Eternal Domination Numbers of $4 \times n$ Grid Graphs

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Abstract

The *eternal domination number* of graph G is the smallest set of mobile guards which can defend G against an infinite sequence of attacks on its vertices. In this paper we give results for the eternal domination numbers of $P_4 \square P_n$.

Keywords. Eternal Domination, Grid Graphs

1 Definitions and Introduction

A *dominating set* of a graph $G = (V, E)$ is a set $D \subseteq V$ such that $\forall u \in V - D$, $\exists v \in D$ such that $uv \in E(G)$. The set of all dominating sets of cardinality q of G will be denoted $\mathbb{D}_q(G)$. The *domination number* of G, denoted as $\gamma(G)$, is the minimum q such that $\mathbb{D}_q(G) \neq \emptyset$.

The "eternal dominating game" is a game between two players, the attacker and defender. The defender is given q guards to protect the graph from attacks made by the attacker. Initially the defender picks a dominating set $D \in \mathbb{D}_q(G)$, and places a guard on each vertex of D. A vertex in D is said to be *defended*. If $u \in D$, a guard is said to be on or at u. On each turn, the attacker picks a vertex $v \in V - D$ to attack. In response to this attack, the defender must defend vertex v by moving a guard to v from an adjacent vertex in D, and must move some subset of guards at other vertices in D to adjacent vertices, in order to re-create another dominating set, say D'. In this case we say that dominating set D has been *transformed* to dominating set D'.

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The two players, attacker and defender, continue to take turns making attacks and corresponding moves to defend the attacks, until the defender is not able to defend an attacked vertex v, or the defender is not able to re-create another dominating set; in which case the attacker wins. If it can be shown that the defender can respond to any sequence of attacks, no matter how long a sequence, by always defending an attacked vertex and re-creating a dominating set, then the defender wins.

Some attacker/defender models limit the number of guards that can move in response to an attack at a vertex v (see [1, 6, 8]). In this paper we focus on the "all guards move" model (which has also been referred to as m-eternal domination), where we place no such limits. Klostermeyer and Mynhardt have recently completed a survey of related results [10].

If G is a graph with domination number $\gamma(G)$ and $q \geq \gamma(G)$, then we define the graph $Dom(G,q)$ that has $\mathbb{D}_q(G)$ as its vertex set, with two dominating sets $D_1, D_2 \in \mathbb{D}_q(G)$ adjacent if there is a function $f : D_1 \to D_2$ such that f is a surjection and $f(u) \in N[u]$ for all $u \in D_1$. An *eternal q-dominating family* is defined to be a subset $\mathcal{E}_q \subseteq \mathbb{D}_q(G)$ having the property that for any dominating set $D \in \mathcal{E}_q$ and any (attacked) vertex $v \in V - D$, there exists a dominating set $D' \in \mathcal{E}_q$ such that $v \in D'$, i.e. v is defended by D' and D transforms to D'. The *eternal domination number* in the all guards move model, which we denote as $\gamma_{all}^{\infty}(G)$, equals the minimum integer q such that G has an eternal q-dominating family.

The *Cartesian product* of graphs G and H is denoted by $G\Box H$. The vertex set of $G\Box H$ is $V(G\Box H) = \{(u,v)|u \in V(G), v \in V(H)\}$, and two vertices (u,v) and (u',v') are adjacent if and only if $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$. When $G = P_m$ and $H = P_n$, these graphs are also known as a *grids* or *grid graphs* of dimensions $m \times n$. Label the vertices of P_m (respectively P_n) in their usual ordering $u_1, u_2, \ldots u_m$ (resp. $v_1, v_2, \ldots v_n$). For simplicity we use (i,j) to denote the vertex (u_i, v_j). We will call the copy of P_m in $P_m\Box P_n$ induced by $(1,i), \ldots, (m,i)$ the i^{th} column of this graph. In this paper, we discuss the eternal domination numbers of grid graphs with $m = 4$.

Goodard, Hedetniemi and Hedetniemi [1] introduced the concept of eternal domination and presented several fundamental bounds and calculations. In [11], Klostermeyer and Mynhardt show the eternal domination number is less than the vertex cover number in any graph of minimum degree at least two having girth at least nine. The same authors have shown there is only one graph with minimum degree two and equal eternal domination number and the analogous eternal vertex cover number [9]. Some variations of this concept have also been studied (see [1, 6, 7, 8]).

Results for the domination number of grid graphs date back to 1983, when Jacobson and Kinch [5] established $\gamma(P_m \Box P_n)$ for $m \leq 4$. Several other results for small values of m were discovered, but the question of a general formula for grid graphs remained open until 2011, when Gonçalves et. al. [3] showed that $\gamma(P_m \Box P_n) = \left\lceil \frac{(m+2)(n+2)}{5} \right\rceil - 4$ for $m, n \geq 16$.

Goldwaser et. al. [2] recently initiated the study of the eternal domination numbers of grid graphs. Specifically, the authors find the eternal domination number of $P_2 \Box P_n$ for every n and the eternal domination number of $P_3 \Box P_n$ for $n \leq 14$ and $n = 19$. For $n > 14$, it is known that

$$\left\lceil \frac{3n+1}{4} \right\rceil = \gamma(P_3 \Box P_n) \leq \gamma_{all}^{\infty}(P_3 \Box P_n) \leq \lceil \frac{8n}{9} \rceil \tag{1}$$

and it is conjectured [2] that $\gamma_{all}^{\infty}(P_3 \Box P_n) = 1 + \lceil \frac{4n}{5} \rceil$. In this paper we show that for any $4 \times n$ grid graph G, $\gamma_{all}^{\infty}(G) \in \{n+1, n(G)+2\}$.

Theorem 1.1 *For every positive integer* n,

$$\gamma_{all}^{\infty}(P_4 \Box P_n) = \begin{cases} n+1 & \text{if } n \text{ is odd or } n \in \{2,6\} \\ n+2 & \text{otherwise.} \end{cases}$$

At the 2012 Graph Protection Workshop in Paris, Finbow and Klostermeyer posed the following problem: Is there a constant c such that $\gamma_{all}^{\infty}(P_m \Box P_n) \leq \gamma(P_m \Box P_n) + c$ for all $m, n \geq 4$. For $m, n \geq 4$, Theorem 1.1, together with (1), implies that

$$\gamma_{all}^{\infty}(P_m \Box P_n) \leq \frac{mn}{4} + o(m+n).$$

2 Notation and Fundamental Bounds

The domination number of a graph is a trivial lower bound on the eternal domination number of a graph. With the exception of a few small cases, this bound is not tight for $4 \times n$ grid graphs. The following results will be used through out the paper to assist in establishing lower bounds.

Theorem 2.1 ([5]) *For every positive integer* n,

$$\gamma(P_4 \Box P_n) = \begin{cases} n+1 & \text{if } n \in \{1,2,3,5,6,9\} \\ n & \text{otherwise.} \end{cases}$$

Lemma 2.2 *For all* $2 \leq k \leq n-1$, *any minimal dominating set of* $P_4 \Box P_n$ *must contain at least* $\gamma(P_4 \Box P_{k-1})$ *vertices in the first* k *columns, and must contain at least* $\gamma(P_4 \Box P_{k-1})$ *vertices in the last* k *columns.*

Proof: The proof is trivial, as no guard outside the first (last) k columns will dominate any vertex in the first (last) $k-1$ columns. Thus the vertices in the first (last) $k-1$ columns must be dominated by guards in the first (last) k columns. ■

Lemma 2.3 *If* $\gamma_{all}^{\infty}(P_4\Box P_n) = j$ *and* $\gamma_{all}^{\infty}(P_4\Box P_q) = k$*, then*

$$\gamma_{all}^{\infty}(P_4\Box P_{n+q}) \leq j + k.$$

Lemma 2.3 is used to establish upper bounds for the eternal domination number by partitioning the graph into smaller graphs for which the eternal domination number is known. In some cases upper bounds are obtained by explicitly providing an eternal q-dominating family. In constructing an eternal q-dominating family of a $4 \times n$ grid graph, we can make use of the symmetries of $P_4\Box P_n$. Therefore we introduce the following notation. Given a dominating set $D \in \mathbb{D}_q(G)$, a reflection of D about the vertical line of symmetry is denoted D_v, while a reflection about the horizontal line of symmetry is denoted D_h. We will use D_{hv} to denote a reflection of D_h about the vertical line of symmetry. The dominating sets D_{vh}, D_{vv} and D_{hh} are defined analogously. It is useful to note that $D_{vv} = D_{hh} = D$. A rotation of a dominating set D by π degrees will be denoted D''. Therefore $D_{vh} = D_{hv} = D''$. When we wish to discuss an arbitrary transformation of rotations and reflections of a dominating set D, we will refer to this dominating set as D_t.

Theorem 2.4 *Given dominating sets* $D, E \in \mathbb{D}_q(P_m\Box P_n)$*, and any arbitrary transformation* t *of rotations and reflections,* D *is adjacent to* E *in* $Dom(P_m\Box P_n, q)$ *if and only if* D_t *is adjacent to* E_t *in* $Dom(P_m\Box P_n, q)$.

Proof: Suppose D is adjacent to E in $Dom(P_m\Box P_n, q)$. Then there is a surjective function $f : D \to E$ such that $f(u) \in N[u]$ for all $u \in D$. For each vertex v, let v_t be the image of v under the transformation t. Define $g : D_t \to E_t$ by $g(u_t) = f(u)_t$ for each $u \in D$. The function g is surjective since f is surjective and t is a bijection. As t is a transformation of rotations and reflections, $N[u_t] = \{v_t | v \in N[u]\}$. Therefore for every $u \in D$, $g(u_t) \in N[u_t]$. It follows that D_t is adjacent to E_t in $Dom(P_m\Box P_n, q)$.

On the other hand, suppose D_t is adjacent to E_t in $Dom(P_m\Box P_n, q)$. Let s be the inverse transformation of t. By the proof of sufficiency, $(D_t)_s$ is adjacent to $(E_t)_s$ in $Dom(P_m\Box P_n, q)$. Hence D is adjacent to E in $Dom(P_m\Box P_n, q)$.

■

3 Two Special Cases

In this section we find the value of $\gamma_{all}^{\infty}(P_4 \Box P_n)$ when $n = 4$ and $n = 6$.

Theorem 3.1 $\gamma_{all}^{\infty}(P_4 \Box P_4) = 6$

Proof: It is known, [2], that $\gamma_{all}^{\infty}(P_2 \Box P_4) = 3$ and therefore by Lemma 2.3, $\gamma_{all}^{\infty}(P_4 \Box P_4) \leq 6$.

Assume $P_4 \Box P_4$ has a dominating set D with cardinality 5. There must be at least one guard in the neighbourhood of each corner. Divide the graph into four quadrants, using the horizontal and vertical lines of symmetry. It follows that, at any time step, there are two guards in one quadrant and one guard in each of the other three quadrants. Assume there are two guards in the upper left quadrant.

Consider the defenders' response to an attack at $(3,3)$. As the guards in the top right and the bottom left quadrant must be adjacent to their corresponding corners, the guard in the bottom right quadrant must move to $(3,3)$. A guard from the top right or bottom left quadrant must move to the neighbourhood of the corner $(4,4)$. By symmetry, we may assume the guard in the top right quadrant moves to $(3,4)$. It follows that a guard from the top left quadrant must move to $(1,3)$. The remaining two guards must move to vertices so that a dominating set is formed. It follows there is a guard on $(2,1)$ and a guard on either $(4,1)$ or $(4,2)$.

If the attacker attacks the corner $(1,4)$, the guard at $(1,3)$ must now move to the attacked vertex, and the guard at $(2,1)$ must move to $(1,1)$ (this is the only guard who can guard $(1,1)$ and $(1,2)$). The guard on either $(4,1)$ or $(4,2)$ must remain in $N[(4,1)]$. Hence the guard at $(3,3)$ must guard $(2,2)$ by either moving to $(3,2)$ or to $(2,3)$. In the former case, the final guard would need to move to a dominating set which would guard both $(4,4)$ and $(2,3)$, which is impossible. In the latter case the final guard must move to a vertex in $\{(3,4),(4,4)\}$ so that both of these two vertices are dominated. It follows that one of the vertices in the bottom left quadrant is unprotected and hence D is not an eternal dominating set. ∎

Theorem 3.2 $\gamma_{all}^{\infty}(P_4 \Box P_6) = 7$

Proof: Since by Theorem 2.1, we know that $\gamma(P_4 \Box P_6) = 7$, it is sufficient to show the three γ-sets A, B and C in Figure 3.1, together with their horizontal reflections, A_h, B_h and C_h, their vertical reflections A_v, B_v and C_v, and the rotations by π, A'', B'' and C'' form an eternal 7-dominating family.

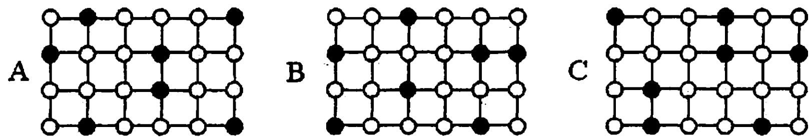

Figure 3.1: Three nonisomorphic dominating sets of an eternal 7-dominating family of $P_4 \Box P_6$

Some of the adjacencies between these dominating sets are given in Table 3.1. The adjacency of two dominating sets, X and Y, is denoted $X \Leftrightarrow Y$, and each guard movement between these dominating sets is symbolized by $x \leftrightarrow y$. Additional adjacencies amongst these dominating sets may be found using Theorem 2.4. For example, A is adjacent to C_h, thus Theorem 2.4 implies that A'' is adjacent to $(C_h)'' = C_v$. These dominating sets induce the subgraph of the $Dom(P_4 \Box P_6, 7)$ given in Figure 3.2.

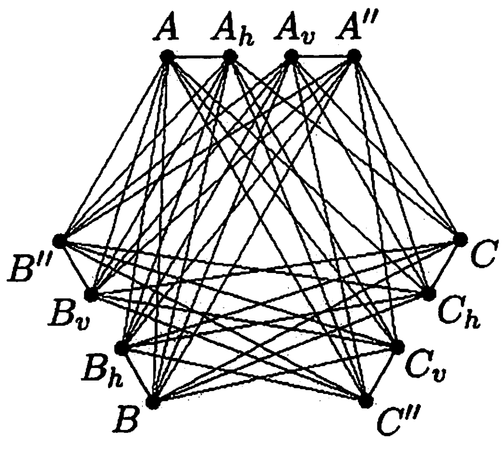

Figure 3.2: The subgraph induced by these twelve dominating sets

Table 3.2 gives the strategy the defender can use to defend $P_4 \Box P_6$ using these twelve dominating sets. Only the strategies for dominating sets A, B and C are listed explicitly. Suppose, for example, the defenders are in dominating set A_h and an attack occurs at $(3, 2)$. To determine the defenders strategy, one looks up the response to an attack (when guards are in dominating set A) to the image of $(3, 2)$ under a horizontal flip (namely

γ-sets	Adj. Vertices
$A \Leftrightarrow A_h$	$(1,2) \leftrightarrow (1,2)$, $(1,6) \leftrightarrow (1,6)$, $(2,1) \leftrightarrow (3,1)$, $(2,4) \leftrightarrow (2,4)$, $(3,4) \leftrightarrow (3,4)$, $(4,2) \leftrightarrow (4,2)$, $(4,6) \leftrightarrow (4,6)$
$A \Leftrightarrow B$	$(1,2) \leftrightarrow (1,3)$, $(1,6) \leftrightarrow (2,6)$, $(2,1) \leftrightarrow (2,1)$, $(2,4) \leftrightarrow (2,5)$, $(3,4) \leftrightarrow (3,3)$, $(4,2) \leftrightarrow (4,1)$, $(4,6) \leftrightarrow (4,5)$
$A \Leftrightarrow B_v$	$(1,2) \leftrightarrow (2,2)$, $(1,6) \leftrightarrow (2,6)$, $(2,1) \leftrightarrow (2,1)$, $(2,4) \leftrightarrow (1,4)$, $(3,4) \leftrightarrow (3,4)$, $(4,2) \leftrightarrow (4,2)$, $(4,6) \leftrightarrow (4,6)$
$A \Leftrightarrow B_h$	$(1,2) \leftrightarrow (1,1)$, $(1,6) \leftrightarrow (1,5)$, $(2,1) \leftrightarrow (3,1)$, $(2,4) \leftrightarrow (2,3)$, $(3,4) \leftrightarrow (3,5)$, $(4,2) \leftrightarrow (4,3)$, $(4,6) \leftrightarrow (3,6)$
$A \Leftrightarrow B''$	$(1,2) \leftrightarrow (1,2)$, $(1,6) \leftrightarrow (1,6)$, $(2,1) \leftrightarrow (3,1)$, $(2,4) \leftrightarrow (2,4)$, $(3,4) \leftrightarrow (4,4)$, $(4,2) \leftrightarrow (3,2)$, $(4,6) \leftrightarrow (3,6)$
$A \Leftrightarrow C_v$	$(1,2) \leftrightarrow (1,3)$, $(1,6) \leftrightarrow (1,6)$, $(2,1) \leftrightarrow (2,1)$, $(2,4) \leftrightarrow (2,3)$, $(3,4) \leftrightarrow (3,5)$, $(4,2) \leftrightarrow (4,2)$, $(4,6) \leftrightarrow (4,5)$
$A \Leftrightarrow C_h$	$(1,2) \leftrightarrow (1,2)$, $(1,6) \leftrightarrow (1,5)$, $(2,1) \leftrightarrow (2,2)$, $(2,4) \leftrightarrow (3,4)$, $(3,4) \leftrightarrow (4,4)$, $(4,2) \leftrightarrow (4,1)$, $(4,6) \leftrightarrow (3,6)$
$A \Leftrightarrow C''$	$(1,2) \leftrightarrow (1,2)$, $(1,6) \leftrightarrow (1,5)$, $(2,1) \leftrightarrow (3,1)$, $(2,4) \leftrightarrow (2,5)$, $(3,4) \leftrightarrow (3,3)$, $(4,2) \leftrightarrow (4,3)$, $(4,6) \leftrightarrow (4,6)$
$B \Leftrightarrow B_h$	$(1,3) \leftrightarrow (2,3)$, $(2,1) \leftrightarrow (1,1)$, $(2,5) \leftrightarrow (1,5)$, $(2,6) \leftrightarrow (3,6)$, $(3,3) \leftrightarrow (4,3)$, $(4,1) \leftrightarrow (3,1)$, $(4,5) \leftrightarrow (3,5)$
$B \Leftrightarrow C$	$(1,3) \leftrightarrow (1,4)$, $(2,1) \leftrightarrow (1,1)$, $(2,5) \leftrightarrow (2,4)$, $(2,6) \leftrightarrow (2,6)$, $(3,3) \leftrightarrow (3,2)$, $(4,1) \leftrightarrow (4,2)$, $(4,5) \leftrightarrow (4,5)$
$B \Leftrightarrow C_v$	$(1,3) \leftrightarrow (1,3)$, $(2,1) \leftrightarrow (2,1)$, $(2,5) \leftrightarrow (3,5)$, $(2,6) \leftrightarrow (1,6)$, $(3,3) \leftrightarrow (2,3)$, $(4,1) \leftrightarrow (4,2)$, $(4,5) \leftrightarrow (4,5)$
$B \Leftrightarrow C_h$	$(1,3) \leftrightarrow (1,2)$, $(2,1) \leftrightarrow (2,2)$, $(2,5) \leftrightarrow (1,5)$, $(2,6) \leftrightarrow (3,6)$, $(3,3) \leftrightarrow (3,4)$, $(4,1) \leftrightarrow (4,1)$, $(4,5) \leftrightarrow (3,5)$
$C \Leftrightarrow C_h$	$(1,1) \leftrightarrow (1,2)$, $(1,4) \leftrightarrow (1,5)$, $(2,4) \leftrightarrow (3,4)$, $(2,6) \leftrightarrow (3,6)$, $(3,2) \leftrightarrow (2,2)$, $(4,2) \leftrightarrow (4,1)$, $(4,5) \leftrightarrow (4,4)$

Table 3.1: Adjacencies between transformations of A, B and C

$(2,2)$). The appropriate response would be the horizontal flip of this dominating set and therefore in this example would be for the guards to move to $(B_v)_h = B''$. ∎

4 $\gamma^{\infty}_{all}(P_4 \Box P_n) = n + 1$ for Odd n

Recall from Theorem 2.1 that $\gamma(P_4 \Box P_n) \geq n$.

Lemma 4.1 *If D is any set of vertices that dominates every vertex of $P_4 \Box P_n$, except possibly vertex $(2,1)$, then $|D| \geq n$.*

Attack / Pos.	A	B	C
$(1,1)$	B_h	B_h	C
$(2,1)$	A	B	B
$(3,1)$	B_h	B_h	A_h
$(4,1)$	B	B	C_h
$(1,2)$	A	A	A_h
$(2,2)$	B_v	C_h	C_h
$(3,2)$	B''	C	C
$(4,2)$	A	A	C
$(1,3)$	B	B	B
$(2,3)$	B_h	B_h	B_h
$(3,3)$	B	B	B
$(4,3)$	B_h	B_h	B_h

Attack / Pos.	A	B	C
$(1,4)$	B_v	C_h	C
$(2,4)$	A	A	C
$(3,4)$	A	A	A_h
$(4,4)$	B''	C	C_h
$(1,5)$	B_h	B_h	C_h
$(2,5)$	B	B	B
$(3,5)$	B_h	B_h	B_h
$(4,5)$	B	B	C
$(1,6)$	A	A	A_h
$(2,6)$	B	B	C
$(3,6)$	B_h	B_h	B_h
$(4,6)$	A	A	A_h

Table 3.2: Guard Movements

Proof: Let D be any set of vertices that dominates every vertex of $P_4 \Box P_n$, except possibly vertex $(2,1)$, and assume $|D| < n$. Consider the graph $G = P_4 \Box P_{n+4}$. Construct a dominating set D' of G such that $D = \{(3,1),(1,2),(4,3),(2,4)\} \cup D^*$, where D^* is the trivial image of D on the last n columns of G. It can be seen that D is a dominating set of $P_4 \Box P_{n+4}$ with $|D| < n+4$, contradicting Theorem 2.1. ∎

Lemma 4.2 *If a dominating set D of $P_4 \Box P_n$ includes a corner, then* $|D| \geq n+1$.

Proof: Consider the smallest counter example. For $n = 4$, the only two dominating sets of cardinality four do not have a guard in any corner and if $n \leq 6$ and $n \neq 4$, then $\gamma(P_4 \Box P_n) = n+1$. Assume $n \geq 7$.

We wish to form a minimum dominating set D of $G = P_4 \Box P_n$ such that $(1,1) \in D$. Define $D' = \{(1,1)\}$ to be a set that initially only contains $(1,1)$. We will add vertices to D' until D' becomes a dominating set. In all cases it will be shown that no matter how D' becomes a dominating set, $|D'| \geq n+1$.

Given two vertices $x, y \notin D'$, we say that y *beats* x or x is *beaten by* y if $N[D' \cup \{x\}] \subset N[D' \cup \{y\}]$. (This terminology was introduced by Hare and Fisher [4].) Therefore, given two vertices $x, y \notin D'$ we will never add vertex x to D' if y beats x.

Initially $D' = \{(1,1)\}$, and we must add a vertex which will dominate vertex $(4,1)$, either $(3,1)$, $(4,1)$, or $(4,2)$.

Case 1: Add $(3,1)$ to D' to dominate $(4,1)$.

Now consider the closed neighbourhood $N[(4,2)]$, which currently contains no vertices in D'. Since vertex $(4,3)$ beats both $(4,1)$ and $(4,2)$, in order to dominate $(4,2)$ we must add either $(3,2)$ or $(4,3)$. If we add $(3,2)$ to D', then each vertex in the second column not in D' is beaten by a vertex in the third column. Hence by Lemma 4.1 and horizontal symmetry, any additions to D' to produce a dominating set D of $P_4 \Box P_n$, will dominate every vertex in the last $n-2$ columns, except possibly vertex $(3,3)$, must contain at least $n-2$ vertices. Therefore, $|D'| \geq n+1$.

However, if we add $(4,3)$ to D' in order to dominate $(4,2)$, then since vertex $(2,3)$ will beat any vertex not in D' in the second column, we will not add any other vertices in the second column in order to produce a dominating set D'. Thus, by the minimality of n, and the fact that we have a vertex $(4,3)$ in the corner of the remaining $n-2$ columns, we must add at least $n-1$ vertices to D' to dominate the last $n-2$ columns. Therefore, $|D'| \leq n-1+2 = n+1$.

Case 2: Add $(4,1)$ to D' to dominate $(4,1)$.

Now consider the closed neighborhood $N[(2,2)]$, which currently contains no vertices in D'. In this case, since vertex $(2,3)$ beats both $(1,2)$ and $(2,1)$, we must add either $(2,2)$, $(2,3)$ or $(3,2)$ to dominate $(2,2)$. But adding vertex $(3,2)$ is equivalent to adding $(2,2)$, by horizontal symmetry, thus, we either add $(2,2)$ or $(2,3)$.

If we add $(2,2)$ to dominate $(2,2)$, then each vertex in the second column not in D' will be beaten by a vertex in the third column. Thus, in order to create a dominating set D', by Lemma 4.1, we must add at least $n-2$ vertices to dominate the last $n-2$ columns. Therefore, D' must have at least $n-2+3 = n+1$ vertices.

On the other hand, suppose we add $(2,3)$ to dominate $(2,2)$. Then in order to dominate $(3,2)$ we will add $(3,3)$ since it beats any other vertex that dominates $(3,2)$. In this case consider dominating next vertex $(1,4)$. Since $(1,5)$ beats any vertex that dominates $(1,4)$ we add $(1,5)$ to D'. But now, by the minimality of n, we will need at least $n-3$ vertices to dominate the last $n-4$ columns, since we have one vertex in the corner. Thus, the final set D' must have at least $n-3+4 = n+1$ vertices.

Case 3: Add $(4,2)$ to D' to dominate $(4,1)$.

In order to dominate $(3,1)$ we add $(3,2)$ since it beats every vertex in N$[(3,1)]$. Now consider dominating $(1,3)$. We can add either $(2,3)$, $(1,3)$ or $(1,4)$.

If we add $(2,3)$, then by Lemma 4.1, we will need at least $n-3$ vertices to dominate the last $n-3$ columns, resulting in at least $n+1$ vertices for D'. If we add $(1,4)$ to D', then by the minimality of n, we will need at least $n-2$ vertices to dominate 6 the last $n-3$ columns, or altogether $n-2+3=n+1$ vertices. Finally, we we add $(1,3)$ to D'. Consider then dominating $(4,4)$, either with $(3,4)$, $(4,4)$ or $(4,5)$. If we add $(3,4)$ then Lemma 4.1 applies to the last $n-4$ columns and we must add at least $n-4$ vertices to dominate the last $n-4$ columns, or $n-4+5=n+1$ vertices. If we add $(4,5)$, then by the minimality of n we will need at least $n-3$ vertices to dominate the last $n-4$ columns, or $n-3+4=n+1$ vertices altogether. Finally, if we use vertex $(4,4)$ to dominate itself, then in order to dominate $(2,4)$ we use $(2,5)$, and again Lemma 4.1 applies. ∎

Theorem 4.3 *For all odd n, $\gamma_{all}^{\infty}(P_4 \Box P_n) = n+1$.*

Proof: By Lemma 4.2, $\gamma_{all}^{\infty}(P_4 \Box P_n) \geq n+1$. For $n=1$ and $n=3$, the results are known (see [2]). For $n=4k+1$, where $k \geq 1$, the four dominating sets in Figure 4.1 are sufficient to eternally dominate $P_4 \Box P_{4k+1}$.

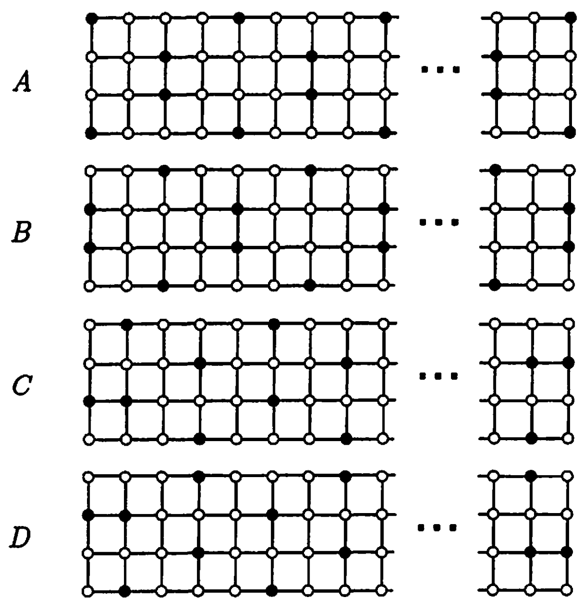

Figure 4.1: An eternal dominating family for a $P_4 \Box P_{4k+1}$

These four dominating sets form a complete subgraph of $Dom(P_4 \Box P_n, n+1)$. To show the adjacency of A and B, move every guard either up or down within it's column. Dominating sets A and C are adjacent, as, from dominating set A, all the guards in the top two rows move one space right, except for the rightmost guard, which is moved down, and all the guards in the bottom two rows move one space left, except for the leftmost guard, which is moved up. Noting that $A'' = A$ and $C'' = D$, Theorem 2.4 implies A is also adjacent to D. Starting at dominating set B, move all guards in the top two rows one space left, and move the leftmost guard down; move all the guards in the bottom two rows one space right, and move the rightmost guard one space up. This will result in dominating set C. Similarly, as $B'' = B$ and $C'' = D$, Theorem 2.4 implies adjacency between B and D. Lastly, we start at dominating set C, and to obtain D, each guard moves within its column as is appropriate. The union of the vertices in these dominating sets is equal to the entire vertex set and hence $\gamma_{all}^{\infty}(P_4 \Box P_n) \leq n+1$.

For $n = 4k+3$, where $k \geq 1$, four dominating sets similar to those used in $n = 4k+1$ are considered. These four dominating sets are shown in Figure 4.2.

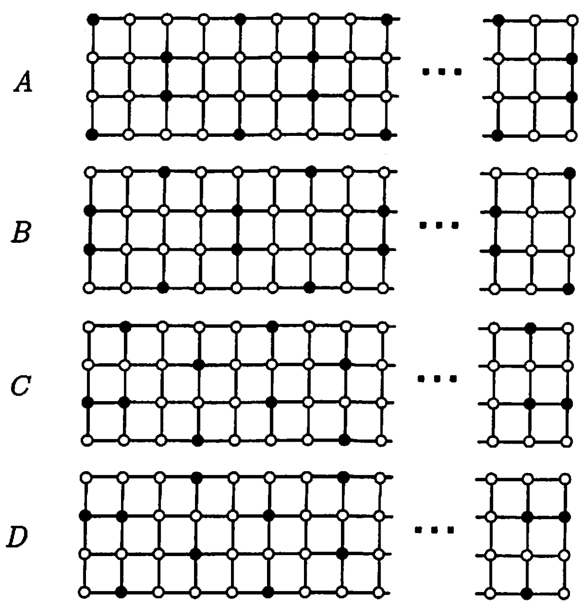

Figure 4.2: An eternal dominating family for a $P_4 \Box P_{4k+3}$

As before, these four dominating sets form a complete subgraph in $Dom(P_4 \Box P_n, n+1)$. Starting from dominating set A, B is obtained by moving guards appropriately within their columns. Again starting at dominating set A, move each guard in the top two rows one space right, except for the rightmost guard, which is moved one space down, and move the guards in the bottom two rows one space left, except for the leftmost guard, which is moved one space up. The result of these movements is C. The equalities $A = A_h = B'' = B_v$ and $C = C_v = D'' = D_h$ and Theorem 2.4 imply adjacencies between all other dominating sets except between C and D. To show this adjacency, move the guard(s) in each column up or down appropriately. The number of guards in each of these dominating sets is $n+1$, and the union of the vertices in these dominating sets is the entire vertex set. ■

5 $\gamma_{all}^{\infty}(P_4 \Box P_n) = n + 2$ for Even $n \geq 8$

If $n = 2$ the result of Theorem 1.1 is known [2] and hence we now focus on the case n is even and $n \geq 8$.

Lemma 5.1 *If D dominates $P_4 \Box P_8$, has 3 guards in the first 3 columns, and $(1,1) \in D$, then $|D| \geq 10$.*

Proof: Let D be a dominating set of $P_4 \Box P_8$ containing vertex $(1,1)$.

Suppose that there are two guards in the first column. Then, as D only has 3 guards in the first three columns, it follows that there is one guard in the second column and 3 guards in the fourth column. By Lemma 2.2, there are at least 4 guards in the last four columns and hence $|D| \geq 10$.

Therefore assume $(1,1)$ is the only vertex in D that is in the first column. It follows that $(3,2), (4,2) \in D$. As there are only 3 vertices in D in the first 3 columns, it must be that $(1,4), (2,4) \in D$. If D contained 6 elements in the first 4 columns, Lemma 2.2 implies $|D| \geq 10$ and hence it must be the case that $(4,5) \in D$. Clearly, there exist vertices $v_1, v_2, v_3, v_4 \in D$ such that $v_1 \in N[(1,8)]$, $v_2 \in N[(4,8)]$, $v_3 \in N[(2,6)]$ and $v_4 \in N[(3,6)]$. It follows that either $|D| \geq 10$ or $v_3 = v_4$ (and hence $v_3 \in \{(2,6), (3,6)\}$). However, in the later case, it is not possible for $\{v_1, v_2, v_3\}$ to dominate the last three columns. Hence, $|D| \geq 10$. ■

Lemma 5.2 *If D is an eternal dominating set of a $P_4 \Box P_n$, for $n \geq 8$, with $(2,1)$, $(4,2)$, $(1,3)$, $(3,4)$ in D and no other vertices from the first four columns in D, then there are at least 2 vertices in the fifth column in D.*

Proof: Consider if there were no guard at $(1,5)$, and the defender's response to an attack on $(1,4)$. The guard at $(1,3)$ responds to the attack forcing the guard at $(1,2)$ to move to $(1,1)$. It follows that the guard at $(4,2)$ must move to a dominating set to dominate $(2,2)$ and $(4,1)$, a contradiction showing $(1,5) \in D$.

If only these guards were in the last five columns, consider an attack on vertex $(4,4)$. The guard at $(3,4)$ defends the attack. If the guard at $(4,2)$ moves to $(4,3)$ or $(3,2)$, then the guard at $(2,1)$ must move to $(3,1)$ and thus the guard at $(1,3)$ must move to protect both $(1,1)$ and $(2,3)$ a contradiction. Hence, the guard at $(4,2)$ doesn't move into $N[(3,3)]$ and therefore the guard at $(1,3)$ moves to $(2,3)$. However, it is not possible for the guards at $(1,2)$ and $(2,4)$ to move so that all the vertices in the first two columns are dominated. ∎

Lemma 5.3 *Given an even $k \geq 2$, any set of guards D in $P_4 \Box P_k$ which dominate each vertex except possibly two vertices in the first column and two vertices in the last column will have $|D| \geq k$.*

Proof: The proof is by induction. When $k = 2$, if the result is false, one guard will need to guard four vertices, two in each column, and this guard can defend at most one vertex in a column that it's not in.

Suppose for some even k, the results holds for all even lengths less than k. If there are no guards in D in the first column, there must be at least two guards in the D in the second column. Moving any additional defenders from the second column to the adjacent vertex in the third column (if this is possible with out having two defenders on the same vertex), the defenders in the last $k-2$ columns, must still defend the last $k-2$ columns, except possibly the two vertices in the third column and two vertices in the last column. By induction, this implies there are at least $k-2$ guards in the last $k-2$ columns and hence $|D| \geq k$. Now consider the case where D has 2 guards in the first column. By induction, we would require at least $k-2$ guards in the last $k-2$ columns and hence $|D| \geq k$. Therefore, by symmetry, there is one guard in the first column and one guard in the last column. The induction hypothesis implies G', the subgraph of $P_4 \Box P_k$ made by deleting the first and kth columns will require at least $k-2$ additional guards and hence $|D| \geq k$. ∎

Corollary 5.4 *Let D be a dominating set of $P_4 \Box P_n$, suppose p and q are integers with the same parity, and let n be even such that $n \geq p+q$. If there exists a subset D' of D with $p+1$ guards in the first p columns and $q+1$*

guards in the last q columns, such that there are at most 2 guards in the pth column and at most 2 guards in the q^{th} last column, then $|D| \geq n+2$.

Proof: Let $k = n - p - q$. The vertices that are unguarded by D' in $P_4 \Box P_n$ form a $P_4 \Box P_k$, such that there are at most 2 vertices guarded on either end column. Form D^* from D/D' by moving (if possible) any guards from D/D' in the p^{th} column to the $p+1^{st}$ column and from the $p+k+1^{st}$ column to the $p+k^{th}$ column. By Lemma 5.3, $|D^*| \geq k$ and hence $|D| \geq k + |D'| = n+2$. ■

Corollary 5.5 *Let n be even, and D be an eternal dominating set of $P_4 \Box P_n$. Let p be even and q odd such that $n \geq p+4$ and $n \geq q+5$. If $D' \subseteq D$ contains at least $p+1$ guards in the first p columns with at most two guards in the pth column and $D^* \subseteq D$ contains at least $q+1$ guards in the first q columns, with at most two guards in the qth column, then $|D| \geq n+2$.*

Proof: By Lemma 2.2, there are at least four guards in the last four columns and it follows from Corollary 5.4 that the result holds or there is exactly one guard in each of the last four columns (since D is dominating). It must be the case that (w.o.l.o.g.) $(2,n), (4,n-1), (1,n-2), (3,n-3) \in D$ and no other vertex in the last four columns is in D. Therefore by Lemma 5.2, there are at least six guards in the last five columns and hence by Corollary 5.4, $|D| \geq n+2$. ■

Theorem 5.6 *For all even $n \geq 8$, $\gamma_{all}^{\infty}(P_4 \Box P_n) = n+2$*

Proof: By Theorem 4.2, $\gamma_{all}^{\infty}(P_4 \Box P_{n-1}) = n$ and it is easy to see that $\gamma_{all}^{\infty}(P_4) = 2$. Therefore by Theorem 2.3, $\gamma_{all}^{\infty}(P_4 \Box P_n) \leq n+2$.

Let n be even and suppose D is an eternal dominating set of $P_4 \Box P_n$, with $(1,1) \in D$ and $|D| \leq n+1$. Suppose first that the guard at $(1,1)$ is the only guard of D in the first column. It follows that $(3,2), (4,2) \in D$. By Corollary 5.5, it must be the case that there are only three guards of D in the first three columns. Hence $(1,4), (2,4) \in D$ and also it follows from Lemma 5.1 that $n > 8$. As there is a guard of D in $N[(4,4)]$, D has a subset of six guards in the first five columns with at most one in the fifth column. Corollary 5.5 implies $|D| \geq n+2$, a contradiction. Thus there are at least two guards from D in the first column.

By Corollary 5.5 there are exactly 2 guards in the first column and no guards in the second column. It follows that there are at least two guards in the third column. Repeated applications of Corollary 5.5, imply that for all even positive integers $k \leq n-4$ there are exactly k guards in the first k columns with two guards in the $k-1^{st}$ column and no guards in the k^{th} column and hence at least two guards in the $k+1^{st}$ column. Let $\{a,b,c\} = \{2,3,4\}$ with $(a,1) \in D$. For D to be dominating, it must be the case that $a \in \{3,4\}$, $(1,l),(a,l) \in D$ for $l \equiv 1 \mod 4$ and $l \leq n-3$ and $(b,l),(c,l) \in D$ for $l \equiv 3 \mod 4$ and $l \leq n-3$.

If $a = 3$, consider the dominating set D^*, the guards response to an attack at $(4,4)$. The guard at $(4,3)$ would be required to move to defend the attack, and hence the guard at $(3,1)$ is required to move to $(4,1)$. As D^* has a guard in the corner, if $n = 8$, then Lemma 5.1 implies there are four guards in D^* in the first three columns, however, moving from D, this is not possible. Hence, $n \geq 10$. The guard at $(2,3)$ must move to dominate $(3,2)$. If this guard were to move to $(3,3)$ only one guard would be able to move to vertices which could dominate $(1,1)$, $(2,1)$, $(1,2)$ and $(2,2)$. Hence, the guard at $(2,3)$ moves to $(2,2)$ and it follows that the guards at $(3,5)$ must move to $(3,4)$. Hence D^* has guards at $(1,4)$ and $(2,2)$ and a guards in $N[(1,1)]$. In addition D^* has guards at $(4,4)$ and $(3,4)$ and a guard in $N[(1,4)]$. This implies D^* has a subset of three guards in the first two columns (with at most two in the second column) and has a subset of six guards in the first five columns (with at most one in the fifth column). By Corollary 5.5, $|D^*| \geq n+2$ and hence $|D| \geq n+2$. We conclude that $a = 4$.

From Lemma 2.2, there are at least three guards in D in the last three columns and hence (as $|D| \leq n+1$) there are exactly three guards of D in the last three columns. Consider the dominating set D', the defenders response from D to an attack at $(1,n)$. By the above arguments for $l \geq 3$: $(1,l),(4,l) \in D'$ for $n-l \equiv 0 \mod 4$, $(2,l),(3,l) \in D'$ for $n-l \equiv 2 \mod 4$ and no vertex of D' is in the l^{th} column if l is odd. As D has three guards in the last three columns and D' has four guards in the last three columns a guard which is in column $n-3$ in D moves to column $n-2$ in D' and hence $(2,n-3),(3,n-3) \in D$. This implies $n \equiv 2 \mod 4$ and therefore $n \geq 10$. As column $n-3$ has no guards in D', both of the guards at $(2,n-3)$ and $(3,n-3)$ in D move to either column $n-4$ or $n-2$ (with at least one of these guards moving to column $n-2$). If both move to column $n-2$, then D' has five guards in the last four columns (with at most one guard in the fourth column) and hence Corollary 5.5 implies $|D'| \geq n+2$, a contradiction. Therefore it must be the case that one guard moves to column $n-4$. This guard must move to either $(2,n-4)$ or $(3,n-4)$ implying column $n-4$ has a guard at this vertex, at $(1,n-4)$ and $(4,n-4)$ in D'. Therefore D' has seven guards in the first six columns and it follows from Corollary 5.5 that $|D| = |D'| \geq n+2$. Thus $\gamma_{all}^{\infty}(P_4 \square P_n) \geq n+2$. ■

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