A new proof of a characterization of (k, l)-colourable chordal graphs

Dennis D.A. Epple *

Abstract

A graph is (k, l)-colourable if its vertex set can be partitioned into k independent sets and l cliques. A graph is chordal if it does not contain any induced cycle of length at least four. A theorem by Hell et. al. states that a chordal graph is (k, l)-colourable if and only if it does not contain $(l+1)K_{k+1}$ as an induced subgraph. Presented here is a short alternative proof of this result, using the characterization of chordal graphs via perfect elimination orderings.

For integers $k, l \geq 0$, a graph is (k, l)-*colourable* if its vertex set can be partitioned into k independent sets and l cliques. Both independent sets and cliques are allowed to be empty. Thus if a graph is (k, l)-colourable then it is also (k', l')-colourable for all $k' \geq k$ and $l' \geq l$.

The concept of (k, l)-colourings is a generalization of both colourings and clique coverings, as a k-colouring can be interpreted as a $(k, 0)$-colouring, whereas an l-clique-covering corresponds to a $(0, l)$-colouring. As well, split graphs are precisely the $(1, 1)$-colourable graphs. It has been shown that determining the (k, l)-colourability of perfect graphs for fixed k and l can be achieved in polynomial time [8]. Further, a characterization of (k, l)-colourable graphs via forbidden subgraphs has been found for two important subclasses of perfect graphs, cographs [2, 3, 4] and chordal graphs [7].

Chordal graphs, defined as graphs without induced cycles of length four or greater, were introduced by Hajnal and Surányi [6], albeit under a different name. Chordal graphs are perfect [1] and any induced subgraph of a chordal graph is chordal again. An important characterization of chordal graphs is the following one by Fulkerson and Gross [5]. For a graph G, a *perfect elimination ordering* is an ordering $v_1, v_2, \ldots, v_n$ of the vertices of

---

*University of Victoria, PO BOX 3060 STN CSC, Victoria, B.C., V8W 3R4, Canada; epple@math.uvic.ca

JCMCC **85** (2013), pp. 341-344

G such that the set of vertices

$$N_>(v_i) = \{v_j \mid j > i, v_iv_j \in E(G)\}$$

forms a clique for each i. The graphs admitting a perfect elimination ordering are precisely the chordal graphs.

The following characterization of (k,l)-colourable chordal graphs has been proven by Hell, Klein, Nogueira and Protti [7] by two different methods. The first one relies on establishing properties of the adjacency graph of all $(k+1)$-cliques of a chordal graph. The second one is based on a greedy-style algorithm applied to a perfect elimination ordering of a chordal graph. Presented here is an alternative direct proof of the result, also based on perfect elimination orderings.

The notation lK_k is used for the disjoint union of l complete graphs of k vertices each.

Theorem 1. *Let G be a chordal graph. Then G is (k,l)-colourable if and only if G does not contain $(l+1)K_{k+1}$ as an induced subgraph.*

Proof. Subsequently, it will be shown that if G is minimally not (k,l)-colourable, then G has to be a disjoint union of K_{k+1}'s. As lK_{k+1} is (k,l)-colourable, while $(l+1)K_{k+1}$ is not, the result then follows.

Suppose that G is minimally not (k,l)-colourable. Let $1,2,\dots,n$ be a perfect elimination ordering of the vertices of G such that the vertices of each component appear consecutively while the order of the components is arbitrary (the concatenation of perfect elimination orderings of the components yields a perfect elimination ordering of the whole graph). Let v be the first vertex such that the set

$$S = \{w \in V(G) \mid w \leq v\}$$

contains a K_{k+1} (such a vertex exists, as G is perfect and not $(k,0)$-colourable, thus contains K_{k+1}). Note that the graph induced by S is not $(k,0)$-colourable, while the graph induced by $S-\{v\}$ can be $(k,0)$-coloured (since it contains no K_{k+1}). Assume without loss of generality that the graph induced by S is connected (otherwise reorder the perfect elimination ordering by starting with the component of v in S, followed by the other vertices of S, both in the same order as before). Using the perfect elimination ordering, it will be shown that $N_>(v)$ is a clique cutset, separating S from $V(G)-S-N_>(v)$. Suppose x is the largest vertex in S having a neighbour x' in $V(G)-S-N_>(v)$. If x has a larger neighbour in S then by the perfect elimination ordering this neighbour is adjacent to x' which is a contradiction to the choice of x. Otherwise take the smallest vertex

y on a shortest path from x to v. Both its neighbours on said path are larger than y, thus have to be adjacent by the perfect elimination ordering, contradicting that the path was a shortest path. Thus $N_>(v)$ separates S from $V(G) - S - N_>(v)$.

Now suppose that either $N_>(v)$ is not empty or $S \neq K_{k+1}$. Then there exists a vertex $u \in N_>(v) \cup (S - \{v\})$ such that the graph induced by $S - \{u\}$ is not $(k, 0)$-colourable. By the minimality of G, the graph $G - u$ is (k, l)-colourable. Since the graph induced by $S - \{u\}$ is not $(k, 0)$-colourable, at least one of its vertices must be covered by a clique in the (k, l)-colouring. The clique can only contain vertices from $S \cup N_>(v) - \{u\}$ since there are no edges between S and $V(G) - S - N_>(v)$. Thus the graph induced by $V(G) - S - N_>(v)$ is $(k, l-1)$-colourable. But then it is possible to construct a (k, l)-colouring of G by adding the clique $\{v\} \cup N_>(v)$ and k-colouring the vertices in $S - v$ (none of those vertices being adjacent to any vertices in $V(G) - S - N_>(v)$), contradicting the assumption. Therefore $S = K_{k+1}$ and S is a component of G.

Hence each component of G that is not equal to K_{k+1} does not contain a K_{k+1}. However, such a component is $(k, 0)$-colourable and can be removed without changing the property of the graph not being (k, l)-colourable and can be removed without changing the property of the graph not being (k, l)-colourable, contradicting the minimality of G. ☐

References

[1] C. Berge, *Les problèmes de coloration en théorie des graphes*, Publ. Inst. Statist. Univ. Paris, 9 (1960), 123-160.

[2] D.D.A. Epple and J. Huang, *A note on the bichromatic number of graphs*, J. Graph Theory, 65 (2010), no. 2, 263-269.

[3] T. Feder and P. Hell and W. Hochstättler, *Generalized Colourings (Matrix Partitions) of Cographs*, Graph Theory in Paris, Trends Math., Birkhäuser, Basel, 2007, 149-167.

[4] R. de Souza Francisco and S. Klein and L.T. Nogueira, *Characterizing (k,l)-partitionable Cographs*, Electronic Notes in Discrete Math., 22 (2005), 277-280.

[5] D.R. Fulkerson and O.A. Gross, *Incidence matrices and interval graphs*, Pacific J. Math., 15 (1965), 835-855.