On Well-Covered Quadrangulations

Arthur S. Finbow
Department of Mathematics and Computing Science
Saint Mary's University
Halifax, Canada B3H 3C3
art.finbow@smu.ca

Bert L. Hartnell
Department of Mathematics and Computing Science
Saint Mary's University
Halifax, Canada B3H 3C3
bert.hartnell@smu.ca

and

Michael D. Plummer
Department of Mathematics Vanderbilt University
Nashville, TN 37240
michael.d.plummer@vanderbilt.edu

Abstract

A graph G is said to be *well-covered* if every maximal independent set of vertices has the same cardinality. A planar (simple) graph in which each face is a quadrilateral is called a *(planar) quadrangulation.* In the present paper we characterize those planar quadrangulations which are well-covered.

1 Introduction

Although it is now well-known that the independent set problem is NP-complete for graphs in general (cf. Karp [13]), for certain interesting subfamilies of graphs the problem becomes polynomially solvable. One such class of graphs are those known as *claw-free* (cf. Minty [14] and Sbihi [19]). Moreover, these graphs can be recognized in polynomial time.

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Another class of graphs for which the independent set problem is trivially solvable is the class of *well-covered* graphs. A graph is said to be *well-covered* (cf. [15]) if every maximal independent set of vertices is maximum. Or in other words, every maximal set of vertices has the same cardinality. But how does one recognize this class? It was shown independently by Chvátal and Slater [5] and by Sankaranarayana and Stewart [18] that the recognition problem for well-covered graphs is co-NP-complete. In contrast, if the graphs are claw-free, then the recognition problem becomes polynomial. (See Tankus and Tarsi [20, 21].)

Various subclasses of well-covered graphs have been characterized. (Cf. [1, 2, 3, 4, 6, 7, 8, 10, 11, 22].) For more comprehensive surveys of well-covered graphs, see [16] and more recently, [12].

A widely studied subclass of graphs are those which are maximal planar and which are commonly called (planar) *triangulations*. Clearly, any triangulation (larger than a single triangle) must have vertex connectivity 3, 4 or 5. In an earlier paper [8] it was shown that there is no 5-connected planar well-covered triangulation. Papers [9] and [10] culminated in the characterization of those 4-connected planar triangulations which are well-covered. There are precisely four of these and they are the four graphs labeled R_6, R_7, R_8, and R_{12} in Figure 1.1.

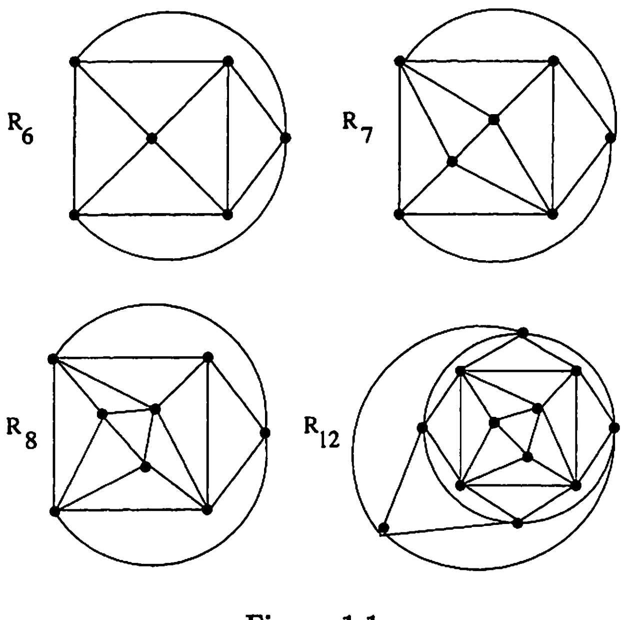


Figure 1.1

In contrast to the 4- and 5-connected cases, there are infinitely many 3-connected planar well-covered triangulations. A complete characterization of this family has only recently been obtained. (Cf. [11].)

As is well-known, a (planar) *quadrangulation* is a plane graph in which all faces are quadrilaterals. In the present paper we characterize precisely which quadrangulations of the plane are well-covered.

Note that in this paper all graphs are finite and simple and if v is a vertex of a graph, $N(v)$ will denote the set of neighbors of the vertex v.

2 Main Results

Lemma 2.1: If G is planar without triangular faces, then $\delta(G) \leq 3$.
Proof: Suppose not; i.e., suppose $\delta(G) \geq 4$. Then $2q = \sum_{v \in V(G)} \deg(v) \geq 4p$, or $q \geq 2p$. Also $4f \leq 2q$, so substituting in Euler's formula, we have $4p - 4q + 4f = 8$ or $q \leq 2p - 4$. So $2p - 4 \geq q \geq 2p$, a contradiction and the Lemma is proved. ■

Lemma 2.2: If G is a quadrangulation of the plane, then G is bipartite.

Proof: Suppose T is the smallest odd cycle in G and suppose that it has r edges. Then all faces interior to T are quadrangles. Suppose there are k such faces. Let $Int(T)$ denote the set of vertices interior to triangle T. Then $4k-r$ counts each edge in $G[Int(T)]$ twice. But this is a contradiction in view of the fact that $4k - r$ is odd. ■

We will make use of the following result of Ravindra [17].

Theorem: Let G be a well-covered bipartite graph. Then G contains a perfect matching and for every perfect matching M of G and every edge $e = xy \in M$, the subgraph $G[N(x) \cup N(y)]$ is a complete bipartite graph.

Lemma 2.3: Let G be a well-covered quadrangulation, M, a perfect matching in G and $e = xy$, an edge of M. If G is not a 4-cycle , then exactly one endvertex of e has degree 2.

Proof: Suppose both x and y have degree at least 3 in G. Then edge e must lie on two distinct faces of G which share exactly edge e in their boundaries. (See Figure 2.1.)

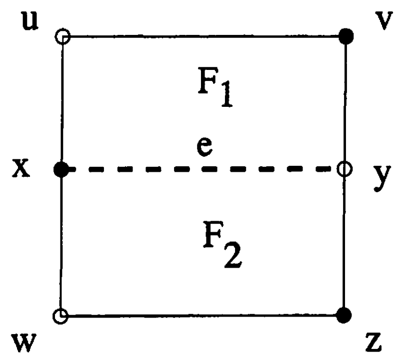


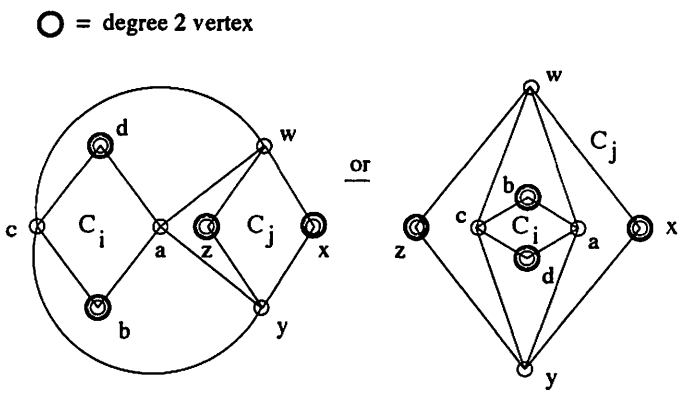
Figure 2.1

But then by planarity, it cannot be the case that both vw and uz are in $E(G)$. Hence $G[N(x) \cup N(y)]$ cannot be a complete bigraph, contradicting Ravindra's theorem. ■

So we may henceforth assume that if G is a well-covered quadrangulation that is not isomorphic to a 4-cycle, then it is bipartite with a perfect matching M such that every edge of M has exactly one endvertex of degree 2. (So exactly half of the vertices of G have degree 2 and the rest have degree at least 3.)

Let WCQ denote the set of all well-covered quadrangulations of the plane. We define a second class of planar quadrangulations, which we shall call WCQ', as follows.

Let $C_1, C_2, \ldots, C_k$ be a set of vertex-disjoint 4-cycles in the plane. We shall refer to these as *basic 4-cycles.* Let Q' be any quadrangulation of the plane spanned by $V(C_1) \cup \cdots \cup V(C_k)$ where each pair of these 4-cycles are joined according to the following recipe: either the two elements of the pair are not joined by any edges or else they are joined as shown in Figure 2.2 below.



(a)

(b)

Figure 2.2

Some examples of graphs belonging to WCQ' are shown in the next four figures.

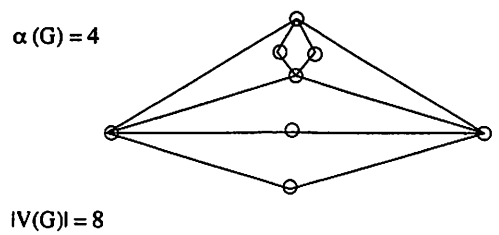


Figure 2.3

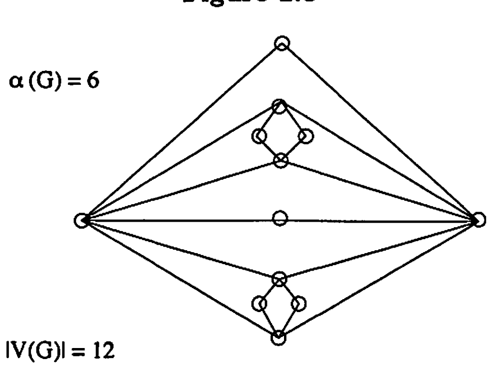


Figure 2.4

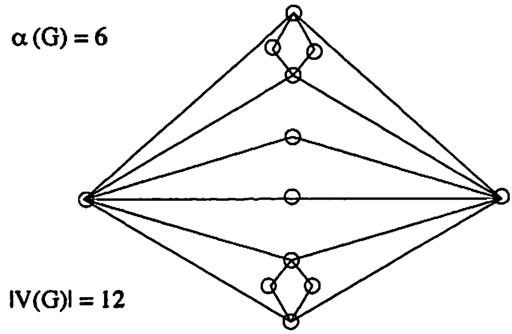


Figure 2.5

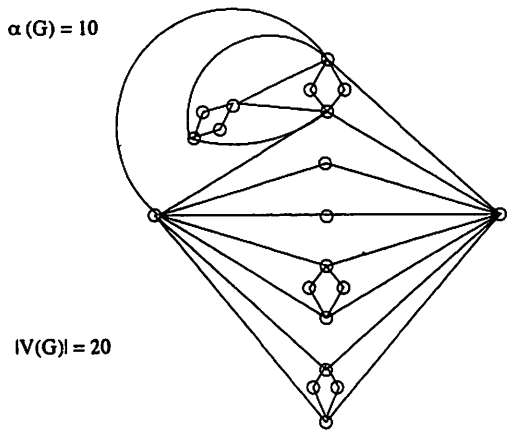


Figure 2.6

Theorem 2.4: $WCQ = WCQ'$.

Proof: Suppose first that Q is a well-covered planar quadrangulation; that is, that $G \in WCQ$. If Q is a 4-cycle, then Q is in the family WCQ' having a single basic 4-cycle.

So suppose Q is not a 4-cycle. (Note then that G cannot have two adjacent vertices of degree 2.) By Ravindra's theorem and Lemma 2.2, Q has a perfect matching M. Choose $e_1 = xy \in M$. By Lemma 2.3 we may suppose that $\deg(y) = 2$ and $\deg(x) \geq 3$. Let the neighbor of y different from x be labeled z. Then vertex z is covered by an edge $e_2 = zw \in M$. Now $\deg(z) \neq 2$, since G is not a 4-cycle. Hence $\deg(w) = 2$ by Lemma 2.3. But then by Ravindra's theorem, the unique neighbor of w not equal to z must be x. So we get the 4-cycle $xyzwx$ shown in Figure 2.7. We shall consider this as our first basic 4-cycle C_1.

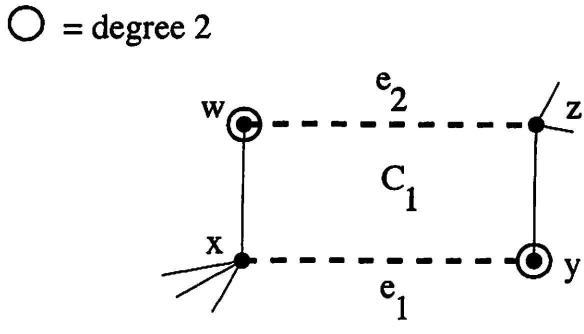


Figure 2.7

Now since G is not a 4-cycle, $|M| \geq 3$, so choose a third edge $e_3 = rs \in M$ where $\deg(s) = 2$. Suppose e_3 is joined to 4-cycle C_1 by an edge incident with s. Say, without loss of generality, that s is adjacent to z. (See Figure 2.8).

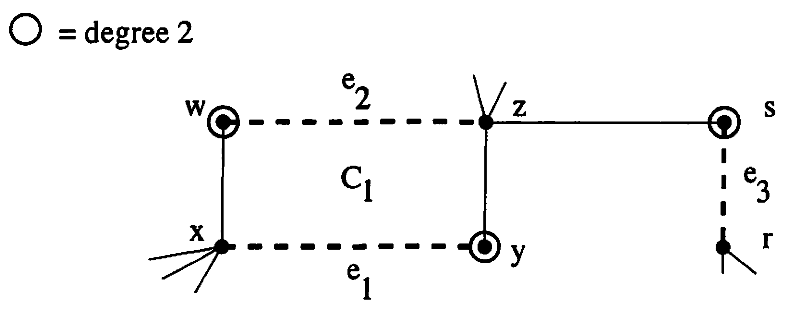


Figure 2.8

Then s is adjacent to x by Ravindra's theorem applied to matching edge e_2. Hence $r = x$, a contradiction.

So s is not adjacent to z and by symmetry, s is not adjacent to x either. So s is adjacent to some new vertex t, $t \notin \{x, y, z, w, r\}$ and $\deg(t) \geq 3$. Let $e_4 = tu$ be the edge of M which covers vertex t. So $\deg(u) = 2$. But then by Ravindra's theorem, u is adjacent to r and we have our second basic cycle $C_2 = rstur$. (See Figure 2.9.)

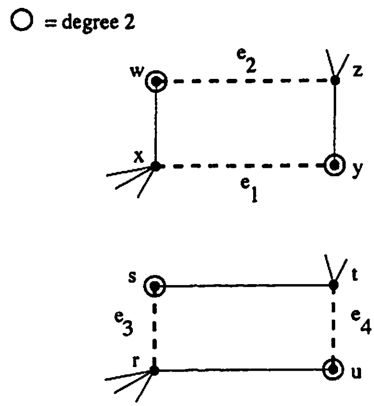


Figure 2.9

Continuing in this way, we eventually arrive at a set of vertex-disjoint 4-cycles which span $V(G)$.

Now consider two different basic 4-cycles C_i and C_j. Suppose the two 4-cycles are joined by an edge. Without loss of generality we may suppose we have the resulting subgraph shown in Figure 2.10.

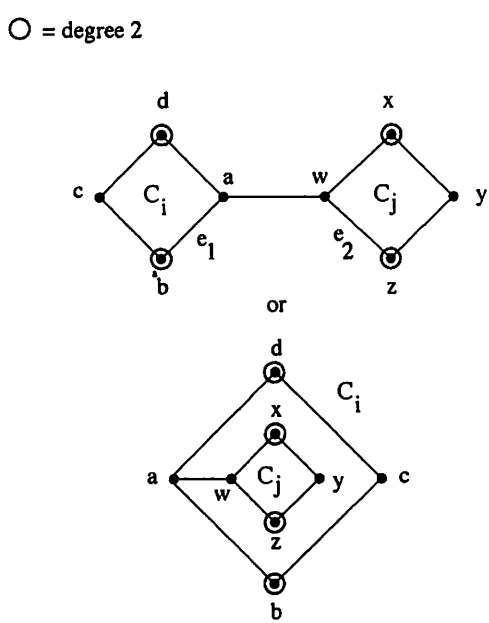


Figure 2.10

Then by Ravindra's theorem, matching edge e_1 forces c to be adjacent to w and edge e_2 forces y to be adjacent to both a and c. We thus arrive at the configuration of Figure 2.2. Thus every planar well-covered quadrangulation is of the type belonging to WCQ' or $WCQ \subseteq WCQ'$.

Conversely, it remains to show that $WCQ' \subseteq WCQ$; that is, any member of WCQ' is well-covered.

If G is a 4-cycle, this is clear. So suppose G is a member of WCQ' and G is not a 4-cycle. We will argue that if I is a maximum independent set of vertices in G then I consists of *exactly two* vertices chosen from each basic 4-cycle. Let us denote the set of vertices of degree 2 in G by A and let $V(G) - A = B$. Let I be maximum independent and consider any basic 4-cycle C_i of G. If I contains a vertex of C_i belonging to A, it must contain both A vertices of C_i, since it is maximal. But then I cannot contain either of the two remaining vertices of C_i.

So suppose I contains a vertex in $V(C_i) \cap B$. Then I cannot contain either A vertex of C_i. Now let C_j be a second basic 4-cycle and suppose it is joined to C_i. It is then joined to C_i precisely as shown in Figure 2.2. By symmetry, we may suppose that B vertex c belongs to I. Then vertices y and w are not in I and hence vertices x and z are in I. But then vertex a must belong to I since I is maximal.

This completes the proof. ∎

Remark 2.5: Note that via Theorem 2.4, it is clear that given a planar quadrangulation, one can check in polynomial time whether or not it is well-covered. Indeed if G is well-covered and connected then either it is simply a 4-cycle or starting with all vertices unlabeled we must be able to label and partition them into 4-cycles as follows:

From the unlabeled vertices select a vertex of degree at least four and label it x_i. It should have exactly two neighbours, label them a_i and b_i, of degree 2. These two vertices must have another common neighbour, label it y_i, of degree at least 4. The neighbours of x_i and y_i must be the same. The vertices x_i, a_i, y_i and b_i form a 4-cycle C_i. We must be able to choose another unlabeled vertex of degree 4 or more and repeat forming a collection of disjoint 4-cycles.

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