On Domination and Digital Convexity Parameters

Ortrud R. Oellermann*
Department of Mathematics and Statistics, University of Winnipeg
515 Portage Avenue, Winnipeg, MB, R3B 2E9, Canada
Email: `o.oellermann@uwinnipeg.ca`

On the occasion of Kieka Mynhardt's Special Birthday

Abstract

Suppose V is a finite set and C a collection of subsets of V that contains $\emptyset$ and V and is closed under taking intersections. Then the ordered pair (V, C) is called a *convexity* and the elements of C are referred to as *convex sets*. For a set $S \subseteq V$, the *convex hull* of S relative to C, denoted by $CH_C(S)$, is the smallest convex set containing S. The *Caratheodory number*, relative to a given convexity, is the smallest integer c such that for any subset S of V and any point $v \in CH_C(S)$ there is a subset F of S with $|F| \leq c$ such that $v \in CH_C(F)$. A subset X of V is said to admit a *Radon partition* if X can be partitioned into two sets X_1, X_2 such that $CH_C(X_1) \cap CH_C(X_2) \neq \emptyset$. The *Radon number* of a convexity is the smallest integer r (if it exists) such that every subset X of V with at least r elements admits a Radon partition. A set S of vertices in a graph G with vertex set V is *digitally convex* if for every vertex $v \in V$, $N[v] \subseteq N[S]$ implies $v \in S$. A set X is *irredundant* if $N[x] - N[X - \{x\}] \neq \emptyset$ for all $x \in X$. The maximum cardinality of an irredundant set is the *upper irredundance number* of G, denoted by $IR(G)$. A set X of vertices in a graph G is a *local irredundant* set for a vertex v of G, if for each $x \in X$, $x \in N[v] - N[X - \{x\}]$ or x is adjacent with a vertex of $N[v] - N[X - \{x\}]$. The *upper local irredundance number* of v, denoted by $l_{IR}(v)$, is the maximum cardinality of a local irredundant set for v. The *upper local irredundance number* of a graph G, denoted by $l_{IR}(G)$, is defined as $l_{IR}(G) = max\{l_{IR}(v) \mid v \in V\}$. We show that for the digital convexity of a graph G: (i) the Caratheodory number equals $l_{IR}(G)$ and

---

*Supported by an NSERC grant CANADA.

(ii) the Radon number is bounded above by $IR(G)+1$ and below by $\beta(G)+1$ where $\beta(G)$ is the independence number of G. For the latter result it is shown that there are classes of graphs for which the lower (respectively, upper) bound is attained while the difference between the upper irredundance number and the independence number can be made as large as we wish. Moreover, there are graphs for which the Radon number of the digital convexity lies strictly between the bounds given in (ii) and does not equal one more than the upper domination number.

Keywords: Digital convexity; Caratheodory number; Radon number; Helly number; Domination; Independence; Irredundance; Local irredundance
AMS subject classification: 05C69

1 Introduction

Fascination with the applications of domination-type parameters has led to an abundance of results in this area. Variations on the classical domination parameters have been introduced in order to adapt domination to specific situations. As a result domination theory has blossomed into a major area of graph theory. An excellent account on domination parameters has been compiled into two survey-type volumes edited by Haynes, Hedetmniemi and Slater [10, 11]. It was proposed in [10] that frameworks, into which domination fits naturally, be studied. It is suggested that: "Each framework provides a unifying theory and a generalized viewpoint that enables one to identify and define new parameters, see relationships among these parameters and develop insight into the computational problems involving these parameters." Numerous frameworks such as hypergraphs and integer programming were discussed in [10]. It was suggested that "other frameworks for the domination number undoubtedly exist, each of which provides a rich understanding of the concept of domination in graphs." Since then other such frameworks have been studied such as the one introduced by Chartrand, Haynes, Henning, and Zhang [3] in terms of stratified graphs. More recently Cáceres, Márquez, Morales, and Puertas [2] introduced another framework for domination based on the 'digital convexity' of a graph which had previously been motivated by its application to image processing and studied in [13, 14]. Several Helly-type parameters are introduced in [2] and their relation to domination parameters studied.

In addition to the Helly number, both the Caratheodory number and the Radon number of Euclidean space are well understood. The object of this paper is to formulate these two parameters for the digital convexity of a graph and to study their connection with domination-type parameters.

2 Preliminaries

For graph theory terminology not given here we follow [4].

2.1 Domination

Let G be a graph with vertex set V and edge set E. The *neighbourhood* of a vertex v of G, denoted by $N(v)$, consists of all vertices adjacent with v in G and the *closed neighbourhood* of v, denoted by $N[v]$, is the set $N(v) \cup \{v\}$. The closed neighbourhood of a set S of vertices is defined as $N[S] = \cup\{N[v] | v \in S\}$. A set S of vertices is a *dominating set* if $N[S] = V$. We use $N^2(v)$ to denote the set of vertices, different from v, that are distance at most 2 from v. The smallest cardinality of a dominating set is the *domination number* of G and is denoted by $\gamma(G)$. The maximum cardinality of a minimal dominating set is the *upper domination number* denoted by $\Gamma(G)$.

A set of vertices is *independent* if no two vertices in the set are adjacent. The *independence number* of G, denoted by $\beta(G)$, is the maximum cardinality of an independent set. A dominating set that is independent is an *independent dominating set*. An independent set of cardinality $\beta(G)$ is necessarily also a dominating set. The smallest cardinality of an independent dominating is the independent domination number and is denoted by $i(G)$.

If S is a minimum dominating set, then for every $v \in S$, $N[v] - N[S - \{v\}] \neq \emptyset$. We say that a vertex $w \in N[v] - N[S - \{v\}]$ is a *private neighbour* of v *relative to* S. A set X is *irredundant* if every $x \in X$ has a private neighbour relative to X, i.e., $N[x] - N[X - \{x\}] \neq \emptyset$ for all $x \in X$. So every minimal dominating set is an irredundant set. The smallest cardinality of a maximal irredundant set is called the *lower irredundance number* of G, denoted by $ir(G)$, and the maximum cardinality of an irredundant set is the *upper irredundance number* of G, denoted by $IR(G)$. Well-known relationships between these invariants first appeared in [5] and can be summarized by the following string of inequalities known as the *domination-chain*:

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta(G) \leq \Gamma(G) \leq IR(G). \qquad (1)$$

It was further shown in [1] that $\gamma(G)/2$ is a lower bound for $ir(G)$ and in [8] that $n - \delta(G)$ is an upper bound for $IR(G)$. Necessary and sufficient conditions are established in [6] for the existence of a graph whose upper and lower domination, independence, and irredundance numbers are six given positive integers.

2.2 Convexity

Suppose V is a finite set and $\mathcal{C}$ a collection of subsets of V that contains $\emptyset$ and V and is closed under taking intersections. Then the ordered pair $(V, \mathcal{C})$ is called a *convexity* and the elements of $\mathcal{C}$ are referred to as *convex sets*. For a set $S \subseteq V$, the *convex hull* of S, denoted by $CH_{\mathcal{C}}(S)$, is the smallest convex set containing S. An extensive treatment of abstract convex structures appears in [15]. Euclidean convexity is usually defined in terms of intervals and has a natural extension to graphs (see [7]).

More recently the digital convexity of a graph was introduced in [14], mainly as a tool in image processing to filter digital images, and studied further in [13]. Let G be a graph with vertex set V and edge set E. The *digital convex hull operator* from 2^V to 2^V, denoted by $CH_{\mathcal{D}}$, maps a subset S of V to the set $CH_{\mathcal{D}}(S) = \{u \in V \mid N[u] \subseteq N[S]\}$. It is not difficult to see that the digital convex hull operator is a closure operator since it satisfies the following axioms:

- $X \subseteq CH_{\mathcal{D}}(X)$ for all $X \subseteq V$;
- $X \subseteq Y$ implies $CH_{\mathcal{D}}(X) \subseteq CH_{\mathcal{D}}(Y)$ for all $X, Y \subseteq V$.
- $CH_{\mathcal{D}}(CH_{\mathcal{D}}(X)) = CH_{\mathcal{D}}(X)$ for all $X \subseteq V$.

If $S \subseteq V$ satisfies $S = CH_{\mathcal{D}}(S)$ we say S is *digitally convex*. The collection of all digitally convex sets contains $\emptyset$ and V and is closed under intersections. Hence this collection of subsets of V is a convexity which we will refer to as the *digital convexity*.

A family $\mathcal{C}$ of sets has the *Helly property* if any subfamily $\mathcal{C}'$ whose elements are pairwise intersecting has non-empty intersection. Suppose $\mathcal{C}$ is a non-empty family of subsets of a finite set V: the *Helly number* of $\mathcal{C}$ is the least positive integer h such that every h-wise intersecting subfamily of $\mathcal{C}$ has non-empty intersection. Note that the Helly number is well-defined, since the integer $h = |\mathcal{C}|$ has the required property. Moreover, it is not difficult to prove that if the Helly number is at least 2, then the Helly number is the largest integer h such that there exists a subfamily $\mathcal{C}'$ of $\mathcal{C}$ with $|\mathcal{C}'| = h$ such that $\mathcal{C}'$ has empty intersection, but every proper subfamily of $\mathcal{C}'$ has non-empty intersection. Suppose that $\mathcal{C}$ is a convexity of a graph G. Then the Helly number of G, with respect to this convexity, is denoted by $h_{\mathcal{C}}(G)$. A set A is *Helly independent* if

$$\cap_{a \in A} CH_{\mathcal{C}}(A - \{a\}) = \emptyset.$$

The Helly number is the minimum cardinality of a maximal Helly independent set. The maximum cardinality of a maximal Helly independent set is called the *upper Helly number* of a graph and is denoted by $HI_C(G)$. In [2] it is shown that for the digital convexity $\mathcal{D}$, $HI_{\mathcal{D}}(G) = \Gamma(G)$. Surprisingly $\gamma(G)$ is not equal to $h_{\mathcal{D}}(G)$. It is shown that $h_{\mathcal{D}}(G) \leq i(G)$ but that neither $\gamma(G)$ nor $ir(G)$ is a lower or upper bound for $h_{\mathcal{D}}(G)$.

3 The Caratheodory Number of the Digital Convexity

For a convexity (V, C) on a finite set V, the *Caratheodory number*, relative to this convexity, is the smallest integer c such that for any subset S of V and any point $v \in CH_C(S)$ there is a subset F of S with $|F| \leq c$ such that $v \in CH_C(F)$. Thus the Caratheodory number of a convexity is the largest integer c such that there is a set S of c points and a point $v \in CH_C(S)$ such that $v \notin CH_C(F)$ for all $c-1$ subsets F of S.

Let G be a graph with vertex set V and let $\mathcal{D}$ be the collection of the digitally convex sets of G. Then the Caratheodory number of the digital convexity of G is denoted by $c_{\mathcal{D}}(G)$. It is not difficult to see (i) that $c_{\mathcal{D}}(K_n) = 1$ for all positive integers n, (ii) that $c_{\mathcal{D}}(C_n) = 2$ for all integers $n \geq 4$ and (iii) that $c_{\mathcal{D}}(T) = \Delta(T)$ for any tree T with maximum degree $\Delta(T)$.

In order to describe the Caratheodory number of the digital convexity for a general graph we introduce the following domination-type concepts. A set X of vertices in a graph G is a *local irredundant* set of a vertex v of G, if for each $x \in X$, $x \in N[v] - N[X - \{x\}]$ or x is adjacent with a vertex of $N[v] - N[X - \{x\}]$. Then every maximal local irredundant set X of v dominates $N[v]$, i.e., $N[v] \subseteq N[X]$. The *upper local irredundance number* of v, denoted by $l_{IR}(v)$, is the maximum cardinality of a local irredundant set for v. The *upper local irredundance number* of a graph G, denoted by $l_{IR}(G)$, is defined as $l_{IR}(G) = max\{l_{IR}(v) \mid v \in V\}$. The next theorem establishes a relationship between the Caratheodory number of the digital convexity and the upper local irredundance number of a graph.

Theorem 3.1. *Let G be a graph with vertex set V. Then $c_{\mathcal{D}}(G) = l_{IR}(G)$.*

Proof. Let $c = c_{\mathcal{D}}(G)$. Then there exists a set X of c points and a vertex $v \in CH_{\mathcal{D}}(X)$ such that for each $x \in X$, $v \notin CH_{\mathcal{D}}(X - \{x\})$, i.e., $N[v] \not\subseteq N[X - \{x\}]$ for all $x \in X$. Hence for each $x \in X$, there is an $x' \in N[v]$ such that either $x' = x$ and x' is not adjacent with any vertex of $X - \{x\}$ or $xx' \in E$ but x' is not adjacent with any vertex of $X - \{x\}$, i.e., $x' \in N[v] - N[X - \{x\}]$. So X is a local irredundant set of v. Hence $c \leq l_{IR}(G)$.

Let $l = l_{IR}(G)$. Then there is a vertex v such that $l_{IR}(v) = l$. So there is a set X of l vertices of G such that for each $x \in X$, $x \in N[v] - N[X - \{x\}]$

or x is adjacent with a vertex of $N[v] - N[X - \{x\}]$. So $N[v] \not\subseteq N[X - \{x\}]$ for all $x \in X$. Hence $v \notin CH_{\mathcal{D}}(X - \{x\})$ for each $x \in X$. So $c_{\mathcal{D}}(G) \geq l$. □

4 The Radon Number of the Digital Convexity

Let $(V, \mathcal{C})$ be a convexity. A subset X of V is said to admit a *Radon partition* if X can be partitioned into two sets X_1, X_2 such that $CH_{\mathcal{C}}(X_1) \cap CH_{\mathcal{C}}(X_2) \neq \emptyset$. The *Radon number* of a convexity space $(V, \mathcal{C})$ is the smallest integer r (if it exists) such that every subset X of V with at least r elements admits a Radon partition. Let $r_{\mathcal{C}}(G)$ be the Radon number of a graph G with vertex set V and convex sets $\mathcal{C}$. To simplify our discussion we will refer to the elements of the partition as the red set and the blue set respectively.

We now turn our attention to the digital convexity of a graph. If $G \cong \overline{K}_n$, then the Radon number of G is not defined. If G has edges, then the Radon number of G is defined since the vertex set of G can be partitioned into a red set and a blue set such that at least one vertex lies in the convex hull of both the red and blue set: to see this let v be a vertex that is not an isolated vertex. Place v in the red set and all the neighbours of v into the blue set and assign the remaining vertices randomly to either the red or blue set. Then v lies in the digital convex hull of both the red and blue set. It is not difficult to see that the Radon number of the complete graph K_n for $n \geq 2$ is 2. Moreover, for $n \geq 3$, the Radon number of the n-cycle is $\lfloor \frac{n}{2} \rfloor + 1$. The next result give bounds on the Radon number of the digital convexity of a graph.

Theorem 4.1. *Let G be a graph with edges. Then the Radon number of the digital convexity of G satisfies:*

$$\beta(G) + 1 \leq r_{\mathcal{D}}(G) \leq IR(G) + 1.$$

Proof. We show first that $r_{\mathcal{D}}(G) \geq \beta(G) + 1$. Let X be a maximum independent set of vertices of G. Suppose that X has a Radon partition X_1, X_2 where the vertices in X_1 will be referred to as the red vertices and those in X_2 as the blue vertices. Then $CH_{\mathcal{D}}(X_1) \cap CH_{\mathcal{D}}(X_2) \neq \emptyset$. Let $v \in CH_{\mathcal{D}}(X_1) \cap CH_{\mathcal{D}}(X_2)$. We show first that v belongs to either the red or the blue set. Suppose this is not the case. Since $N[v] \subseteq N[X_1]$ and $N[v] \subseteq N[X_2]$, v is adjacent with both a red vertex, call it x_1, and a blue vertex, call it x_2. Since $N[v] \subseteq X_2$ it is necessarily the case that x_1 is adjacent with a blue vertex. This is not possible since the union of the red and blue vertices form the independent set X. Hence v belongs to X, say v is a red vertex. Since $N[v]$ is contained in the closed neighbourhood of the blue vertices, i.e., in $N[X_2]$, v must be adjacent with a blue vertex. This is not possible since $\{v\} \cup X_2$ is an independent set of vertices.

For the upper bound let X be a set of $IR(G) + 1$ vertices. Then X is not an irredundant set. So X contains a vertex v with the property that v does not have a private neighbour relative to X. So $N[v] \subseteq N[X - \{v\}]$. In particular v is adjacent with some vertex of X and each neighbour of v is either in X or is adjacent with a vertex of X. If we assign v to the red set X_1 and each vertex of $N^2(v) \cap X$ to the blue set X_2, then $N[v] \subseteq N[X_1]$ and $N[v] \subseteq N[X_2]$. Hence $v \in CH_{\mathcal{D}}(X_1) \cap CH_{\mathcal{D}}(X_2)$; completing the proof of the upper bound. □

It was shown in [12] that if G is a chordal graph, then $\beta(G) = IR(G)$. This result and the above thus yield the following.

Corollary 4.2. *If G is a chordal graph with edges, then $r_{\mathcal{D}}(G) = \beta(G) + 1$.*

Corollary 4.3. *If T is a non-trivial tree, then $r_{\mathcal{D}}(T) = \beta(T) + 1$.*

The next result addresses the sharpness of the bounds in Theorem 4.1. To this end we consider the *generalized Petersen graph* $P(n, k)$ defined as follows:

$$V(P(n,k)) = \{a_i, b_i | 0 \le i \le n-1\},$$
$$E(P(n,k)) = \{a_i a_{i+1}, a_i b_i, b_i b_{i+k} | 0 \le i \le n-1\}.$$

The cycle $a_0 a_1 \ldots a_{n-1}$ will be referred to as the *outer cycle* of the generalized Petersen graph. It was shown in [9], that for every positive integer s, $\beta(P(6s, 2s)) = 5s - 1$. The graph $P(6, 2)$ is shown in Fig. 1.

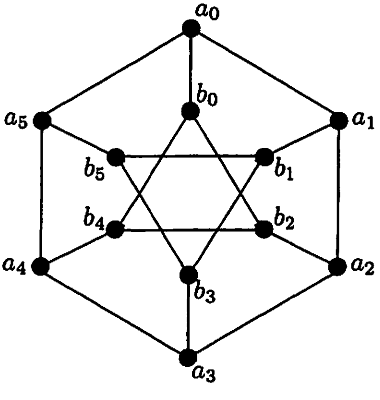

Figure 1: The Generalized Petersen graph $P(6, 2)$.

Theorem 4.4. *For every positive integer M there is*

1. *a graph G such that $r_{\mathcal{D}}(G) = \beta(G) + 1$ and $IR(G) - \beta(G) \geq M$*
2. *a graph H such that $r_{\mathcal{D}}(H) = IR(H) + 1$ and $IR(H) - \beta(H) \geq M$.*

Proof. (1) We show that there are graphs for which the lower bound of Theorem 4.1 is attained while the difference between the upper and lower bounds given in this theorem can be as large as we wish. Let $t \geq 4$ be an integer and let G_t be the graph obtained from the Cartesian product $K_t \times K_2$ by deleting an edge that joins vertices from each of the copies of K_t. Then $\beta(G_t) = 2$ and $IR(G_t) = t - 1$. If X is a set of two vertices from distinct copies of K_t, then X does not have a Radon partition. So $r_{\mathcal{D}}(G_t) \geq 3$. If X is any set of three vertices, then X contains at least two vertices from the same copy of K_t. Let x, y be two such vertices and let z be the vertex of degree $t-1$ in the same copy of K_t as x and y (possibly z equals x or y). Assign x to the red set X_1 and y to the blue set X_2 and the remaining vertex is assigned to either X_1 or X_2. Then $N[z]$ is contained in both $N[X_1]$ and $N[X_2]$, i.e., $z \in CH_{\mathcal{D}}(X_1) \cap CH_{\mathcal{D}}(X_2)$. Hence X has a Radon partition. So $r_{\mathcal{D}}(G_t) = 3$. Thus these graphs achieve the lower bound of Theorem 4.1. Moreover the difference between the Radon number and the upper irredundance number of these graphs can be made arbitrarily large by choosing t large enough. Thus the difference between the upper bound of Theorem 4.1 and the Radon number can be arbitrarily large.

(2) We show here that there are graphs for which the upper bound of Theorem 4.1 is attained while the difference between the upper and lower bounds of this theorem can be made as large as we wish. We begin by showing that the upper irredundance number of $P(6s, 2s) = 6s$. To see this observe first of all that if X consists of the vertices on the outer cycle, then X is an irredundant set since each vertex a_i of X has a private neighbour b_i relative to X. So $IR(P(6s, 2s)) \geq 6s$. Observe that for $0 \leq i \leq 2s - 1$ the induced subgraph $N_i = \langle\{a_i, b_i, a_{i+2s}, b_{i+2s}, a_{i+4s}, b_{i+4s}\}\rangle$ is a net, see Fig 2. Moreover these $2s$ nets are vertex and edge disjoint. If X is a set of at least $6s + 1$ vertices, then at least one of these N_i's contains at least

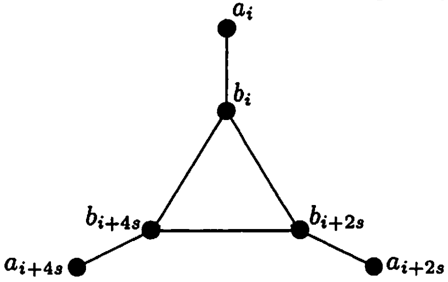

Figure 2: The net.

four vertices. Consequently X has a vertex that does not have a private neighbour relative to X. So $IR(P(6s,2s)) \leq 6s$.

By Theorem 4.1, $r_{\mathcal{D}}(P(6s,2s)) \leq 6s+1 = IR(P(6s,2s))+1$. To see that $6s+1$ is also a lower bound for $r_{\mathcal{D}}(P(6s,2s))$ let X be the set of vertices of the outer $6s$-cycle. Then X does not have a Radon partition. To see this suppose that X_1, X_2 is a partition of X into red and blue vertices, respectively. If $v \in V(P(6s,2s)) - X$, i.e., if v is not on the outer cycle, then v cannot be adjacent with a red and a blue vertex since v is adjacent with exactly one vertex in X. If $v \in X$, then the unique neighbour of v in $V(G) - X$ cannot be adjacent with both a red and a blue vertex. So no vertex of $V(G)$ can be in $CH_C(X_1) \cap CH_C(X_2)$. Hence $r_{\mathcal{D}}(G) \geq 6s+1$. By choosing s large enough the difference between the independence number and irredundance number of $P(6s,2s)$ can be made as large as we wish. Thus the difference between the lower bound of Theorem 4.1 and the Radon number can be arbitrarily large for graphs belonging to the class of generalized Petersen graphs of the type $P(6s,2s)$.

□

For both the graphs G_t and the generalized Petersen graphs $P(6s,2s)$ described in the proof of the previous theorem, the digital Radon number is one more than the upper domination number. It is thus natural to ask if this is the case for all non-empty graphs. We show that this need not be the case by showing that there are graphs for which the digital Radon number exceeds the upper domination number plus 1.

Let G be the graph shown in Fig. 3. So $V(G) = \{1,2,3,4,5,\} \cup \{a,b,c,d,r,\ s,t,x,y,z\}$ and the vertices of G are joined in such a way that $K = \langle\{3,4,5\}\rangle \cong K_3$, $L = \langle\{a,b,c,d\}\rangle \cong K_4$, $M = \langle\{r,s,t,x,y,z\}\rangle \cong K_6$ and $N = \langle\{v,1,2\}\rangle \cong K_3$; and the remaining edges of G are in $\{vb, vc, vd, 1x,\ 2y, 3r, 4s, 5t, b3, b4, c4, c5, d3, d5\}$. Since $\{1,3,a,z\}$ is an independent set $\Gamma(G) \geq \beta(G) \geq 4$. Let S be any minimal dominating set. If $|S| > 4$, then S contains at least two vertices from one of the four cliques K, L, M, or N. Moreover, S must contain at least one vertex from M; otherwise z would not be dominated, and one vertex from L; otherwise a would not be dominated. Now S cannot contain two vertices of N; otherwise, 1 or 2, say 1, is in S and does not have a private neighbour in S, contradicting the minimality of S. Similarly S cannot contain more than one vertex of K. So S contains at least two vertices of L or at least two vertices of M.

Suppose S contains at least two vertices of L. Then $a \notin S$; otherwise a does not have a private neighbour. So S contains at least two vertices from $\{b,c,d\}$. We may assume $b, c \in S$; the other cases can be argued similarly. Then S does not contain any vertex from K; otherwise neither b nor c has a private neighbour. So S contains at least two vertices of M neither of which can be z. Since 3 is the private neighbour of b and 4 the private neighbour

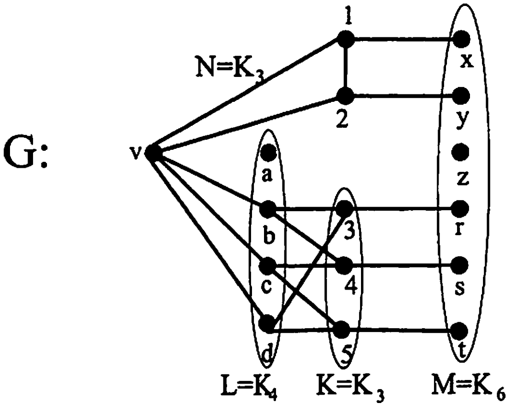

Figure 3: A graph G for which $r_{\mathcal{D}}(G) \neq \Gamma(G) + 1$

of c, neither r nor s belongs to S. Since $c5 \in E(G)$, $t \notin S$; otherwise t does not have a private neighbour. Hence $x, y \in S$. However, then S does not contain any vertex of N; otherwise neither x nor y has a private neighbour. Thus $|S| \leq 4$, a contradiction.

Thus S contains at most one vertex from L. Hence S contains at least two vertices from M. Suppose that S contains a vertex from K. Then S does not contain any of the vertices in $\{r, s, t\}$ since these vertices would not have a private neighbour. We may assume $3 \in S$. Then x, y are the only vertices of S that are in M and since each of x and y has a private neighbour, S contains no vertex of N. However, then $|N| \leq 4$. Suppose now that S contains no vertex from K. Then S contains at least three vertices from M. If S contains either x or y, then S does not contain any vertex of N. Since S is a dominating set, S must contain one of b, c or d, say b. But then S cannot contain r or s. Thus S contains at most four vertices. Hence $\Gamma(G) = 4$.

Since $\{1, 2, 3, 4, 5, a\}$ is an irredundant set, $IR(G) \geq 6$. We now show that the Radon number of G is 6. Let $X = \{1, 2, 3, 4, 5\}$. Let X_1, X_2 be any partition of X. We will refer to the vertices in X_1 as the red vertices and those in X_2 as the blue vertices. We show that no vertex of G can belong to $I = CH_{\mathcal{D}}(X_1) \cap CH_{\mathcal{D}}(X_2)$. Since $z \notin N[X]$, no vertex of M belongs to I. Similarly no vertex of L belongs to I. Since each vertex of X has a private neighbour (relative to X) that belongs to M, such a private neighbour cannot be adjacent with both a red and a blue vertex. Hence no vertex of X is in I. It remains to show that v cannot belong to I. If v is in I, then every neighbour of v is adjacent with both a red and a blue vertex. We may assume 3 is red. Since b must be adjacent with a blue vertex, 4 is blue. Since c must be adjacent with a red vertex, 5 is red. But then d is not adjacent with a blue vertex. Thus X does not have a Radon partition. So $r_{\mathcal{D}}(G) \geq 6$.

We now show that $r_{\mathcal{D}}(G) \leq 6$ by showing that if X is a set of vertices without a Radon partition, then $|X| \leq 5$. Observe that such a set X contains at most one vertex from L and at most one vertex from M: If X contains at least two vertices from L, then one of these can be coloured red and another one blue and the remaining vertices can be coloured randomly red or blue. Thus a is in the convex hull of both the red and the blue vertices. The case where X contains at least two vertices from M can handled similarly.

Suppose first that X contains v. Then X does not contain vertices from both L and $\{1,2\}$; otherwise, it is readily seen that X has a Radon partition. Suppose X contains at least one vertex from $\{1,2\}$. If X contains two of the vertices from K, then it can again be shown that X has a Radon partition. So $|X| \leq 5$. Suppose that X contains neither 1 nor 2. If X contains every vertex of K, then either $|X| \leq 5$ or X contains a vertex from M. In the latter case X has a Radon partition. Suppose that X contains at most two vertices from K. Then $|X| \leq 5$.

Suppose that $v \notin X$. If X contains both 1 and 2, then we may assume that X does not contain a vertex from M, otherwise X has a Radon partition. Moreover, X either contains at most two vertices from K or no vertex from L, otherwise X has a Radon partition. So $|X| \leq 5$ in this case also. So X contains at most one vertex from $\{1,2\}$. If X contains every vertex from K, then X contains no vertex from M; otherwise X has a Radon partition. So either X contains every vertex from K and no vertex from M or X contains at most two vertices from K. In either case $|X| \leq 5$. So $r_{\mathcal{D}}(G) = 6$. So for this graph the Radon number of the digital convexity does not equal either the upper or lower bound of Theorem 4.1 nor does it equal $\Gamma(G) + 1$.

5 Concluding Remarks

In this paper we showed that the Caratheodory number of the digital convexity of a graph equals the upper local irredundance number of the graph. We showed that the Radon number for the digital convexity is bounded above by one more than the upper domination number and below by one more than the independence number. We illustrated sharpness of both bounds and showed that the Radon number of the digital convexity my lie strictly between these two bounds and at the same time not equal one more than the upper domination number. It is not clear whether, for a given nonempty graph G, $r_{\mathcal{D}}(G) - 1$ can be placed in the domination-chain given in

(1). If so, we have shown that it should appear between $\Gamma(G)$ and $IR(G)$.

References

[1] B. Bollobás and E.J. Cockayne, Graph-theoretic parameters concerning domination, independence and irredundance, *J. Graph Theory* 3 (1979) 241-249.

[2] J. Cáceres, A. Márquez, M. Morales, and M.L. Puertas, Towards a new framework for domination, *Computers and Mathematics with Applications* 62 (2011) 44–50.

[3] G. Chartrand, T.W. Haynes, M.A. Henning and P. Zhang, Stratification and domination in graphs, *Discrete Math.* 272 (2003) 171–185.

[4] G. Chartrand, L. Lesniak, and P. Zhang, *Graphs and Digraphs*, Chapman and Hall/CRC, Boca Raton (2010).

[5] E. Cockayne, S.T. Hedetniemi, and D.J. Miller, Properties of hereditary hypergraphs and middle graphs, *Canad. Math. Bull.* 21 (1978) 461–468.

[6] E. Cockayne and C.M. Mynhardt, The sequnce of upper and lower domination, independence and irredundance numbers of a graph, *Discrete Math.* 122 (1993) 89–102.

[7] M. Farber and R.E. Jamison, Convexity in graphs and hypergraphs, *SIAM J. Alg. Discrete Meth.* 7 (1986) 433–444.

[8] O. Favaron, Two relations between the parameters of independence and irredundance, *Discrete Math.* 70 (1988) 17–20.

[9] J. Fox, R. Gera, and P. Stanica, The Independence Number for the Generalized Petersen Graphs, *Ars Combin.* 103 (2012) 439–451.

[10] T.W. Haynes, S.T. Hedtniemi, and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York (1998).

[11] T.W. Haynes, S.T. Hedtniemi, and P.J. Slater, *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York (1998).

[12] M.S. Jacobson and K. Peters, Chordal graphs and upper irredundance, upper domination and independence, *Discrete Math.* 86 (1990) 59–69.

[13] R.E. Jamison and J.L. Pfaltz, Closure systems and their structure, *Inform. Sci.* 139 (2001) 275-286.

[14] J.L. Pfaltz and A. Rosenfeld, Sequential operations in digital picture processing, *J. ACM* 13(4) (1996) 471–494.

[15] M.J.L. Van de Vel, *Theory of convex structures*, North-Holland, Amsterdam (1993).