$,

$$x_i = -\frac{1}{a_{ii}} \sum_{j \in N(i)} a_{ij} x_j, \text{ for } i \in L, \tag{2}$$

and hence (1a) holds. It is useful to note that since L is an independent set of vertices, $A[L]$ is diagonal and is assumed to be invertible. To show (1b), observe that the (j,k) entry of $A_{12}^T A[L]^{-1} A_{12}$ equals

$$\sum_{\substack{t\in N(j)\cap N(k)\\ t\in L}} \frac{a_{jt}a_{tk}}{a_{tt}}. \tag{3}$$

Thus, the jth row of $A/A[L]x(\ell)$ is

$$\left(a_{jj} - \sum_{t\in N_L(j)} \frac{a_{jt}^2}{a_{tt}}\right) x_j + \sum_{\substack{k=\ell+1\\ k\neq j}}^{n} \left(a_{jk} - \sum_{\substack{t\in N(j)\cap N(k)\\ t\in L}} \frac{a_{jt}a_{tk}}{a_{tt}}\right) x_k. \tag{4}$$

Let r_j denote the jth row of A and let A' be the matrix obtained from A by performing the elementary row operations $-\frac{a_{ji}}{a_{ii}} r_i + r_j \to r_j$, for all $i \in L$ and $j = \ell+1, \ldots, n$, with $a_{ji} \neq 0$. Then, the jth row of $A'x$ is precisely (4). This holds for all $j \in L^c$, and since $A'[L^c, \{1,\ldots,n\}]x(\ell) = 0$, then $A/A[L]x(\ell) = 0$.

For the converse, suppose (1a) and (1b) hold. From the partitioned form of A, assume

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

From (1a), we have $y_1 = 0$. Hence $A_{11}x_1 = -A_{12}x_2$. Using (1b), we have

$$0 = A_{22}x_2 - A_{12}^T A_{11}^{-1} A_{12} x_2 = A_{22}x_2 - A_{12}^T A_{11}^{-1}(-A_{11}x_1) = y_2. \qquad \square$$

For a symmetric matrix $A = [a_{ij}] \in M_n$, the notation $\Gamma(A)$ is used to denote the simple graph on vertices $\{1,\ldots,n\}$, where there is an edge between vertices i and j $(i \neq j)$ if and only if $a_{ij} \neq 0$; a_{ii} is immaterial in the determination of $\Gamma(A)$. For a given graph G on n vertices and $A \in S(G)$, let L_1 be a maximal set of independent vertices of G where the corresponding diagonal entries of A are nonzero. Apply Theorem 2.1 to obtain the matrix $A/A[L_1]$ of order $n - |L_1|$ with the same nullity as of A. This process can be repeated until either the resulting matrix is of order 2×2, and hence has nullity at most two, or the resulting matrix is of the form

$$\begin{bmatrix} 0 & B \\ B^t & C \end{bmatrix},$$

where a maximal set of independent vertices, say L_s with $|L_s| = s$, is labeled first, $0 \in M_s$, and C has zero diagonal entries. In the case of trees and cycles, we can label the vertices (isolating independent sets) so that the resulting graph at each step of this method is well-defined, and is in fact of the same type, and therefore, a similar argument applies on the resulting graph. For example, if T is a tree, then the vertices of T can be labeled so that under the conditions of Theorem 2.1, $T\backslash L$

is a tree, and for any $A \in S(T)$, the matrix $A/A[L]$ is in $S(T\backslash L)$.

3. Trees

In this section, Theorem 2.1 is applied to the case of trees to present alternate and elementary proofs for several known results. This is one motivation for applying our unifying approach with regard to the spectra of acyclic matrices.

Suppose T is a tree on $n \geq 3$ vertices, that is, T is a connected graph on n vertices with no cycles. Let L_1 be the set of pendent vertices of T, and L_i denote the set of pendent vertices of $T\backslash \cup_{j=1}^{i-1} L_j$. Label the vertices of T so that the vertices of L_1 come first, and the vertices in L_{i-1} come before the vertices in L_i. With this labeling, if $A \in S(T)$, then A is of the following form where the block $A_{ij} \in M_{|L_i|\times|L_j|}$ denotes the adjacency of vertices in L_i to the vertices in L_j

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1k} \\ A_{12}^T & A_{22} & \dots & A_{2k} \\ \vdots & & \ddots & \vdots \\ A_{1k}^T & A_{2k}^T & \dots & A_{kk} \end{bmatrix}.$$

We call such a labeling a *pendent labeling*. Since each L_i, for $i = 1, \dots, k-1$, is the set of pendent vertices of a tree, all of the diagonal blocks A_{ii}, $i = 1, \dots, k-1$, are in fact diagonal matrices. The last set of vertices, L_k, obtained from the pendent labeling of a tree T, contains either one or two vertices. Otherwise, there is a tree with $n \geq 3$ vertices where all of the vertices are pendent, which is impossible. Throughout this work, trees are labeled with the pendent labeling unless otherwise stated.

Lemma 3.1. Consider a tree T with the pendent labeling and let $A \in S(T)$. If $A[L_1]$ is invertible, then

1. $A[L_1^c, L_1]A[L_1]^{-1}A[L_1, L_1^c]$ is a diagonal matrix.
2. The diagonal entry of $A[L_1^c, L_1]A[L_1]^{-1}A[L_1, L_1^c]$ corresponding to the vertex t of the original tree T is equal to $\sum_{i \in N_p(t)} \frac{a_{ti}^2}{a_{ii}}$.

Proof.

1. Since a set of pendent vertices is an independent set, Theorem 2.1 can be applied. On the other hand, if t is a pendent vertex, then $t \notin N(j) \cap N(k)$ for any $j, k, j \neq k$. This implies that all of the terms in (3) are zero, except possibly the diagonal entries.

2. The sum $\sum_{i \in N_p(t)} \frac{a_{ti}^2}{a_{ii}}$ is equal to the sum in (3) in the proof of Theorem 2.1, for $j = k$. □

The next result is a basic consequence of Lemma 3.1 and will be an important fact needed in this section.

Corollary 3.2. Under the hypotheses of Lemma 3.1, the matrices $A/A[L_1]$ and $A(L_1)$ have equal off-diagonal entries. Therefore, $A/A[L_1] \in S(T\backslash L_1)$.

Let $A^{(0)} = A$, $A^{(1)} = A/A[L_1]$, and $A^{(s)} = A^{(s-1)}/A^{(s-1)}[L_s]$ for $s = 1, \ldots, k-1$. Using Corollary 3.2, $A^{(s)} \in S(T_s)$, for each $s \in \{0, 1, \ldots, k-1\}$, where T_s is obtained from the tree T by deleting the vertices in $\cup_{i=1}^{s} L_i$. We let $T_0 = T$.

Lemma 3.3. Suppose a tree T has vertices partitioned as $L_1, \ldots, L_k$ with the pendent labeling. If all upper left blocks $A^{(s)}[L_{s+1}]$ of the Schur complements $A^{(s)}$, $s = 0, 1, \ldots, k-1$, are invertible, then $\dim\text{Nul}(A) = 0$. That is, A is invertible.

Proof. Consider the equation $Ax = 0$. By repeating the proof of Theorem 2.1 for each $A^{(s)}[L_{s+1}]$, $s = 1, 2, \ldots, k-1$, we know that all of the entries of x can be written as a linear combination of the entries of x corresponding to the vertices in L_k. But L_k has either one or two entries. So the last equation $A^{(k-1)}x[L_k] = 0$ implies that either $x_n = 0$, or $x_{n-1} = x_n = 0$, depending on the size of L_k. By backward substitution, this implies that $x = 0$. Thus, $Ax = 0$ has only the trivial solution, which means A is invertible. □

For a vector $x = [x_i] \in \mathbb{R}^n$, the *support* of x, denoted by sup x, is the set of indices $i \in \{1, 2, \ldots, n\}$, where $x_i \neq 0$. If $S \subset \mathbb{R}^n$ is a set of vectors, then the *support* of S, denoted by sup S, is the set of indices $i \in \{1, 2, \ldots, n\}$, where $x_i \neq 0$ for some $x \in S$. It is not difficult to verify that, if S is a subspace of $\mathbb{R}^n$ and sup $S = \{1, \ldots, n\}$, then S must contain a vector x in which each coordinate of x is nonzero. Such a vector is called a *totally nonzero* vector. Our first fact deals with the case that there is a totally nonzero null vector for a matrix in $S(T)$.

Theorem 3.4. [14, Thm. 1] For a tree T on $n \geq 3$ vertices and $A \in S(T)$, if sup Nul$(A) = \{1, 2, \ldots, n\}$, then $\dim\text{Nul}(A) = 1$.

Proof. Consider a pendent labeling for T and suppose $x \in \text{Nul}(A)$ is a totally nonzero vector. There is such a vector since sup Nul$(A) = \{1, 2, \ldots, n\}$. Using the pendent labeling, for each $i \in L_1$, the ith row of the equation $Ax = 0$ is of the form $a_{ii}x_i + a_{ii'}x_{i'} = 0$. Since $x_i a_{ii'} x_{i'} \neq 0$, we have $a_{ii} \neq 0$, so the diagonal submatrix $A[L_1]$ has nonzero diagonal entries, and therefore, it is invertible, and $A^{(1)} = A/A[L_1] \in S(T\backslash L_1) = S(T_1)$, by Corollary 3.2. Since x is totally nonzero, repeating the above argument for $s = 2, \ldots, k-2$, we deduce that each of the submatrices $A^{(s)}[L_{s+1}]$ of the Schur complements $A^{(s)}$, $s = 0, 1, \ldots, k-2$ are invertible, and $A^{(s)} \in S(T_s)$, by Corollary 3.2. Moreover, using Theorem 2.1 repeatedly, the entries of x corresponding to the vertices in $L_1 \cup \ldots \cup L_{k-1}$ are all nonzero scalar multiples of the entries of L_k. So $\dim\text{Nul}(A) \leq |L_k| \leq 2$. If $|L_k| = 1$, using the fact that A is not invertible, the proof is complete. If $|L_k| = 2$, then the last system of equations obtained by the above process is $A^{(k-1)}x[L_k] = 0$, where $A^{(k-1)}$ is a 2×2 nonzero matrix and $x[L_k]$ is a totally nonzero vector with 2 entries. Therefore, the entries of $x[L_k]$ are multiples of each other, that is $x_{n-1} = \alpha x_n$, for some $\alpha \neq 0$. Thus every entry of x can be written as a nonzero scalar multiple of x_n, which implies $\dim\text{Nul}(A) = 1$. □

See also [14] for additional results making use of the notion of support on the null space of trees and the corresponding nullities.

According to the previous theorem totally nonzero null vectors forces main diagonal entries associated with pendent vertices to be nonzero. As an example consequence, it follows that the (0,1) adjacency matrix of a tree can never have a totally nonzero null vector. In particular, there must exist an i with $1 \leq i \leq n$ such that $x_i = 0$ for all null vectors x. Equivalently, the ith standard basis vector must be in the range of this (0,1) adjacency matrix associated with a tree. Along these lines, it makes sense to consider those $A \in S(T)$ with nonzero main diagonal. For instance, the subcollection of positive semidefinite matrices in $S(T)$.

Using Theorem 2.1, restricted to the positive semidefinite case in $S(T)$ allows us to recover a known fact which previously relied on rather powerful results (see also [9]). Recall that a real symmetric matrix is *positive semidefinite* if it has nonnegative eigenvalues.

Corollary 3.5. Suppose T is a tree on $n \geq 3$ vertices. If $A \in S(T)$ is a positive semidefinite matrix, then $\dim\text{Nul}(A) \leq 1$.

Proof. Let $A \in S(T)$ be a positive semidefinite matrix. It is clear that if $a_{ii} = 0$, then $a_{ij} = a_{ji} = 0$, for all $j = 1, \ldots, n$. Therefore, the diagonal entries of A cannot be zero (otherwise there would be zero entries corresponding to some edges of T, which contradicts $A \in S(T)$). This implies that $A[L_1]$ is invertible, and thus, the Schur complement $A/A[L_1]$, is also a positive semidefinite matrix (see [10, Thm. 7.7.6]). Now, using Theorem 2.1 repeatedly, we have $\dim\text{Nul}(A) \leq 1$. □

If A is a symmetric matrix with an eigenvalue λ, the algebraic multiplicity of λ is denoted by $\text{mult}_A(\lambda)$. Now that the positive semidefinite case has been studied, we have the next result as a basic consequence.

Corollary 3.6. Let $A \in S(T)$ with spectrum $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. If $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$, then $\text{mult}_A(\lambda_1) = \text{mult}_A(\lambda_n) = 1$.

Proof. The matrix $A' = A - \lambda_1 I \in S(T)$ has eigenvalues $0 \leq \lambda_2 - \lambda_1 \leq \ldots \leq \lambda_n - \lambda_1$, so it is positive semidefinite. Therefore, using Corollary 3.5, $\text{mult}_A(\lambda_1) = \text{dimNul}(A') = 1$. To show that $\text{mult}_A(\lambda_n) = 1$, note that the matrix $A'' = -A + \lambda_n I \in S(T)$ has eigenvalues $0 \leq \lambda_n - \lambda_{n-1} \leq \ldots \leq \lambda_n - \lambda_1$, so it is positive semidefinite, and hence Corollary 3.5 implies the desired result. □

It has long been known that the eigenvalues associated with a path are distinct (see [6]). Recall that a *path*, P_n, consists of vertices $v_1, v_2, \ldots, v_n$ and edges $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n$. In fact, it is easy to verify that if 1 and n are pendent vertices of P_n, then for any $A \in S(P_n)$, from the equations $Ax = 0$ and $x_1 = 0$, it follows that $x = 0$. Hence using null vectors only, we may also deduce that for any $A \in S(P_n)$, the eigenvalues of A have multiplicity one (that is, they are all simple).

Restricting matrices in $S(T)$ to contain a totally nonzero null vector implies that the dimension of the null space was at most one. From this we were able to recover some results on the nullities of positive definite matrices associated with trees.

A natural place to move forward is to consider other graphs with large collections of independent vertices, such as cycles. However, even for cycles, there exist matrices with nullity more than one, but they have totally nonzero null vectors.

4. Cycles

A *cycle* C_n consists of vertices $v_1, v_2, \ldots, v_n$ and edges $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n$, v_nv_1. In this section, we review a well known result on the nullity of matrices in $S(C_n)$. In the case for which all of the diagonal entries of $A \in S(C_n)$ are zero, the structure of the null vectors of $A \in S(C_n)$ are explicitly described. For the cases for which all of the diagonal entries are nonzero, or there are both zero and nonzero diagonal entries, an algorithm is provided that will implicitly determine the structure of the null vectors. It will be advantageous to introduce different labeling schemes on the vertices for each case.

The first type of labeling of C_n, is the usual method of ordering the vertices of a cycle. That is, they are ordered as $v_1, v_2, \ldots, v_n$, where $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n$, v_nv_1 are the edges of C_n. We call this labeling a *consecutive labeling*. The next result offers a basic upper bound on the nullity of a cycle, which is known, see for instance [5]. However, our approach relies heavily on eigenvector structure as in Section 3.

Theorem 4.1. For the cycle C_n on n vertices, $\dim\mathrm{Nul}(A) \leq 2$. Therefore, the multiplicity of each eigenvalue of a cycle is at most two.

Proof. Suppose C_n is labeled by the consecutive labeling, and let $A = [a_{ij}] \in S(C_n)$. Consider the linear system of equations $Ax = 0$ for a vector $x \in \mathbb{R}^n$. Then, the submatrix $A' = A[\{2,3,\ldots,n-1\},\{1,2,\ldots,n\}] \in M_{n-2,n}$ has linearly independent rows, therefore, it has rank $n-2$. Hence $\dim\mathrm{Nul}(A) \leq 2$. □

In order to explicitly describe the null vectors of $A \in S(C_n)$, we consider two base cases: (i) all of the diagonal entries of A are zero (discussed in Subsection 4.1); (ii) all of the diagonal entries of A are nonzero (discussed in Subsection 4.2). Each case is described below and is then extended algorithmically, with some limitations, to the case of arbitrary main diagonal in Subsection 4.3.

4.1. Zero Diagonal Entries

Using the consecutive labeling, the following results characterize the eigenvectors of a matrix $A \in S(C_n)$, when all of the diagonal entries are assumed to be zero. In the following results, addition in subscripts is taken modulo n.

Lemma 4.2. Let $n = 2k \geq 4$, and consider $A \in S(C_n)$, with $a_{ii} = 0$, for all $i = 1,2,\ldots,n$. Then A is singular if and only if

$$\frac{a_{12}}{a_{1,2k}} = (-1)^k \prod_{i=1}^{k-1} \frac{a_{2i,2i+1}}{a_{2i+1,2i+2}}$$

and in this case $\dim\mathrm{Nul}(A) = 2$. Moreover, every nonzero null vector x of A is either totally nonzero, or satisfies the following property

$$x_i = 0 \iff x_{i+1} \neq 0, \text{for all } i = 1,\ldots,n.$$

Proof. Suppose C_n is labeled consecutively. Using rows $r_2, r_4, \ldots, r_{2k-2}, r_{2k}$, and rows $r_1, r_3, \ldots, r_{2k-1}$, respectively, each of the components of a null vector x can be written in terms of x_1 or x_2 as

$r_2:$	$x_3 = -\frac{a_{12}}{a_{23}}x_1$	$r_1:$	$x_{2k} = -\frac{a_{12}}{a_{1,2k}}x_2$
$r_4:$	$x_5 = \frac{a_{34}a_{12}}{a_{45}a_{23}}x_1$	$r_3:$	$x_4 = -\frac{a_{23}}{a_{34}}x_2$
$\vdots$		$r_5:$	$x_6 = \frac{a_{45}a_{23}}{a_{56}a_{34}}x_2$
$r_{2k-2}:$	$x_{2k-1} = (-1)^{k-1}\prod_{i=1}^{k-1}\frac{a_{2i-1,2i}}{a_{2i,2i+1}}x_1$	$\vdots$	
$r_{2k}:$	$x_{2k-1} = -\frac{a_{1,2k}}{a_{2k-1,2k}}x_1$	$r_{2k-1}:$	$x_{2k} = (-1)^{k-1}\prod_{i=1}^{k-1}\frac{a_{2i,2i+1}}{a_{2i+1,2i+2}}x_2.$

If A is not invertible, then there is a nonzero null vector of A, thus either $x_1 \neq 0$ or $x_2 \neq 0$, which in turn implies either rows r_{2k-2} and r_{2k} give the same value for x_{2k-1} or rows r_1 and r_{2k-1} give the same value for x_{2k}, or perhaps both equations hold. That is, at least one of the equalities below must hold

$$\frac{a_{1,2k}}{a_{2k-1,2k}} = (-1)^k \prod_{i=1}^{k-1} \frac{a_{2i-1,2i}}{a_{2i,2i+1}} \tag{5}$$

or

$$\frac{a_{12}}{a_{1,2k}} = (-1)^k \prod_{i=1}^{k-1} \frac{a_{2i,2i+1}}{a_{2i+1,2i+2}}. \tag{6}$$

However, by direct computations we can show that (5) and (6) are, in fact, equivalent. Hence if A is not invertible, then (5) (equivalently (6)) holds. Clearly, if (5) holds, then A is not invertible as well. The equivalence between (5) and (6), also implies that if A is not invertible, then every null vector of A is of the form $x = x_1 g_1 + x_2 g_2$, where

$$g_1 = \begin{bmatrix} 1 \\ 0 \\ -\frac{a_{12}}{a_{23}} \\ 0 \\ \vdots \\ 0 \\ -\frac{a_{1,2k}}{a_{2k-1,2k}} \\ 0 \end{bmatrix}, g_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -\frac{a_{23}}{a_{34}} \\ \vdots \\ 0 \\ \frac{a_{12}}{a_{1,2k}} \end{bmatrix}.$$

Since g_1, g_2 are linearly independent vectors, the nullity of A is two. To show the last statement, we consider the following cases.

1. If $x_1 = x_2 = 0$, then $x = 0$.
2. If $x_1 = 0, x_2 \neq 0$ (or $x_2 = 0, x_1 \neq 0$), then the vector x is a scalar multiple of g_2 (or g_1), which has the properties in the statement of the theorem.
3. If $x_1 x_2 \neq 0$, then by the structure of vectors g_1 and g_2, the vector x is totally nonzero. □

By a similar method to the proof of Lemma 4.2, we can extend this to the odd case as follows (using the same notation as above)

$$\frac{(-1)^{k-1} a_{1,2k+1}}{a_{12}} \prod_{i=1}^{k} \frac{a_{2i-1,2i}}{a_{2i,2i+1}} = \frac{(-1)^k a_{1,2k+1}}{a_{2k,2k+1}} \prod_{i=1}^{k-1} \frac{a_{2i+1,2i+2}}{a_{2i,2i+1}}.$$

This implies that $(-1)^{k-1} = (-1)^k$, which is a contradiction. Therefore, $x_1 = x_2 = 0$. This implies that $x = 0$, which means A is invertible (see also [13], where cycle expansions of determinants could also be considered).

Lemma 4.3. Let $n = 2k+1$, $k \geq 1$, $A \in S(C_n)$, and $a_{ii} = 0, i = 1, 2, \ldots, n$. Then A is invertible, that is $\dim\mathrm{Nul}(A) = 0$.

Describing the nullities of matrices in $S(C_n)$ with zero main diagonal via explicit descriptions of the their null vectors leads to a better understanding of the null space structure of cycles and perhaps even beyond to include graphs that contain cycles (such as unicyclic graphs), and perhaps to an elementary proof of the converse to Corollary 3.5.

Note that, in the case of n even, when the nullity of A is two, there is a totally nonzero null vector. That is, when $\sup\mathrm{Nul}(A) = \{1, 2, \ldots, n\}$, we have $\dim\mathrm{Nul}(A) = 2$. Thus, Theorem 3.4, which holds for trees, is not valid for the case of an even cycle.

4.2. Nonzero Diagonal Entries

Suppose G is a graph and let v be a vertex of G. For $A \in S(G)$, with $a_{vv} \neq 0$, the graph $\Gamma(A/a_{vv})$ is well defined where A/a_{vv} is the Schur complement of a_{vv} in A. Suppose u and w are adjacent to v. Then, using (3) in the proof of Theorem 2.1, the (u, w) entry of $A/a_{vv} = [a_{ij}^{(v)}]$ equals $a_{uw} - \frac{a_{uv}a_{vw}}{a_{vv}}$. Figures 1 and 2 describe the relationship between the graph G and $\Gamma(A/a_{vv})$ for two neighbors of v; adjacencies among non-neighbors of v are unchanged from G to $\Gamma(A/a_{vv})$.

The following remarks are immediate consequences of the above definition.

1. In general, the entry $a_{uw}^{(v)} = a_{uw} - \frac{a_{uv}a_{vw}}{a_{vv}}$ can be both zero and nonzero depending on the matrix A. Therefore, if there is an edge between u and w in G, there may or may not be an edge between u and w in the graph $\Gamma(A/a_{vv})$. That is, it is not possible to predict the graph $\Gamma(A/a_{vv})$ in general.

2. If $N(v)$, the set of neighbors of v, forms an independent set in G, then $\Gamma(A/a_{vv})$ is the graph obtained from G by deleting v and all of its incident edges, and adding all edges between pairs of vertices in $N(v)$.

Regarding Remark 1, when studying some specific families of matrices, it is possible to predict the graph $\Gamma(A/A[L])$. For instance, if A is a weighted or generalized Laplacian matrix associated with a graph G (see [16] for a definition), then using Lemma 4.5, the (u, w) entry of $A/A[L]$ is nonzero, if u and w have a common neighbor $v \in L$. Therefore, for any vertex $v \in L$, every pair of vertices

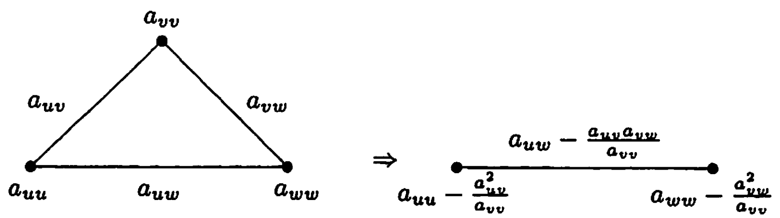


Figure 1: $\Gamma(A/a_{vv})$, when there is an edge between two neighbors of v

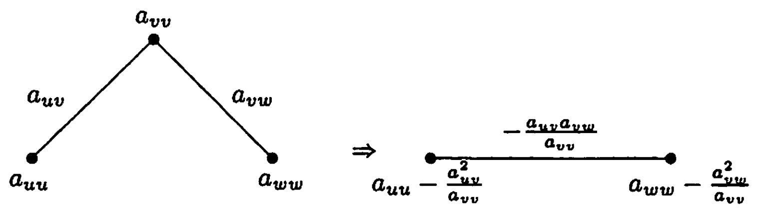


Figure 2: $\Gamma(A/a_{vv})$, when there is no edge between two neighbors of v

$u, w \in N(v)$ are connected by an edge in $\Gamma(A/A[L])$. This follows from the following fact.

Proposition 4.4. Let G be a graph on n vertices and suppose v_1 and v_2 are not adjacent. Then for $A \in S(G)$ with $a_{v_1v_1}a_{v_2v_2} \neq 0$,

$$(A/A[v_1])/(A/A[v_1])[v_2] = A/A[v_1, v_2].$$

Proof. Since v_1 and v_2 are not adjacent, the vertices of G can be labeled so that A is of the following form

$$A = \begin{bmatrix} a_{v_1v_1} & 0 & r_1 \\ 0 & a_{v_2v_2} & r_2 \\ r_1^T & r_2^T & A_{22} \end{bmatrix}.$$

It is straightforward to see that

$$(A/A[v_1])/(A/A[v_1])[v_2] = A_{22} - \frac{1}{a_{v_1v_1}} r_1^T r_1 - \frac{1}{a_{v_2v_2}} r_2^T r_2 = A/A[v_1, v_2]. \quad \square$$

Lemma 4.5. Suppose $A \in M_n$ is a (weighted) Laplacian matrix associated with a connected graph G on $n \geq 4$ vertices, and let L be an independent set of vertices in G. Then, for $v \in L$ and $u, w \in N(v)$, the (u, w) entry of $A/A[L]$ is nonzero. Moreover, $A/A[L]$ is a Laplacian matrix associated with $\Gamma(A/A[L])$.

Proof. Consider a labeling of vertices for which the vertices v, u, w appear in the given order. We know that A is an irreducible singular M-matrix (see [11, Section 2.5]), so rank$A = n-1$, and every proper principal submatrix of A is a nonsingular M-matrix. Suppose $L = \{v\}$ and $u, w \in N(v)$. Then the principal submatrix of A

$$A[v,u,w] = \begin{bmatrix} a_{vv} & a_{vu} & a_{vw} \\ a_{uv} & a_{uu} & a_{uw} \\ a_{wv} & a_{wu} & a_{ww} \end{bmatrix}$$

is an invertible M-matrix, and hence is inverse positive (see [11, Section 2.5]). In particular, the $(2,3)$ entry of $A[v,u,w]^{-1}$ is positive, i.e. $\frac{(-1)}{\det A[v,u,w]}(a_{vv}a_{uw} - a_{uv}a_{vw}) > 0$. Thus, $a_{uw} - \frac{a_{uv}a_{vw}}{a_{vv}} < 0$, which means the (u, w) entry of $A/A[L]$ is negative. Straightforward computations show that the diagonal entry in each row (or column) of $A/A[L]$ equals to the sum of the off diagonal entries of that row (or column), thus $A/A[L]$ is a Laplacian matrix associated with $\Gamma(A/A[L])$. If L has more than one vertex, then using Proposition 4.4 repeatedly completes the proof. □

For any weighted Laplacian matrix A of a given graph G, Spielman in [16] studied algorithms that provide a fast method to solve the linear system of equations $Ax = b$. Various methods are investigated in [16] which either provide the exact solution for the linear system or an approximation of the solution. One method to give the exact solution of the linear system $Ax = b$ is Gaussian elimination. In order to perform a fast Gaussian elimination on a positive semidefinite matrix A, one may find a permutation matrix P such that Cholesky factorization of P^TAP may be computed easily; see [16]. We note that finding such a permutation matrix is equivalent to a re-labeling of the vertices of G. Lemma 4.5 implies that performing Cholesky factorization on a maximal independent set of vertices in G, reduces the homogeneous linear system $Ax = 0$ to one of smaller size, namely $A/A[L]x(L) = 0$. Moreover, in this case $A/A[L]$ is a Laplacian matrix for the graph $\Gamma(A/A[L])$. This graph is obtained from G by deleting the vertices of L and joining every pair of vertices $\{u, w\}$ in $G \backslash L$ with a common neighbor $v \in L$ by an edge. Clearly, this process can be repeated on the graph $\Gamma(A/A[L])$.

Returning to cycles, consider a vertex v in the cycle C_n, $n \geq 4$ with $a_{vv} \neq 0$. Suppose u, w are neighbors of v. Since there is no edge between u and w, using Figure 2 and Theorem 2.1, the symmetric matrix $A/a_{vv} = [a_{ij}^{(v)}]$ is in $S(C_{n-1})$

with

$$a_{ij}^{(v)} = \begin{cases} -\frac{a_{uv}a_{vw}}{a_{vv}}, & \text{if } (i,j) = (u,w) \\ a_{uu} - \frac{a_{uv}^2}{a_{vv}}, & \text{if } (i,j) = (u,u) \\ a_{ww} - \frac{a_{vw}^2}{a_{vv}}, & \text{if } (i,j) = (w,w) \\ a_{ij}, & \text{otherwise.} \end{cases}$$

Using Theorem 2.1, this process can be performed simultaneously on a maximal independent set of vertices, say L_1, with $a_{vv} \neq 0$, for all $v \in L_1$, and from the above identities the resulting graph is a cycle, that is $\Gamma(A/A[L_1]) = C_{n-|L_1|}$. Now, if $n \geq 4$, and all of the diagonal entries of A are nonzero, then the vertices can be labeled so that Theorem 2.1 can be applied repeatedly on the largest invertible diagonal submatrix of each resulting matrix. We call such a labeling an *alternate labeling*, and define it as follows: consider the vertices of C_n lying on the circumference of a circle where the edges are part of the circumference. Begin by labeling some vertex of C_n, 1. Moving in a clockwise direction from vertex i, name the second unlabeled vertex $i+1$. Repeat this process until all of the vertices but one are labeled. The last unlabeled vertex is then labeled n. Figure 3 shows an alternate labeling of C_7. Having this labeling, a matrix $A \in S(G)$ can

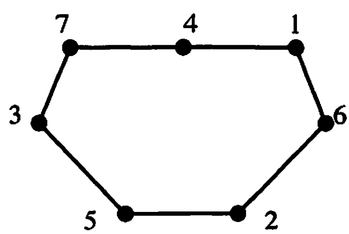


Figure 3: An alternate labeling of C_7

be partitioned in the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix}$$

where $A_{11} \in M_{\lfloor \frac{n}{2} \rfloor}$ is a diagonal matrix. Let $L = \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$. In the case of nonzero diagonal entries, $A_{11} = A[L]$ is invertible, thus using Theorem 2.1, $\text{dimNul}(A) = \text{dimNul}(A/A[L])$. Moreover, in the equation $Ax = 0$, each x_i, $i \in L$ is a linear combination of two variables x_r and x_s where $r, s \in \{\lfloor \frac{n}{2} \rfloor + 1, \ldots, n\}$, and the linear system $Ax = 0$, can be reduced to $A/A[L]x(\lfloor \frac{n}{2} \rfloor) = 0$. Now, if the diagonal entries of the matrix $A/A[L]$ are nonzero, the above process can be repeated since the inherited labeling is precisely in the desired from. Furthermore, if at each stage the resulting diagonal block corresponding to an independent set of vertices is invertible, then Theorem 2.1 can be reapplied until the resulting graph has either 3 or 4 vertices, depending on n. This provides an algorithm to compute the nullity of a matrix $A \in S(C_n)$, when the diagonal entries of both the original matrix and resulting matrices at each step are nonzero, an example for this case is the class of positive semidefinite matrices in $S(C_n)$. The algorithm has the following properties: (i) Theorem 2.1 can be applied to a maximal independent set of vertices; (ii) if the resulting graph at each step has at least 3 vertices, then it is still a cycle; (iii) the resulting graph at each step, has already been labeled so that Theorem 2.1 can be applied to a maximal independent set of vertices.

Note that, here at the first step, only $\lfloor \frac{n}{2} \rfloor$ of the diagonal entries corresponding to the vertices in L need to be nonzero, and not all of the vertices of C_n. Similarly, at each step we only need nonzero diagonal entries for the corresponding independent set.

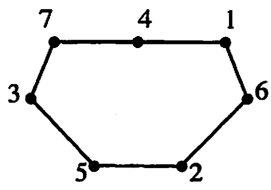
4.3. Both Zero and Nonzero Diagonal Entries

In the previous two subsections, the null vector structure and corresponding nullity for the extreme cases of either all main diagonal entries are zero or all main diagonal entries are nonzero were described.

A natural progression would then be to consider the situation of a matrix in $S(C_n)$ with arbitrary main diagonal. We could proceed as in Subsection 4.2 by reducing the size of the cycle according to independent vertices with nonzero main diagonal entries (we alluded to this in Subsection 4.2, when we noted the algorithm would still apply as long as there were a sufficient number of nonzero main diagonal entries).

The procedure, unfortunately, is tedious to carry out since at each stage we need to check for nonzero diagonal entries associated with collections of independent vertices, and such conditions cannot be verified combinatorially before hand. One strategy would then be to reduce as much as possible the number of vertices to produce a smaller sized cycle, and not disturbing the nullity. From here another inductive technique may be applied. To this end, we are still pursuing other algorithmic ideas in hopes of getting a better handle on nullities with an eye towards maximum nullity of a graph. The following example illustrates the algorithm in the case of both zero and and nonzero diagonal entries for cycles.

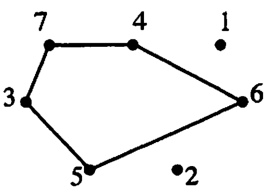
Example 4.6. Consider $A \in S(C_7)$ with three nonzero diagonal entries lying on a path on three vertices of C_7. Following the alternate labeling given in Figure 3, and without loss of generality suppose $a_{11}a_{22}a_{66} \neq 0$. Then $A \in S(C_7)$ is of the following from



$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \left[\begin{array}{ccccccc} a_{11} & 0 & 0 & a_{14} & 0 & a_{16} & 0 \\ 0 & a_{22} & 0 & 0 & a_{25} & a_{26} & 0 \\ 0 & 0 & 0 & 0 & a_{35} & 0 & a_{37} \\ a_{14} & 0 & 0 & 0 & 0 & 0 & a_{47} \\ 0 & a_{25} & a_{35} & 0 & 0 & 0 & 0 \\ a_{16} & a_{26} & 0 & 0 & 0 & a_{66} & 0 \\ 0 & 0 & a_{37} & a_{47} & 0 & 0 & 0 \end{array}\right] \end{array}.$$

Figure 4: C_7 and $A \in S(C_7)$ with $a_{11}a_{22}a_{66} \neq 0$

For $L_1 = \{1, 2\}$, we have



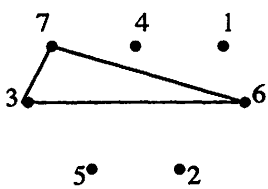
$$A^{(1)} = \begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccc} 3 & 4 & 5 & 6 & 7 \end{array} \\ \left[\begin{array}{ccccc} 0 & 0 & a_{35} & 0 & a_{37} \\ 0 & -\frac{a_{14}^2}{a_{11}} & 0 & -\frac{a_{14}a_{16}}{a_{11}} & a_{47} \\ a_{35} & 0 & -\frac{a_{25}^2}{a_{22}} & -\frac{a_{25}a_{26}}{a_{22}} & 0 \\ 0 & -\frac{a_{14}a_{16}}{a_{11}} & -\frac{a_{25}a_{26}}{a_{22}} & a_{66} - \frac{a_{16}^2}{a_{11}} - \frac{a_{26}^2}{a_{22}} & 0 \\ a_{37} & a_{47} & 0 & 0 & 0 \end{array}\right] \end{array}$$

Figure 5: $L_1 = \{1, 2\}$ and $A^{(1)} = A/A[L_1] \in S(C_5)$

with

$$\begin{aligned} x_1 &= -\tfrac{a_{14}}{a_{11}}x_4 - \tfrac{a_{16}}{a_{11}}x_6 \\ x_2 &= -\tfrac{a_{25}}{a_{22}}x_5 - \tfrac{a_{26}}{a_{22}}x_6. \end{aligned}$$

Then $L_2 = \{4, 5\}$ and



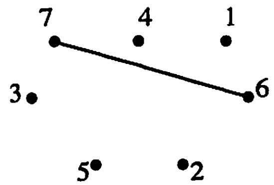
$$A^{(2)} = \begin{array}{c} \\ 3 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccc} 3 & 6 & 7 \end{array} \\ \left[\begin{array}{ccc} \frac{a_{22}a_{35}^2}{a_{25}^2} & -\frac{a_{26}a_{35}}{a_{25}} & a_{37} \\ -\frac{a_{26}a_{35}}{a_{25}} & a_{66} & -\frac{a_{16}a_{47}}{a_{14}} \\ a_{37} & -\frac{a_{16}a_{47}}{a_{14}} & \frac{a_{11}a_{47}^2}{a_{14}^2} \end{array}\right] \end{array}$$

Figure 6: $L_2 = \{4, 5\}$ and $A^{(2)} = A^{(1)}/A^{(1)}[L_2] \in S(C_3)$

with

$$\begin{aligned} x_4 &= -\tfrac{a_{16}}{a_{14}}x_6 + \tfrac{a_{11}a_{47}}{a_{14}^2}x_7 \\ x_5 &= \tfrac{a_{22}a_{35}}{a_{25}^2}x_3 - \tfrac{a_{26}}{a_{25}}x_6. \end{aligned}$$

Finally, $L_3 = \{3\}$ and



$$A^{(3)} = \begin{array}{c} \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cc} \quad 6 \quad & \quad 7 \quad \end{array} \\ \begin{bmatrix} a_{66} - \frac{a_{26}^2}{a_{22}} & -\frac{a_{16}a_{47}}{a_{14}} + \frac{a_{25}a_{26}a_{37}}{a_{22}a_{35}} \\ -\frac{a_{16}a_{47}}{a_{14}} + \frac{a_{25}a_{26}a_{37}}{a_{22}a_{35}} & \frac{a_{11}a_{47}^2}{a_{14}^2} - \frac{a_{25}^2 a_{37}^2}{a_{22}a_{35}^2} \end{bmatrix} \end{array}$$

Figure 7: $L_3 = \{3\}$ and $A^{(3)} = A^{(2)}/A^{(2)}[L_3] \in S(K_2)$

with

$$x_3 = \frac{a_{25}a_{26}}{a_{22}a_{35}}x_6 - \frac{a_{25}^2 a_{37}}{a_{22}a_{35}^2}x_7.$$

By substituting backward we have

$$\begin{aligned} x_5 &= \frac{a_{37}}{a_{35}}x_7 \\ x_2 &= -\frac{a_{25}a_{37}}{a_{22}a_{35}}x_7 - \frac{a_{26}}{a_{22}}x_6 \\ x_1 &= -\frac{a_{47}}{a_{14}}x_7. \end{aligned}$$

Hence every null vector of A is of the from

$$x = \begin{bmatrix} -\frac{a_{47}}{a_{14}}x_7 \\ -\frac{a_{26}}{a_{22}}x_6 - \frac{a_{25}a_{37}}{a_{22}a_{35}}x_7 \\ \frac{a_{25}a_{26}}{a_{22}a_{35}}x_6 - \frac{a_{25}^2 a_{37}}{a_{22}a_{35}^2}x_7 \\ -\frac{a_{16}}{a_{14}}x_6 + \frac{a_{11}a_{47}}{a_{14}^2}x_7 \\ \frac{a_{37}}{a_{35}}x_7 \\ x_6 \\ x_7 \end{bmatrix} = x_6 \begin{bmatrix} 0 \\ -\frac{a_{26}}{a_{22}} \\ \frac{a_{25}a_{26}}{a_{22}a_{35}} \\ -\frac{a_{16}}{a_{14}} \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_7 \begin{bmatrix} -\frac{a_{47}}{a_{14}} \\ -\frac{a_{25}a_{37}}{a_{22}a_{35}} \\ -\frac{a_{25}^2 a_{37}}{a_{22}a_{35}^2} \\ \frac{a_{11}a_{47}}{a_{14}^2} \\ \frac{a_{37}}{a_{35}} \\ 0 \\ 1 \end{bmatrix}.$$

Solving the equation $A^{(3)}x = 0$ given by

$$\begin{bmatrix} a_{66} - \frac{a_{26}^2}{a_{22}} & -\frac{a_{16}a_{47}}{a_{14}} + \frac{a_{25}a_{26}a_{37}}{a_{22}a_{35}} \\ -\frac{a_{16}a_{47}}{a_{14}} + \frac{a_{25}a_{26}a_{37}}{a_{22}a_{35}} & \frac{a_{11}a_{47}^2}{a_{14}^2} - \frac{a_{25}^2 a_{37}^2}{a_{22}a_{35}^2} \end{bmatrix} \begin{bmatrix} x_6 \\ x_7 \end{bmatrix} = 0,$$

will completely describe the null space of A.

Acknowledgment: We thank both the Editor and the anonymous referee for carefully reading of a previous version and for the constructive suggestions which led to significant improvements of this manuscript.

References

[1] S. Akbari, E. Ghorbani, and A. Mahmoodi. Nowhere-Zero Eigenvectors of Graphs. *Linear and Multilinear Algebra*, DOI:10.1080/03081087.2012.675331, 2012.

[2] F. Barioli and S. M. Fallat. On two conjectures regarding an inverse eigenvalue problem for acyclic symmetric matrices. *Electronic Journal of Linear Algebra*, 11: 41–50, 2004.

[3] R.A. Brualdi and H.J. Ryser. *Combinatorial Matrix Theory*, Cambridge, New York, 1991.

[4] D. Cvetkovic, M. Doob and H. Sachs, *Spectra of Graphs*, Academic Press, New York, 1979.

[5] S. Fallat and L. Hogben. The minimum rank of symmetric matrices described by a graph: A survey. *Linear Algebra and its Applications* 426: 558–582, 2007.

[6] M. Fiedler. A characterization of tridiagonal matrices. *Linear Algebra and Its Applications*, 2: 191–197, 1969.

[7] M. Fiedler. Eigenvectors of Acyclic Matrices. *Czechoslavak Mathematical Journal*, 25: 607–618, 1969.

[8] L. Hogben. Spectral graph theory and the inverse eigenvalue problem of a graph. *Electronic Journal of Linear Algebra*, 14:12-31, 2005.

[9] H. van der Holst. Graphs whose positive semi-definite matrices have nullity at most two. *Linear Algebra and its Applications*, 375: 1–11, 2003.

[10] R.A. Horn, and C.R. Johnson, *Matrix Analysis*, Cambridge University Press, New York, 1985.

[11] R. A. Horn and C. R. Johnson, Topics in Matrix Analysis, Cambridge University Press, New York, 1991.

[12] C. R. Johnson and A. Leal Duarte. The maximum multiplicity of an eigenvalue in a matrix whose graph is a tree. *Linear and Multilinear Algebra* 46: 139–144, 1999.

[13] J. Maybee, D. Olesky, P. van den Driessche, and G. Weiner, Matrices, Digraphs, and Determinants. *SIAM J. Matrix Anal. Appl.* 10: 500–519, 1989.

[14] P.M. Nylen, Null-space structure of tree-patterned matrices. *Linear Algebra and Its Applications* 279: 153-161, 1998.

[15] S. Parter. On the eigenvalues and eigenvectors of a class of matrices. *Journal of the Society for Industrial and Applied Mathematics*, 8 : 376–388, 1960.

[16] D. A. Spielman. Algorithms, Graph Theory and Linear Equations in Laplacian Matrices *Proceedings of the International Congress of Mathematicians*, Hyderabad, India, 2010.

[17] G. Wiener. Spectral multiplicity and splitting results for a class of qualitative matrices. *Linear Algebra and Its Applications*, 61: 15–29, 1984.