Distinguishing Chromatic Numbers of Wreath Products

C.J. Hodgins
Continuing Academic Learner Services
Southern Alberta Institute of Technology
Calgary, AB, T2M 0L4, Canada
cameron.hodgins@sait.ca

K. Seyffarth
Department of Mathematics and Statistics
University of Calgary
Calgary, AB, T2N 1N4, Canada
kseyffar@math.ucalgary.ca

Abstract

The *distinguishing chromatic number* of a graph G is the least integer, $\chi_D(G)$, for which G has a colouring of its vertices so that adjacent vertices receive different colours, and the identity is the only automorphism of G that preserves vertex colours. Our focus is on determining distinguishing chromatic numbers of wreath products of graphs, extending the work of Tang. We prove that if C_n is a cycle with n vertices and P_n is a path with n vertices, then $\chi_D(C_n[G])$ and $\chi_D(P_n[G])$ can be found for any connected graph G. We also obtain an upper bound on $\chi_D(T[G])$ when T is a tree and G is any connected graph. Some of our results depend on the notion of *inequivalent* colourings. Cheng introduces inequivalent colourings, and provides a formula for computing the number of inequivalent distinguishing k-colourings of a rooted tree. We add to this work by obtaining an expression for computing the number of inequivalent distinguishing k-colourings of a cycle.

1 Introduction

A *colouring* of a graph G is an assignment of colours (labels) to the vertices of G so that adjacent vertices receive different colours. Although some

JCMCC **85** (2013), pp. 299-319

authors refer to such a labelling as a proper colouring, we use the simpler term colouring. A colouring of G is a *distinguishing colouring* if the only automorphism of G that preserves the colours of the vertices is the identity, and the *distinguishing chromatic number* of G, denoted $\chi_D(G)$, is the minimum number of colours required to produce a distinguishing colouring of G. Collins and Trenk [5] introduce the notion of the distinguishing chromatic number, compute $\chi_D(G)$ for various classes of graphs, and provide $\chi_D(G)$ analogues of Brooks' Theorem for trees and for connected graphs. Their work has led numerous researchers to investigate the distinguishing chromatic number (see [2, 3, 4, 7, 8, 9, 10, 12, 13]), often for particular classes of graphs.

Unless otherwise specified, we use the notation and terminology of [1]. The *wreath product* of graph H with G, denoted $H[G]$, has vertex set $V(H) \times V(G)$; two vertices (u, v) and (u', v') are adjacent in $H[G]$ if and only if either $uu' \in E(H)$, or $u = u'$ and $vv' \in E(G)$. In essence, $u \in V(H)$ corresponds to a copy G_u of G, and an edge $uv \in E(H)$ corresponds to a complete bipartite graph between $V(G_u)$ and $V(G_v)$.

Let G be a graph and c a colouring of G. An automorphism $g \in \mathrm{Aut}(G)$ is called *colour preserving* if, for every $u \in V(G)$, $c(g(u)) = c(u)$, and we say c *fixes* a vertex $u \in V(G)$ if and only if any colour preserving automorphism $g \in \mathrm{Aut}(G)$ has the property that $g(u) = u$ More generally, c fixes an induced subgraph G' of G if and only if any colour preserving automorphism $g \in \mathrm{Aut}(G)$ has the property that $g(G') = G'$.

In this paper, we are concerned with computing $\chi_D(H[G])$ when H is a path, cycle or tree, and G is an arbitrary connected graph. We obtain exact values, in terms of $\chi_D(G)$, when H is a path or a cycle, and an upper bound when H is a tree. The values depend on the number of *inequivalent* distinguishing k-colouring of G, a concept introduced by Cheng [2]. The last section of this paper focuses on obtaining expressions for computing the number of inequivalent distinguishing k-colourings of trees and cycles.

Two colourings of a graph G, c_1 and c_2 are *equivalent* if for some $g \in \mathrm{Aut}(G), c_1(u) = c_2(g(u))$ for each $u \in V(G)$. Otherwise, the colourings c_1 and c_2 are *inequivalent.* Denote by $\mathscr{C}(G, k)$ the set of all k-colourings of G using colours from a set S with $|S| = k$, and let $\chi_D(G, k)$ be the number of equivalence classes of $\mathscr{C}(G, k)$ that contain only distinguishing k-colourings of G. Then $\chi_D(G, k)$ is the number of *inequivalent distinguishing k-colourings* of G. There are situations in which it is easy to see that two colourings of a graph G are inequivalent.

Remark 1. *Let c_1 and c_2 be colourings of a graph G, and suppose that there exists a colour f and a vertex $u \in V(G)$ so that $c_1(u) = f$ and $c_2(x) \neq f$ for any $x \in V(G)$. Then c_1 and c_2 are inequivalent colourings of G.*

The more interesting inequivalent colouring are those in which the image of c_1 is equal to the image of c_2.

Let G be a graph and c a vertex colouring of G. We write (G, c) for the graph G along with the colouring c, and say that (G, c) is a *colouring* of G. If F is a subgraph of G, then (F, c) is the subgraph F along with the colouring c restricted to F.

Lemma 1. *If c is a distinguishing colouring of $H[G]$, and $u \in V(H)$, then (G_u, c) is a distinguishing colouring of G_u.*

Proof. Let $u \in V(H)$ and let $g \in \mathrm{Aut}(G)$ be a nontrivial colour preserving automorphism of G, i.e., $(g(G_u), c)$ is isomorphic to (G_u, c) as a coloured graph. Define $g^* \in \mathrm{Aut}(H[G])$ as the automorphism of $H[G]$ that maintains the automorphism g on $V(G_u)$, and acts as the identity on all other vertices of $H[G]$. Then $(g^*(H[G]), c)$ is isomorphic to $(H[G], c)$ as a coloured graph, so g^* is a nontrivial automorphism of $H[G]$ that preserves colours, contradicting the fact that c is a distinguishing colouring of $H[G]$. □

Remark 2. *Suppose that H is a graph with at least one edge, and let $uv \in E(H)$. For any graph G, uv corresponds to a complete bipartite graph between the copies G_u and G_v of G in $H[G]$. Thus, in a distinguishing colouring of $H[G]$, G_u and G_v must be coloured by disjoint sets of colours.*

A lower bound for $\chi_D(H[G])$ follows immediately from Remark 2 and Lemma 1.

Lemma 2. *For any graph H with at least one edge and for any graph G,*

$$\chi_D(H[G]) \geq 2\chi_D(G).$$

2 Wreath Products with Cycles

We denote by C_n a cycle of length n, and write $C_n = v_0 v_1 \ldots v_{n-1} v_0$ to indicate that $V(C_n) = \{v_0, v_1, \ldots, v_{n-1}\}$ and $E(C_n) = \{v_i v_{i+1} \mid 0 \leq i \leq n-1\}$ (subscripts modulo n). Let I_m denote an independent set of size m, i.e., a set of m vertices, no two joined by an edge. Tang [11] finds exact values for $\chi_D(C_n[I_m])$ for all $n \geq 3$ and $m \geq 1$. In this section we extend Tang's work and find the distinguishing chromatic numbers of wreath products of cycles with arbitrary connected graphs. Before doing so, we first establish an important lemma.

Lemma 3. *Let G and H be connected graphs, and suppose that for all $p, q \in V(H)$ with $pq \in E(G)$, $N_H(p)\backslash\{q\} \neq N_H(q)\backslash\{p\}$. If $g \in Aut(H[G])$, then for any $u \in V(H)$, g maps G_u to G_w for some $w \in V(H)$.*

Proof. Suppose to the contrary that for some $u \in V(H)$, and $x, y \in V(G_u)$ with $xy \in E(G_u)$, that there is a $g \in \mathrm{Aut}(H[G])$ such that $g(x) \in V(G_a)$ and $g(y) \in V(G_b)$ for some $a, b \in V(H)$, $a \neq b$. Since $xy \in E(G_u)$, $g(x)g(y)$ is an edge of $H[G]$, and thus $ab \in E(H)$. The condition on H guarantees that there exists a vertex $c \in V(H)$ such that c is adjacent to exactly one of a and b; without loss of generality, we may assume that $ca \in E(H)$ and $cb \notin E(H)$. Therefore, in $H[G]$, the $|V(G)|$ vertices in $V(G_c)$ are adjacent to $g(x)$, but not adjacent to $g(y)$. However, in $H[G]$, the only vertices adjacent to x and not y are in $V(G_u)\backslash\{x,y\}$, since all vertices in $V(G_z)$, $z \in N_H(u)$, are adjacent to both x and y. Thus there are at most $|V(G)|-2$ vertices of G adjacent to x but not y, and hence there is no bijective map between vertices adjacent to x but not y and vertices adjacent to $g(x)$ but not $g(y)$, a contradiction.

Therefore any pair of adjacent vertices in G_u must be mapped to the same copy of G in $H[G]$ under g, i.e., g maps G_u to G_w for some $w \in V(H)$. □

Remark 3. *If there exists $pq \in E(H)$ with $N_H(p)\backslash\{q\} = N_H(q)\backslash\{p\}$, then if G has $|V(G)| \geq 2$ and has either a dominating vertex or is a complete bipartite graph, then Lemma 3 does not hold.*

Lemma 3 implies that if G is connected and $n \geq 4$, then any automorphism of $C_n[G]$ permutes copies of G using an automorphism of C_n (i.e., a rotation or a reflection). We rely on this to prove our results about the distinguishing chromatic number of $C_n[G]$ for $n \geq 3$ and G connected. Results for n even and n odd are stated separately.

Theorem 4. *For all even $n \geq 4$ and any connected graph G,*

$$\chi_D(C_n[G]) = \begin{cases} 2\chi_D(G) & \text{if } \chi_D(G,\chi_D(G)) \geq 2, \\ 2\chi_D(G)+1 & \text{if } \chi_D(G,\chi_D(G)) = 1 \text{ and } n \geq 6, \\ 2\chi_D(G)+2 & \text{if } \chi_D(G,\chi_D(G)) = 1 \text{ and } n = 4. \end{cases}$$

Proof. Let n be even, $n \geq 4$, and suppose $C_n = v_0v_1\cdots v_{n-1}v_0$. Furthermore, suppose G is a connected graph with $\chi_D(G,\chi_D(G)) \geq 2$. Let S_0 and S_1 be disjoint sets (of colours), each containing $\chi_D(G)$ elements. Since $\chi_D(G,\chi_D(G)) \geq 2$, there exist inequivalent distinguishing $\chi_D(G)$-colourings c_0 and c_1 of G that have image S_0. There also exist inequivalent distinguishing $\chi_D(G)$-colourings d_0 and d_1 of G that have image S_1. We obtain a distinguishing colouring c of $C_n[G]$ by defining the colouring on the copies of G in $C_n[G]$ as follows. Set

$$\begin{aligned}
&(G_{v_0}, c) = (G, c_0);\\
&(G_{v_1}, c) = (G, d_0);\\
&(G_{v_i}, c) = (G, c_1), \text{ for even } i,\ 2 \le i \le n-2;\\
&(G_{v_i}, c) = (G, d_1), \text{ for odd } i,\ 3 \le i \le n-1.
\end{aligned}$$

Since the set of colours used to colour $G_{v_1}, G_{v_3}, \ldots, G_{v_{n-1}}$ is disjoint from the set used to colour $G_{v_0}, G_{v_2} \ldots G_{v_{n-2}}$, the result is a colouring. Let $g \in \mathrm{Aut}(C_n[G])$ be a colour preserving automorphism of $C_n[G]$. Then g permutes copies of G using an automorphism of C_n, and since G_{v_0} and G_{v_1} are the only copies of G having colourings c_0 and d_0 respectively, g must fix both G_{v_0} and G_{v_1}. The only automorphism of C_n that fixes v_0 and v_1 is the identity, implying that g is the identity on $C_n[G]$, and thus c is a distinguishing colouring of $C_n[G]$. Therefore $\chi_D(C_n[G], c) \le 2\chi_D(G)$. By Lemma 2, $\chi_D(C_n[G], c) = 2\chi_D(G)$.

For the second part of the theorem, suppose that $\chi_D(G, \chi_D(G)) = 1$ and $n \ge 6$. To obtain a distinguishing colouring of $C_n[G]$ using $2\chi_D(G) + 1$ colours, we use four colourings of G, defined on four different colour sets. Let

$$\begin{aligned}
&S_0 = \{0, 1, \ldots, \chi_D(G) - 1\},\\
&S_1 = \{1, 2, \ldots, \chi_D(G)\},\\
&S_2 = \{0, 1, \ldots, \chi_D(G) - 2, \chi_D(G)\},\\
&S_3 = \{\chi_D(G) + 1, \chi_D(G) + 2, \ldots, 2\chi_D(G) - 1, 2\chi_D(G)\}.
\end{aligned}$$

Consider the colourings of G, $(G, c_0), (G, c_1), (G, c_2), (G, c_3)$ with images S_0, S_1, S_2, S_3, respectively. We define a distinguishing colouring c of $C_n[G]$, as follows.

$$\begin{aligned}
&(G_{v_0}, c) = (G, c_0);\\
&(G_{v_1}, c) = (G, c_3);\\
&(G_{v_2}, c) = (G, c_1);\\
&(G_{v_i}, c) = (G, c_3) \text{ for } i \text{ odd},\ 3 \le i \le n-1;\\
&(G_{v_i}, c) = (G, c_2) \text{ for } i \text{ even},\ 4 \le i \le n-2.
\end{aligned}$$

Since G_{v_0} and G_{v_2} are the only copies of G to have the colourings c_0 and c_1 respectively, any colour preserving $g \in \mathrm{Aut}(C_n[G])$ that permutes the copies of G and preserves colours must fix both G_{v_0} and G_{v_2}, However the only $g \in \mathrm{Aut}(C_n[G])$ that fixes G_{v_0} and G_{v_1} is the one that fixes all copies of G, i.e., the identity. Thus c is a distinguishing colouring of

$C_n[G]$ with $2\chi_D(G)+1$ colours. Combining this with Lemma 2 yields $\chi_D(C_n[G]) = 2\chi_D(G)$ or $\chi_D(G) = 2\chi_D(G)+1$.

To prove that $2\chi_D(G)$ colours are not sufficient for a distinguishing colouring of $C_n[G]$, suppose $C_n[G]$ has a distinguishing colouring with $2\chi_D(G)$ colours. By Remark 2, we may assume that G_{v_0} is coloured by a set S_0 of $\chi_D(G)$ colours, G_{v_1} is coloured with a set S_1 of $\chi_D(G)$ colours, and that $S_0 \cap S_1 = \emptyset$. For $i \in \{0,1\}$, let c_i be a distinguishing colouring of G_{v_i} having image S_i. Since $\chi_D(G, \chi_D(G)) = 1$, c_0 and c_1 are uniquely determined. It follows from Remark 2 that there is a unique colouring, c, of $C_n[G]$ with the $2\chi_D(G)$ colours of $S_0 \cup S_1$, and that c satisfies

$$(G_{v_i}, c) = (G, c_0) \text{ for } i \text{ even;}$$
$$(G_{v_i}, c) = (G, c_1) \text{ for } i \text{ odd.}$$

However this colouring is not distinguishing. Consider the automorphism $g \in \mathrm{Aut}(C_n[G])$ defined by

$$g(G_{v_i}) = G_{v_{i+2}},$$

where the subscripts are taken modulo n. Then g is a nontrivial colour preserving automorphism of G, and thus $\chi_D(C_n[G]) = 2\chi_D(G)+1$.

For the final part of the theorem, suppose $\chi_D(G, \chi_D(G)) = 1$ and $n = 4$. Define four colour sets as follows.

$$\begin{aligned}
S_0 =& \{0, 1, \dots, \chi_D(G)-1\},\\
S_1 =& \{\chi_D(G), \chi_D(G)+1, \dots, 2\chi_D(G)-1\},\\
S_2 =& \{0, 1, \dots, \chi_D(G)-2, 2\chi_D(G)\},\\
S_3 =& \{\chi_D(G), \chi_D(G)+1, \dots, 2\chi_D(G)-2, 2\chi_D(G)+1\}.
\end{aligned}$$

Since $|S_i| = \chi_D(G)$ for each i, $0 \le i \le 3$, G has a distinguishing colouring c_i whose image is S_i. Furthermore, $(G, c_0), (G, c_1), (G, c_2), (G, c_3)$ are inequivalent because $S_i \neq S_j$ for all i, j where $i \neq j$ and $i, j \in \{0,1,2,3\}$.

Let $C_4 = v_0v_1v_2v_3v_0$, and colour the copies of G in $C_4[G]$ as follows.

$$\begin{aligned}
(G_{v_0}, c) &= (G, c_0);\\
(G_{v_1}, c) &= (G, c_1);\\
(G_{v_2}, c) &= (G, c_2);\\
(G_{v_3}, c) &= (G, c_3).
\end{aligned}$$

The sets S_0, S_1, S_2 and S_3 used to defined c_0, c_1, c_2 and c_3, respectively, have the property that each pair of sets $\{S_i, S_{i+1}\}, 0 \le i \le 3$ (subscripts modulo 4) is disjoint, ensuring that c is a colouring of $C_4[G]$. Since the four colourings of the copies of G are inequivalent, there is no colour preserving

$g \in \mathrm{Aut}(C_4[G])$ that permutes copies of G. As a result c is a distinguishing colouring of $C_4[G]$. This shows that $\chi_D(C_4[G]) \leq 2\chi_D(G)+2$. By Lemma 2, $\chi_D(C_4[G]) = 2\chi_D(G)$, $\chi_D(C_4[G]) = 2\chi_D(G)+1$, or $\chi_D(C_4[G]) = 2\chi_D(G)+2$.

Consider $C_4[G]$ where $C_4 = v_0v_1v_2v_3v_0$, and let c be a distinguishing colouring of $C_4[G]$. Then (G_{v_0}, c) and (G_{v_2}, c) are inequivalent, and (G_{v_1}, c) and (G_{v_3}, c) are also inequivalent. If not, then there is a colour preserving automorphism of $C_4[G]$ that interchanges G_{v_1} and G_{v_3}, and fixes G_{v_0} and G_{v_2}, or there is a colour preserving automorphism that interchanges G_{v_0} and G_{v_2}, and fixes G_{v_1} and G_{v_3}, respectively.

Suppose $(C_4[G], c)$ has a distinguishing colouring with the $2\chi_D(G)+1$ colours $\{0, 1, \ldots, 2\chi_D(G)\}$. Let S_0, S_1, S_2, S_3 denote the images of c restricted to $G_{v_0}, G_{v_1}, G_{v_2}, G_{v_3}$, respectively. Then for each i, $0 \leq i \leq 3$, (subscripts taken modulo 4)

$$|S_i| \geq \chi_D(G), \tag{1}$$

$$S_i \cap S_{i+1} = \emptyset, \tag{2}$$

$$S_i \neq S_{i+2}. \tag{3}$$

We conclude from this that $|S_i| = \chi_D(G)$ for all i, for if $|S_i| = \chi_D(G)+1$ then there would be only one possible colour set disjoint from S_i. This would then imply that $S_{i+1} = S_{i-1}$, which violates (3). Without loss of generality, we may assume that

$$S_0 = \{0, 1, \ldots, \chi_D(G)-1\},$$
$$\text{and } S_1 = \{\chi_D(G), \chi_D(G)+1, \ldots, 2\chi_D(G)-1\}.$$

Since $S_1 \cap S_2 = \emptyset$, $S_2 \subset \{0, 1, \ldots, \chi_D(G)-1, 2\chi_D(G)\}$. Also since $S_2 \neq S_0$, $2\chi_D(G) \in S_2$. Similarly, $S_3 \cap S_0 = \emptyset$, implies that $S_3 \subset \{\chi_D(G), \chi_D(G)+1, \ldots, 2\chi_D(G)-1, 2\chi_D(G)\}$. Since $S_3 \neq S_1$, $2\chi_D(G) \in S_3$, and so $2\chi_D(G) \in S_2 \cap S_3$, a contradiction. Thus $(C_4[G], c)$ does not have a distinguishing colouring with $2\chi_D(G)+1$, implying that $\chi_D(C_4[G]) > 2\chi_D(G)+1$. Therefore, $\chi_D(C_4[G]) = 2\chi_D(G)+2$. □

When C_n has odd length, the number of colours required to produce a distinguishing colouring of $C_n[G]$ depends on n.

Theorem 5. *For all odd $n \geq 3$ and any connected graph G,*

$$\chi_D(C_n[G]) = 2\chi_D(G) + \left\lceil \chi_D(G) / \binom{\frac{n-1}{2}}{} \right\rceil.$$

Proof. Let $C_n = v_0v_1 \cdots v_{n-1}v_0$, and let

$$q = \left\lceil \chi_D(G) / \left(\frac{n-1}{2}\right) \right\rceil.$$

Define the following $2\lceil \chi_D(G)/q \rceil + 1$ sets of colours, whose entries are taken modulo $(2\chi_D(G) + q)$. Let

$$S_k = \{k\chi_D(G), k\chi_D(G) + 1, \ldots, (k+1)\chi_D(G) - 1\},$$

for $0 \leq k \leq 2\lceil \chi_D(G)/q \rceil$. We leave it to the reader to verify that $S_k \cap S_{k+1} = \emptyset$ for $0 \leq k \leq 2\lceil \chi_D(G)/q \rceil - 1$, $S_0 \cap S_{2\lceil \chi_D/q \rceil} = \emptyset$, and $S_j \neq S_k$ for $0 \leq j < k \leq 2\lceil \chi_D(G)/q \rceil$.

Since $|S_k| = \chi_D(G)$ for each k, $0 \leq k \leq 2\lceil \chi_D(G)/q \rceil$, G has a distinguishing colouring whose image is S_k. Let (G, c_k) denote a distinguishing colouring of G whose image is S_k. By Remark 1,

$$(G, c_0), (G, c_1), \ldots, (G, c_{2\lceil \chi_D(G)/q \rceil})$$

are inequivalent distinguishing colourings of G. We obtain a distinguishing colouring c of $C_n[G]$ by defining colourings for the copies of G in $C_n[G]$. Set

$$(G_{v_k}, c) = (G, c_k)$$

for $0 \leq k \leq 2\lceil \chi_D(G)/q \rceil$, and for $\lceil \chi_D(G)/q \rceil \leq j \leq (n-3)/2$, let

$$\begin{aligned}(G_{v_{2j+1}}, c) &= (G, c_{2\lceil \chi_D(G)/q \rceil - 1});\\ (G_{v_{2j+2}}, c) &= (G, c_{2\lceil \chi_D(G)/q \rceil}).\end{aligned}$$

Since G_{v_0} and G_{v_1} are the only copies of G to have the colourings c_0 and c_1, respectively, any colour preserving $g \in \mathrm{Aut}(C_n[G])$ that permutes the copies of G and preserves colours must fix both G_{v_0} and G_{v_1}. However the only $g \in \mathrm{Aut}(C_n[G])$ that fixes G_{v_0} and G_{v_1} is the one that fixes all copies of G, i.e., the identity. Thus c is a distinguishing colouring of $C_n[G]$.

To prove that $C_n[G]$ has no distinguishing colouring with fewer than

$$2\chi_D(G) + \left\lceil \chi_D(G) / \left(\frac{n-1}{2}\right) \right\rceil$$

colours, suppose the contrary, that

$$\chi_D(C_n[G]) < 2\chi_D(G) + \left\lceil \chi_D(G) / \left(\frac{n-1}{2}\right) \right\rceil.$$

Let c be a distinguishing colouring of $C_n[G]$ with $\chi_D(C_n[G])$ colours. Since $\chi_D(C_n[G])$ and $\lceil \chi_D(G)/(n-1/2) \rceil$ are both integers,

$$\begin{aligned}\chi_D(C_n[G]) &< 2\chi_D(G) + \chi_D(G) / \left(\frac{n-1}{2}\right),\\ \frac{n-1}{2} &< \frac{(n-1)\cdot\chi_D(G)}{\chi_D(C_n[G])} + \frac{\chi_D(G)}{\chi_D(C_n[G])},\\ \frac{n-1}{2} &< \frac{n\cdot\chi_D(G)}{\chi_D(C_n[G])}.\end{aligned}$$

Since n is odd, $(n-1)/2 = \lfloor n/2 \rfloor$, so

$$\frac{n \cdot \chi_D(G)}{\chi_D(C_n[G])} > \left\lfloor \frac{n}{2} \right\rfloor .$$

Each of the n copies of G must be coloured with at least $\chi_D(G)$ colours from a set of $\chi_D(C_n[G])$ colours, so by the Pigeonhole Principle, there is some colour that appears on at least $n \cdot \chi_D(G)/\chi_D(C_n[G])$ copies of G. However, $n \cdot \chi_D(G)/\chi_D(C_n[G]) > \lfloor n/2 \rfloor$ implies that such a colour appears on more then half of the copies of G. Thus, there exists some i such that (G_{v_i}, c) and $(G_{v_{i+1}}, c)$ have a colour in common, contradicting Remark 2. □

3 Wreath Products with Paths and Trees

We denote by P_n a path of length $(n-1)$, and write $P_n = v_0v_1 \ldots v_{n-1}$ to indicate that $V(P_n) = \{v_0, v_1, \ldots, v_{n-1}\}$ and $E(P_n) = \{v_iv_{i+1} \mid 0 \leq i \leq n-1\}$. If G is connected and $n \geq 3$, then by Lemma 3 any automorphism of $P_n[G]$ permutes copies of G using an automorphism of P_n. This is key to proving the following theorem.

Theorem 6. *For any connected graph G and any $n \geq 2$,*

$$\chi_D(P_n[G]) = \begin{cases} 2\chi_D(G) & \text{if } n \text{ is even,} \\ 2\chi_D(G) & \text{if } \chi_D(G, \chi_D(G)) \geq 2 \text{ and } n \text{ is odd,} \\ 2\chi_D(G)+1 & \text{if } \chi_D(G, \chi_D(G)) = 1 \text{ and } n \text{ is odd.} \end{cases}$$

Proof. Let $n = 2k \geq 2$, and let $P_{2k} = v_0v_1 \cdots v_{2k-1}$, be a path with an even number of vertices. By Lemma 2, it suffices to give a distinguishing colouring of $P_{2k}[G]$ with $2\chi_D(G)$ colours. Suppose (G, c_0) and (G, c_1) are distinguishing colourings of G with disjoint images $S_0 = \{0, 1, \ldots, \chi_D(G)-1\}$ and $S_1 = \{\chi_D(G), \chi_D(G)+1, \ldots, 2\chi_D(G)-1\}$, respectively.

Define the colouring c on $P_{2k}[G]$ by colouring the copies of G in $P_{2k}[G]$ as follows, $0 \leq i \leq k-1$:

$$(G_{v_{2i}}, c) = (G, c_0);$$
$$(G_{v_{2i+1}}, c) = (G, c_1).$$

Then for all i, G_{v_i} and $G_{v_{i+1}}$ are coloured with disjoint colour sets. Suppose $g \in \mathrm{Aut}(P_{2k}[G])$ is a nontrivial colour preserving automorphism of $P_{2k}[G]$. Since the only nontrivial automorphism of P_{2k} is the one that interchanges v_i with v_{2k-i-1}, it follows from Lemma 3 that g interchanges G_{v_i} with $G_{v_{2k-i-1}}$. In particular, when $i = k-1$, g interchanges $G_{v_{k-1}}$ with G_{v_k}. However the image c restricted to $G_{v_{k-1}}$ is different from the image of c restricted to G_{v_k}, so g cannot interchange $G_{v_{k-1}}$ and G_{v_k}, a contradiction.

Therefore, c is a distinguishing colouring of $P_{2k}[G]$ with $2\chi_D(G)$ colours, showing that $\chi_D(P_{2k}[G]) = 2\chi_D(G)$.

Now suppose that $P_{2k-1} = v_0v_1\cdots v_{2k-2}$, $k \geq 2$, is a path with an odd number of vertices and assume $\chi_D(G, \chi_D(G)) \geq 2$. Let (G, c_0) and (G, c_1) be inequivalent distinguishing colourings of G having image $S_0 = \{0, 1, \ldots, \chi_D(G) - 1\}$, and let (G, d_0) be a distinguishing colouring of G having image $S_1 = \{\chi_D(G), \chi_D(G) + 1, \ldots, 2\chi_D(G) - 1\}$.

Define the colouring c on $P_{2k-1}[G]$ by colouring the copies of G in $P_{2k-1}[G]$ as follows.

$$\begin{aligned}(G_{v_0}, c) &= (G, c_1);\\ (G_{v_{2i-1}}, c) &= (G, d_0), 1 \leq i \leq k-1;\\ (G_{v_{2i}}, c) &= (G, c_0), 1 \leq i \leq k-1.\end{aligned}$$

Then for all j, $0 \leq j \leq 2k-3$, G_{v_j} and $G_{v_{j+1}}$ are coloured with disjoint colour sets. Suppose $g \in \mathrm{Aut}(P_{2k-1}[G])$ is a nontrivial colour preserving automorphism of $P_{2k-1}[G]$. Since the only nontrivial automorphism of P_{2k-1} is the one that interchanges v_i with v_{2k-i-2} $(0 \leq i \leq k-1)$, it follows from Lemma 3 that g interchanges G_{v_i} with $G_{v_{2k-i-2}}$. In particular, when $i = 0$, g interchanges G_{v_0} with $G_{v_{2k-2}}$. However (G_{v_0}, c) and $(G_{v_{2k-2}}, c)$ are inequivalent colourings of G, so g cannot interchange them, a contradiction. Therefore, c is a distinguishing colouring of $P_{2k-1}[G]$, showing that $\chi_D(P_{2k-1}[G]) \leq 2\chi_D(G)$. By Lemma 2, $\chi_D(P_{2k-1}[G]) = 2\chi_D(G)$.

Finally, suppose $P_{2k-1} = v_0v_1\cdots v_{2k-2}$ is a path with an odd number of vertices, and assume $\chi_D(G, \chi_D(G)) = 1$. Define three different colour sets as follows.

$$\begin{aligned}S_0 &= \{0, 1, \ldots, \chi_D(G) - 1\},\\ S_1 &= \{\chi_D(G), \chi_D(G) + 1, \ldots, 2\chi_D(G) - 1\},\\ S_2 &= \{0, 1, \ldots, \chi_D(G) - 2, 2\chi_D(G)\}.\end{aligned}$$

Let (G, c_0), (G, c_1) and (G, c_2) be distinguishing colourings of G with images S_0, S_1 and S_2, respectively. Define the colouring c on $P_{2k-1}[G]$ by colouring the copies of G in $P_{2k-1}[G]$ as follows:

$$\begin{aligned}(G_{v_0}, c) &= (G, c_0);\\ (G_{v_{2i-1}}, c) &= (G, c_1), 1 \leq i \leq k-1;\\ (G_{v_{2i}}, c) &= (G, c_2), 1 \leq i \leq k-1.\end{aligned}$$

Then for all j, $0 \leq j \leq 2k-3$, G_{v_j} and $G_{v_{j+1}}$ are coloured with disjoint colour sets. Suppose $g \in \mathrm{Aut}(P_{2k-1}[G])$ is a nontrivial colour preserving automorphism of $P_{2k-1}[G]$. Since the only nontrivial automorphism of P_{2k-1} is the one that interchanges v_i with v_{2k-i-2}, it follows from Lemma 3 that g interchanges G_{v_0} with $G_{v_{2k-2}}$. However, this mapping is not colour preserving because the image of c restricted to G_{v_0} is different from the image of c restricted to $G_{v_{2k-2}}$. Therefore c is a distinguishing colouring of $P_{2k-1}[G]$ and hence $\chi_D(P_{2k-1}[G]) \leq 2\chi_D(G)+1$. By Lemma 2, it follows that $\chi_D(P_{2k-1}[G]) = 2\chi_D(G)$ or $\chi_D(P_{2k-1}[G]) = 2\chi_D(G)+1$.

Suppose that $\chi_D(P_{2k-1}[G]) = 2\chi_D(G)$. By Remark 2, G_{v_0} and G_{v_1} must be coloured with disjoint sets of colours, S_0 and S_1, respectively. However, $|S_0|, |S_1| \geq \chi_D(G)$, so $|S_0 \cup S_1| \geq 2\chi_D(G)$. Since the number of colours allowed is exactly $2\chi_D(G)$, it follows that $|S_0| = |S_1| = \chi_D(G)$. For $i \in \{0,1\}$, define c_i to be a distinguishing colouring of G having image S_i. Since $\chi_D(G, \chi_D(G)) = 1$, c_i is unique, and by Remark 2, it follows that the only colourings of $P_{2k-1}[G]$ with these $2\chi_D(G)$ colours are

$$(G_{v_{2i}}, c) = (G, c_0), 0 \leq i \leq k-1;$$
$$(G_{v_{2i+1}}, c) = (G, c_1), 0 \leq i \leq k-2;$$

or

$$(G_{v_{2i}}, c) = (G, c_1), 0 \leq i \leq k-1;$$
$$(G_{v_{2i+1}}, c) = (G, c_0), 0 \leq i \leq k-2.$$

In either case, there is a colour preserving automorphism $g \in \mathrm{Aut}(P_{2k-1}[G])$ that interchanges G_{v_i} with $G_{v_{2k-i-2}}$, for $0 \leq i \leq k-1$. Therefore, c is not a distinguishing colouring of $P_{2k-1}[G]$, and thus $\chi_D(P_{2k-1}[G]) = 2\chi_D(G)+1$. □

The next result generalizes Theorem 6 to trees. In what follows, assume that a rooted tree T is drawn in the plane in such a way that the children of each vertex are ordered from left to right.

Theorem 7. *Let T be a tree with $|V(T)| \geq 3$ and root v_0. Define a bipartition $X = \{v_0, v_1, \ldots, v_{n-1}\}$, $Y = \{u_1, u_2, \ldots, u_{m-1}\}$ on $V(T)$. Then for any connected graph G, $\chi_D(T[G]) \leq r+p$, where r is the smallest integer such that*

$$\chi_D(G,r) \geq \max\{d(v_0), d(v_1)-1, d(v_2)-1, \ldots,, d(v_{n-1})-1\},$$

and p the smallest integer such that $\chi_D(G,p) \geq \max\{d(u_i) \mid 1 \leq i \leq m-1\}$.

Proof. Choose the smallest r and p such that

$$\chi_D(G,r) \geq \max\{ d(v_0), d(v_1)-1, d(v_2)-1, \ldots, d(v_{n-1})-1\},$$

and

$$\chi_D(G,p) \geq \max\{d(u_i) \mid 1 \leq i \leq m-1\}.$$

Let

$$\{(G,c_0),(G,c_1),\ldots,(G,c_{\chi_D(G,r)-1})\}$$

be pairwise inequivalent distinguishing colourings of G using colour set $S_0=\{0,1,\ldots,r-1\}$, and let

$$\{(G,d_0),(G,d_1),\ldots,(G,d_{\chi_D(G,p)-1})\}$$

be pairwise inequivalent distinguishing colourings of G using colour set $S_1=\{r,r+1,\ldots,r+p-1\}$. In what follows, copies of G in $T[G]$ that correspond to vertices of X are coloured using (G,d_i), $0\le i\le \chi_D(G,p)-1$, and those corresponding to vertices of Y are coloured with (G,c_j), $0\le j\le \chi_D(G,r)-1$. To obtain a distinguishing colouring c of $T[G]$, begin by setting $(G_{v_0},c)=(G,d_{\chi_D(G,p)-1})$. We then discard the colouring $(G,d_{\chi_D(G,p)-1})$, ensuring that any colour preserving $g\in \mathrm{Aut}(T[G])$ fixes G_{v_0}. Next, colour the copies of G corresponding to the children of v_0, from left to right, with colourings (G,c_j), $j=0,1,\ldots,d(v_0)-1$. The choice of r insures that $\chi_D(G,r)\ge d(v_0)$, so the colourings of the children of v_0 are pairwise inequivalent. We complete the distinguishing colouring of $T[G]$ by colouring the remaining copies of G in $T[G]$ as follows.

If for some $u\in Y$, and some j, $0\le j\le \chi_D(G,p)-1$, $(G_u,c)=(G,c_j)$, then the copies of G corresponding to the children of u are coloured, from left to right, with colourings (G,d_i), $i=0,1,\ldots,d(u)-2$. The choice of p insures that $\chi_D(G,p)\ge d(u)$, and hence the only colourings that may be required are (G,d_j), $0\le j\le \chi_D(G,p)-2$.

If for some $v\in X$, $v\ne v_0$, and some k, $(G_v,c)=(G,d_k), 0\le k\le \chi_D(G,r)-2$, then the copies of G corresponding to the children of v are coloured, from left to right, with colourings (G,c_j), $j=0,1,\ldots,d(v)-2$. The choice of r insures that $\chi_D(G,r)\ge d(v)-1$.

This colouring c of $T[G]$ has the following properties:

1. If $xy\in E(T)$ and x is the parent of y, then G_x is coloured before G_y.

2. If $y_1,y_2,\ldots,y_t$ are the children of x in T, then

   $$(G_{y_1},c),(G_{y_2},c),\ldots,(G_{y_t},c)$$

   are pairwise inequivalent distinguishing colourings of G. The fact that $S_0\cap S_1=\emptyset$, and because copies of G corresponding to vertices in X are coloured using S_0, while vertices in Y are coloured using S_1, insures that c is, in fact, a colouring of $T[G]$.

As a consequence of these properties, if $y_1,y_2,\ldots,y_t$ are the children of $x\in T$, and if a colour preserving automorphism $g\in \mathrm{Aut}(T[G])$ fixes G_x, then g also fixes $(G_{y_1},c),(G_{y_2},c),\ldots,(G_{y_t},c)$. It now follows by induction on the number of levels in the tree that, once G_{v_0} has been coloured so that any colour preserving $g \in \mathrm{Aut}(T[G])$ fixes G_{v_0}, the resulting colouring c of $T[G]$ is distinguishing. □

4 Inequivalent Distinguishing Colourings of Trees and Cycles

Inequivalent colourings are introduced by Cheng [2], and an expression for the number of inequivalent distinguishing k-colourings of a *rooted* tree is obtained. Let T be a tree and $r \in V(T)$. We denote by $T(r)$ the tree T rooted at r. The automorphism group of $T(r)$ is defined as

$$\mathrm{Aut}(T(r)) = \{g \in \mathrm{Aut}(T) \mid g(r) = r\}.$$

A colouring c of $T(r)$ is distinguishing provided that the only colour preserving automorphism $g \in \mathrm{Aut}(T(r))$ is the identity.

Theorem 8. *[2, Theorem 7] Let $T(r)$ be a tree rooted at vertex r, and let $\mathcal{T}$ be the collection of all subtrees of $T(r)$ rooted at the children of r. Suppose that $\mathcal{T}$ consists of q isomorphism classes, and that the i^{th} isomorphism class contains m_i copies of the rooted tree T_i. Then for any $k \in \mathbb{N}$,*

$$\chi_D(T(r), k) = k \prod_{i=1}^{q} \binom{\chi_D(T_i, k)(k-1))/k}{m_i}.$$

This result was obtained independently by Hodgins [6, Theorem 4.2.1], who gives the following expressions for the number of inequivalent distinguishing k-colourings of an (unrooted) tree, based on the tree having one centre or two centres.

Corollary 9. *[6, Corollary 4.2.1] Let T be a tree with a single vertex w as its center, and let $\mathcal{T}$ be the collection of all subtrees of T rooted at the children of w. Suppose that $\mathcal{T}$ consists of q isomorphism classes, and that the i^{th} isomorphism class contains m_i copies of the rooted tree T_i. Then for any $k \in \mathbb{N}$,*

$$\chi_D(T, k) = k \prod_{i=1}^{q} \binom{\chi_D(T_i, k)(k-1))/k}{m_i}.$$

Corollary 10. *[6, Theorem 4.2.2] Let T be a tree with adjacent centres u and v, and let $T'(u)$ and $T'(v)$ denote the rooted subtrees of $T \backslash \{uv\}$ rooted at u and v, respectively. Then for any $k \in \mathbb{N}$*

$$\chi_D(T, k) = \begin{cases} \frac{k-1}{k} \chi_D(T'(u), k) \cdot \chi_D(T'(v), k) & \text{if } T'(u) \ncong T'(v); \\ \frac{k-1}{2k} \chi_D(T'(u), k) \cdot \chi_D(T'(v), k) & \text{if } T'(u) \cong T'(v). \end{cases}$$

(Here, $\ncong$ and $\cong$ mean not isomorphic and isomorphic, respectively, as rooted trees.)

In the remainder of this section, we derive expressions for the number of inequivalent distinguishing k-colourings of C_n, a cycle on n vertices. The expressions depend on the parity of the cycle, and use chromatic polynomials of cycle and paths. The *chromatic polynomial* of a graph G is denoted $P(G,x)$, and the value of $P(G,x)$ at a positive integer k, denoted $P(G,k)$, is the number of k-colourings of G. The basic technique for expressing $\chi_D(C_n,k)$ is to count the number of k-colourings of C_n that are not distinguishing, and subtracting this from $P(C_n,k)$, the number of k-colourings of C_n.

Suppose $C_n = v_0v_1\cdots v_{n-1}v_0$, $n \geq 3$. If a colouring of C_n is not distinguishing, then C_n has a nontrivial colour preserving automorphism, and such an automorphism of C_n is either a reflection or a rotation. First suppose that a colouring c of C_n is preserved by a nontrivial rotation. Then there exists an integer p, $1 < p < n$, such that $p|n$ and such that

$$c(v_i) = c(v_{i+\frac{n}{p}})$$

for all i, $0 \leq i \leq n-1$. (Here and in what follows, subscripts are taken modulo n.) By induction, it follows that for each i, $0 \leq i \leq n-1$,

$$c(v_i) = c(v_{i+j\frac{n}{p}})$$

for all integers j. Thus, such a colouring c is completely defined by the colours of the vertices of the path $v_0v_1\cdots v_{n/p-1}$.

Definition 1. *Let $C_n = v_0v_1\cdots v_{n-1}v_0$, and suppose p is an integer such that $p|n$ and $1 < p < n$. We define a set of k-colourings of C_n as follows: $c \in \mathcal{R}^p_{n,k}$ if and only if $c(v_i) = c(v_{i+n/p})$ for all i, $0 \leq i \leq p-1$. Then $\mathcal{R}^p_{n,k}$ consists of all k-colourings of C_n that are preserved by a rotation through n/p.*

The following technical lemma is used to simplify the expression for $\chi_D(C_n,r)$.

Lemma 11. *Let $\mathcal{R}^p_{n,k}$ denote the set of all k-colourings of C_n that are preserved by rotation through n/p for some $1 < p < n$. If $1 < p' < p$ and $p'|p$, then $\mathcal{R}^p_{n,k} \subseteq \mathcal{R}^{p'}_{n,k}$.*

Proof. Let $p = qp'$ where $q > 1$, and let $c \in \mathcal{R}^p_{n,k}$. Then

$$c(v_i) = c(v_{i+\frac{n}{p}})$$

for $i \in \mathbb{Z}$ (throughout subscripts are taken modulo n), and hence

$$c(v_i) = c(v_{i+j\frac{n}{p}})$$

for $j \in \mathbb{Z}$. In particular, for $j = q$, we have

$$c(v_i) = c(v_{i+q\frac{n}{p}}) = c(v_{i+\frac{n}{p'}}),$$

since $p = qp'$. Therefore, $c \in \mathcal{R}^{p'}_{n,k}$. □

Corollary 12. *Let $\mathcal{R}^p_{n,k}$ denote the set of all k-colourings of C_n that are preserved by rotation through n/p for some $1 < p < n$. If $p = p_1p_2$ for $1 < p_i < p$, $i \in \{1, 2\}$ then $\mathcal{R}^{p_1}_{n,k} \cap \mathcal{R}^{p_2}_{n,k} = \mathcal{R}^p_{n,k}$.*

Proof. By Lemma 11, $\mathcal{R}^p_{n,k} \subseteq \mathcal{R}^{p_1}_{n,k}$. Similarly $\mathcal{R}^p_{n,k} \subseteq \mathcal{R}^{p_2}_{n,k}$. Therefore

$$\mathcal{R}^p_{n,k} \subseteq \mathcal{R}^{p_1}_{n,k} \cap \mathcal{R}^{p_2}_{n,k}.$$

Now suppose $c \in \mathcal{R}^{p_1}_{n,k} \cap \mathcal{R}^{p_2}_{n,k}$. Then for $i, j, m \in \mathbb{Z}$ (throughout subscripts are taken modulo n),

$$c(v_i) = c(v_{i+j\frac{n}{p_1}}) = c(v_{i+m\frac{n}{p_2}}).$$

This implies

$$\begin{aligned} c(v_i) &= c(v_{i+j\frac{n}{p_1}+m\frac{n}{p_2}}) \\ &= c(v_{i+\frac{mnp_1+jnp_2}{p_1p_2}}) \\ &= c(v_{i+(mp_1+jp_2)\frac{n}{p}}). \end{aligned}$$

Since $(mp_1 + jp_2) \in \mathbb{Z}$, $c \in \mathcal{R}^p_{n,k}$, and thus $\mathcal{R}^{p_1}_{n,k} \cap \mathcal{R}^{p_2}_{n,k} \subseteq \mathcal{R}^p_{n,k}$. It follows that $\mathcal{R}^{p_1}_{n,k} \cap \mathcal{R}^{p_2}_{n,k} = \mathcal{R}^p_{n,k}$. □

Remark 4. *Notice that $|\mathcal{R}^p_{n,k}| = P(C_{n/p}, k)$. A colouring $c \in \mathcal{R}^p_{n,k}$ is determined by $c(v_i)$, $0 \leq i \leq n/p - 1$. In addition, $c(v_{n/p-1}) = c(v_{n-1}) \neq c(v_0)$, so there is a one-to-one correspondence between elements of $\mathcal{R}^p_{n,k}$ and k-colourings of $C_{n/p}$.*

Suppose that $n \geq 3$ is odd and that c is a colouring of C_n that is not distinguishing. Then any nontrivial colour preserving automorphism $g \in \text{Aut}(C_n)$ must be a rotation, since a reflection would require a pair of adjacent vertices $u, v \in V(C_n)$ to have $g(u) = v$ and $g(v) = u$, which is impossible since $c(u) \neq c(v)$ and g preserves colours.

Theorem 13. *Let n be odd, $n = p_1^{j_1} p_2^{j_2} \dots p_q^{j_q}$, for distinct primes $p_i > 2$, and $j_i \in \mathbb{N}$, $1 \leq i \leq q$. Then*

$$\chi_D(C_n, k) = \frac{P(C_n, k) - \left|\bigcup_{i=1}^{q} \mathcal{R}_{n,k}^{p_i}\right|}{2n}.$$

Proof. Let $C_n = v_0 v_1 \cdots v_{n-1} v_0$, and suppose c is a non-distinguishing k-colouring of C_n. Then $c \in \mathcal{R}_{n,k}^{p}$ for some p such that $p|n$ and $1 < p < n$. Thus the set of non-distinguishing k-colourings of C_n can be described by

$$\bigcup_{\substack{p|n \\ 1<p<n}} \mathcal{R}_{n,k}^{p}.$$

Suppose $p = p_1^{t_1} \times p_2^{t_2} \times \cdots \times p_q^{t_q}$, $0 \leq t_i \leq j_i$ and $1 \leq i \leq q$. Then, as a consequence of Corollary 12,

$$\bigcup_{\substack{p|k \\ 1<p<n}} \mathcal{R}_{n,k}^{p} = \bigcup_{i=1}^{q} \mathcal{R}_{n,k}^{p_i}.$$

The number of k-colourings of C_n is $P(C_n, k)$, so the number of distinguishing k-colourings of C_n is

$$P(C_n, k) - \left|\bigcup_{i=1}^{q} \mathcal{R}_{n,k}^{p_i}\right|. \tag{4}$$

To find the number of inequivalent distinguishing k-colourings of C_n we divide by $2n$, the order of the dihedral group D_n. Therefore

$$\chi_D(C_n, k) = \frac{P(C_n, k) - \left|\bigcup_{i=1}^{j} \mathcal{R}_{n,k}^{p_i}\right|}{2n}.$$

$\square$

By using $|\mathcal{R}_{n,k}^{p}| = P(C_{n/k}, r)$, and the Principle of Inclusion-Exclusion, one can expand (4) and obtain an expression for $\chi_D(C_n, k)$ in terms of the chromatic polynomials of cycles. This can be evaluated to give the exact value of $\chi_D(C_n, k)$.

Before finding an expression for the number of inequivalent distinguishing k-colourings for a cycle of even length, it is necessary to first characterize non-distinguishing k-colourings of even length cycles that are preserved by reflection through a line containing antipodal vertices of the cycle, in addition to being preserved by a nontrivial rotation. Note that for such a k-colouring, we may assume, without loss of generality, that the vertices of C_{2n} are labelled so that there is a line of reflection through v_0 and v_n.

Definition 2. *Let $C_{2n} = v_0v_1 \cdots v_{2n-1}v_0$ and suppose p is an integer such that $p|n$ and $1 < p \leq n$. We define $\mathcal{F}^p_{2n,k}$ as the set of all k-colourings of C_{2n} for which*

$$c(v_i) = c(v_{-i}) \text{ and}$$
$$c(v_i) = c(v_{i+2n/p})$$

(throughout, subscripts are taken modulo $2n$). Then $\mathcal{F}^p_{2n,k}$ consists of all k-colourings of C_{2n} that are preserved by rotation through $2n/p$ and by reflection in the line containing v_0 and v_n.

A colouring c in $\mathcal{F}^p_{2n,k}$ is determined by the colours of the vertices on the path $v_0v_1 \dots v_{n/p}$. The reasons for this are twofold. First suppose $v_0, v_1, \dots, v_{n/p}$ have been coloured (i.e., labelled so that adjacent vertices have different colours). Since there is a reflection in the line containing v_0 and v_n it follows that

$$\begin{aligned} c(v_{2n-1}) &= c(v_1) \\ c(v_{2n-2}) &= c(v_2) \\ \vdots\ \ &= \ \ \vdots \\ c(v_{(2n-\frac{n}{p})+2}) &= c(v_{\frac{n}{p}-2}) \\ c(v_{(2n-\frac{n}{p})+1}) &= c(v_{\frac{n}{p}-1}) \\ c(v_{(2n-\frac{n}{p})}) &= c(v_{\frac{n}{p}}). \end{aligned}$$

Thus the colours of $v_{2n-n/p}, v_{2n-n/p+1}, \dots, v_{2n-1}$ are forced by the reflection. Furthermore, since c is preserved by rotation through $2n/p$, it follows that

$$\begin{aligned} c(v_{\frac{n}{p}+1}) &= c(v_{(2n-\frac{n}{p})+1}) &&= c(v_{\frac{n}{p}-1}) \\ c(v_{\frac{n}{p}+2}) &= c(v_{(2n-\frac{n}{p})+2}) &&= c(v_{\frac{n}{p}-2}) \\ c(v_{\frac{n}{p}+3}) &= c(v_{(2n-\frac{n}{p})+3}) &&= c(v_{\frac{n}{p}-3}) \\ \vdots\ \ &= \ \ \vdots &&= \ \ \vdots \\ c(v_{\frac{2n}{p}-1}) &= c(v_{2n-1}) &&= c(v_1). \end{aligned}$$

Now we have coloured $v_0, v_1, \dots, v_{2n/p-1}$. Since the colouring is preserved by rotation through $2n/p$, it follows by induction that for each i, $0 \leq i \leq 2n-1$,

$$c(v_i) = c(v_{i+j\frac{2n}{p}}),$$

for all $j \in \mathbb{Z}$, where subscripts are taken modulo $2n$.

Lemma 14. *Let $\mathcal{F}^p_{2n,k}$ denote the set of all k-colourings of C_{2n} that are preserved by rotation through $2n/p$ for some $1 < p \le n$, and a reflection in the line through v_0 and v_n. If $1 < p' < p$ and $p'|p$, then $\mathcal{F}^p_{2n,k} \subseteq \mathcal{F}^{p'}_{2n,k}$.*

Proof. Let $p = qp'$ where $q > 1$, and let $c \in \mathcal{F}^p_{2n,k}$. Then $c(v_i) = c(v_{i+2n/p})$ for all i, $0 \le i \le 2n-1$ and $c(v_i) = c(v_{-i})$ (throughout, subscripts are taken modulo $2n$). This implies that $c(v_i) = c(v_{i+j2n/p})$ for all $j \in \mathbb{Z}$. In particular, for $j = q$, we have

$$c(v_0) = c(v_{i+q\frac{2n}{p}}) = c(v_{i+\frac{2n}{p'}}).$$

Since $c \in \mathcal{F}^p_{2n,k}$, $c(v_i) = c(v_{-i})$ for all i, $0 \le i \le 2n-1$, where subscripts are taken modulo $2n$. Therefore, $c \in \mathcal{F}^{p'}_{2n,k}$. □

A key point from the proof of Lemma 14 is that, independent of the value of p, any $c \in \mathcal{F}^p_{2n,k}$ has the property that $c(v_i) = c(v_{-i})$.

Corollary 15. *Let $\mathcal{F}^p_{2n,k}$ denote the set of all k-colourings of C_{2n} that are preserved by rotation through $2n/p$ for some $1 \le p \le n$, and by reflection in the line containing v_0 and v_n. If $p = p_1p_2$ for $1 < p_i < p, i \in \{1,2\}$ then $\mathcal{F}^{p_1}_{2n,k} \cap \mathcal{F}^{p_2}_{2n,k} = \mathcal{F}^p_{2n,k}$.*

Proof. By Lemma 11, $\mathcal{F}^p_{2n,k} \subseteq \mathcal{F}^{p_1}_{2n,k}$. Similarly $\mathcal{F}^p_{2n,k} \subseteq \mathcal{F}^{p_2}_{2n,k}$. Therefore

$$\mathcal{F}^p_{2n,k} \subseteq \mathcal{F}^{p_1}_{2n,k} \cap \mathcal{F}^{p_2}_{2n,k}.$$

Now suppose $c \in \mathcal{F}^{p_1}_{2n,k} \cap \mathcal{F}^{p_2}_{2n,k}$. First note that

$$c(v_i) = c(v_{-i}).$$

Secondly,

$$c(v_i) = c(v_{i+j\frac{2n}{p_1}}) = c(v_{i+m\frac{2n}{p_2}}),$$

for any $m, j \in \mathbb{Z}$ (all subscripts are taken modulo $2n$). This implies that

$$\begin{aligned} c(v_i) &= c(v_{i+j\frac{2n}{p_1}+m\frac{2n}{p_2}}) \\ &= c(v_{i+\frac{2mnp_1+2njp_2}{p_1p_2}}) \\ &= c(v_{i+(mp_1+jp_2)\frac{2n}{p}}). \end{aligned}$$

Since $(mp_1 + jp_2) \in \mathbb{Z}$, $c \in \mathcal{F}^p_{2n,k}$, and thus $\mathcal{F}^{p_1}_{2n,k} \cap \mathcal{F}^{p_2}_{2n,k} \subseteq \mathcal{F}^p_{2n,k}$. We conclude that $\mathcal{F}^{p_1}_{2n,k} \cap \mathcal{F}^{p_2}_{2n,k} = \mathcal{F}^p_{2n,k}$. □

Remark 5. *Notice that* $|\mathcal{F}^p_{2n,k}| = P(P_{n/p+1}, k)$. *A colouring* $c \in \mathcal{F}^p_{2n,k}$ *is determined by* $c(v_i), 0 \leq i \leq n/p$. *As a result, there is a one-to-one correspondence between elements of* $\mathcal{F}^p_{2n,k}$ *and k-colourings of* $P_{n/p+1}$.

Theorem 16. *Let* $2n = p_0^{j_0} p_1^{j_1} p_2^{j_2} \dots p_q^{j_q}$, *for distinct primes* p_i *with* $p_0 = 2$, *and* $j_i \in \mathbb{N}$, $0 \leq i \leq q$. *Then,*

$$\chi_D(C_{2n}, k) = \frac{P(C_{2n}, k) - \left|\bigcup_{i=0}^{q} \mathcal{R}^{p_i}_{2n,k}\right| - P(P_{n+1}, k) + \left|\bigcup_{i=\delta}^{q} \mathcal{F}^{p_i}_{2n,k}\right|}{4n},$$

where $\delta = 1$ *if* $j_0 = 1$, *and* $\delta = 0$ *if* $j_0 > 1$.

Proof. Suppose $C_{2n} = v_0 v_1 \dots v_{2n-1} v_0$ and that c is a non-distinguishing k-colouring of C_{2n}. Then c is preserved by a rotation, a reflection or both. If c is preserved by a rotation, then $c \in \mathcal{R}^p_{2n,k}$ for some $p|2n$ and $1 < p < 2n$. If, in addition, c is preserved by reflection, then $c \in \mathcal{F}^p_{2n,k}$. If c is preserved by a reflection then, without loss of generality we may assume that there is a line of reflection through v_0 and v_n and that

$$c(v_i) = c(v_{-i})$$

for all i, $1 \leq i \leq n-1$ and $n+1 \leq i \leq 2n-1$ (subscripts taken modulo $2n$). This is equivalent to the number of ways to k-colour the path $P_{n+1} = v_0 v_1 \dots v_n$ and reflect the colours in line through v_0 and v_n. The number of k-colourings of P_{n+1} is simply $P(P_{n+1}, k)$. Using inclusion-exclusion, the number of k-colourings of C_{2n} that are not distinguishing is

$$\left|\bigcup_{\substack{p|2n \\ 1<p<2n}} \mathcal{R}^p_{2n,k}\right| + P(P_{n+1}, k) - \left|\bigcup_{\substack{p|n \\ 1<p\leq n}} \mathcal{F}^p_{2n,k}\right|. \tag{5}$$

In the first and last terms of this expression, we know that if $p|2n$, then $p = p_0^{t_0} p_1^{t_1} p_2^{t_2} \cdots p_q^{t_q}$ for some $0 \leq t_i \leq j_i$ and $0 \leq i \leq q$. As a consequence of Lemma 11,

$$\left|\bigcup_{\substack{p|2n \\ 1<p<2n}} \mathcal{R}^p_{2n,k}\right| = \left|\bigcup_{i=0}^{q} \mathcal{R}^{p_i}_{2n,k}\right|.$$

If $j_0 = 1$, then $2 \not| n$, and Lemma 14 implies that

$$\left|\bigcup_{\substack{p|n \\ 1<p\leq n}} \mathcal{F}^p_{2n,k}\right| = \left|\bigcup_{i=1}^{q} \mathcal{F}^{p_i}_{2n,k}\right|;$$

if $j_0 > 1$, then $2|n$, and Lemma 14 implies that

$$\left| \bigcup_{\substack{p|n \\ 1<p\leq n}} \mathcal{F}^{p}_{2n,k} \right| = \left| \bigcup_{i=0}^{q} \mathcal{F}^{p_i}_{2n,k} \right|.$$

To find the number of inequivalent distinguishing colourings of C_{2n} we divide by $4n$, the order of the dihedral group D_{2n}. It follows that

$$\chi_D(C_{2n},k) = \frac{P(C_{2n},k) - \left|\bigcup_{i=0}^{q} \mathcal{R}^{p_i}_{2n,k}\right| - P(P_{n+1},k) + \left|\bigcup_{i=\delta}^{q} \mathcal{F}^{p_i}_{2n,k}\right|}{4n}, \quad (6)$$

where $\delta = 1$ if $j_0 = 1$, and $\delta = 0$ if $j_0 > 1$. □

It follows from Remark 4 and Remark 5 that $|\mathcal{R}^{p}_{2n,k}|$ and $|\mathcal{F}^{p}_{2n,k}|$ can be computed using chromatic polynomials for cycles and paths, evaluated at k:

$$|\mathcal{R}^{p}_{2n,k}| = P(C_{2n/p},k)$$

and

$$|\mathcal{F}^{p}_{2n,k}| = P(P_{n/p},k).$$

Therefore the expression (6) can be evaluated to give an exact value of $\chi_D(C_{2n},k)$.

Acknowledgement. The authors would like to thank the referee the helpful comments and suggestions that improved this paper.

References

[1] J.A. Bondy and U.S.R. Murty, *Graph Theory*, Springer, GTM 244, 2008.

[2] C.T. Cheng, On computing the distinguishing and distinguishing chromatic numbers of interval graphs and other results, *Discrete Math.* **309** (2009), no. 16, 5169-5182.

[3] J.O. Choi, S.G. Hartke and H. Kaul, The distinguishing chromatic number of Cartesian products of graphs, *SIAM J. Discrete Math.* **24** (2010), no. 1, 82-100.

[4] K.L. Collins, M. Hovey and A.N. Trenk, Bounds on the distinguishing chromatic number, *Electron. J. Combin.* **16** (2009), #R88.

[5] K.L. Collins and A.N. Trenk, The distinguishing chromatic number, *Electron. J. Combin.* **13** (2006), #R16.

[6] C.J. Hodgins, The Distinguishing Chromatic Number of Wreath Products, Master's Thesis, University of Calgary, 2010.

[7] J. Jerebic, and S. Klavžar, The distinguishing chromatic number of Cartesian products of two complete graphs, *Discrete Math.* **310** (2010), no. 12, 1715-1720.

[8] C. Laflamme and K. Seyffarth, Distinguishing chromatic numbers of bipartite graphs, *Electron. J. Combin.* **16** (2009), #R76.

[9] A.K. Lal and B. Bhattacharjya, Breaking the symmetries of the book graph and the generalized Petersen graph, *SIAM J. Discrete Math.* **23** (2009), no. 3, 1200-1216.

[10] S. Negami and S. Sakurai, Distinguishing chromatic numbers of planar graphs, *Yokohama Math. J.* **55** (2010), no. 2, 179-188.

[11] D.K. Tang, The Distinguishing Chromatic Number of the Graph $C_n \circ I_m$, Honor's Thesis, Wesleyan University, 2007.

[12] M.E. Watkins and X. Zhou, Distinguishability of locally finite trees, *Electron. J. Combin.* **14** (2007), #R29.

[13] J. Weigand and M.S. Jacobson, Distinguishing and distinguishing chromatic numbers of generalized Petersen graphs, *AKCE J. Graphs Combin.* **5** (2008), no. 2, 199-211.