A new lower bound for the irredundance number of a tree

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Abstract

In this note we prove that for any tree T, $\gamma_{\leq 2}(T) \leq \gamma_\gamma(T) \leq ir(T) \leq \gamma(T)$, where $\gamma_{\leq 2}(G)$ is the distance-2 domination number, $ir(T)$ is the (lower) irredundance number, $\gamma(T)$ is the domination number, and $\gamma_\gamma(T)$, newly defined here, equals the minimum cardinality of a set of vertices that dominates a minimum dominating set of T.

Dedicated to Prof. Christina M. (Kieka) Mynhardt on the occasion of her 60th birthday.

1 Introduction

Let $G = (V, E)$ be a graph of order $n = |V|$ and let $v \in V$ be an arbitrary vertex. The *open neighborhood* of v is the set $N(v) = \{u \in V | uv \in E\}$, while the *open neighborhood of a set* $S \subseteq V$ is the set $N(S) = \bigcup_{u \in S} N(u)$. Similarly, the *closed neighborhood* of a vertex v is the set $N[v] = N(v) \cup \{v\}$, and the *closed neighborhood of a set* $S \subseteq V$ is the set $N[S] = \bigcup_{u \in S} N[u]$.

A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is a *dominating set* if every vertex in $V - S$ is adjacent to at least one vertex in S, or equivalently if

JCMCC **85** (2013), pp. 345-352

$N[S] = V$. The *domination number* $\gamma(G)$ of a graph G equals the minimum cardinality of a dominating set in G. A dominating set $S \subseteq V$ of cardinality $\gamma(G)$ is called a *γ-set* of G. We say that a set $S \subseteq V$ *dominates* another set $S' \subseteq V$ if $S' \subseteq N[S]$.

Define $\gamma_\gamma(G)$ to equal the minimum cardinality of a set S that dominates a γ-set of G.

A *distance-2 dominating set* is a set of vertices $S \subseteq V$ having the property that every vertex in $V - S$ is within distance-2 of at least one vertex in S. The *distance-2 domination number* $\gamma_{\leq 2}(G)$ equals the minimum cardinality of a distance-2 dominating set in G.

A set $S \subseteq V$ is called *irredundant* if for every vertex $u \in S$, $N[u] - N[S - \{u\}] \neq \emptyset$, that is, the closed neighborhood $N[u]$ of u contains a vertex that is not contained in the closed neighborhood $N[S - \{u\}]$ of the set S minus the vertex u. Any vertex in the set $N[u] - N[S - \{u\}]$ is called a *private neighbor* of u. Note that it is possible that $u \in N[u] - N[S - \{u\}]$, in which case we say that vertex u is *its own private neighbor*. The minimum cardinality of a maximal irredundant set is denoted $ir(G)$, and is called the *(lower) irredundance number* of G. Any maximal irredundant set $R \subseteq V$ of cardinality $ir(G)$ is called an *ir-set* of G.

The reader is referred to the two texts on domination in graphs by Haynes, Hedetniemi and Slater [7, 8] for a comprehensive discussion of topics relating to dominating sets and irredundant sets in graphs. Many papers on irredundance in graphs have been published, among which the following have relevance to this paper: [1, 2, 3, 4, 5, 6].

2 A new lower bound for the irredundance number of a tree

It is well known that for any graph G, $ir(G) \leq \gamma(G)$, since every γ-set is a maximal irredundant set and $ir(G)$ equals the minimum cardinality of a maximal irredundant set. It is easy to see that any ir-set in a graph is a distance-2 dominating set. Therefore, we have the following inequality chain:

$$\gamma_{\leq 2}(G) \leq ir(G) \leq \gamma(G).$$

There are several graph parameters other than $\gamma_{\leq 2}(G)$ that provide lower bounds for $ir(G)$. These will be discussed in the last section of this paper. The following result provides a new lower bound for the irredundance number of a tree.

Theorem 1 *For any tree* T, $\gamma_{\leq 2}(T) \leq \gamma_\gamma(T) \leq ir(T)$.

Proof. Let T be any tree and let T be rooted at a leaf labeled r, called the *root of* T. Let A be any γ-set of T containing no leaves of T and let R be any ir-set of T. Furthermore, define the following six sets of vertices:

1. $N = \{x \in R :$ the only private neighbor of x is the parent of $x\}$
2. $M = \{x \in R : x$ is a private neighbor of itself and no child of x is a private neighbor of $x\}$
3. $G = \{w \in V : w$ is the grandparent of some vertex $u \in N$, or is the parent of a vertex in N if it has no grandparent (in this case the parent is the root r of $T)\}$.
4. $P = \{v \in V : v$ is the parent of a vertex $x \in M$, or $v = r$, the root of T, if $r \in M\}$.
5. $S = (R - (N \cup M)) \cup G \cup P$.

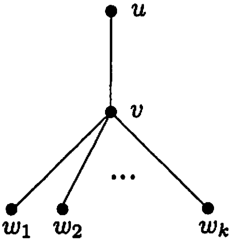
Given these sets, we will show that the set S dominates the γ-set A and that $|S| \leq |R| = ir(T)$, and therefore, $\gamma_\gamma(T) \leq |S| \leq ir(T)$.

First, notice that $|S| \leq |R| = ir(T)$. This follows since every vertex that is deleted from R that is an element of N is replaced by a vertex in the set G, and every vertex that is deleted from R that is an element of M is replaced by a vertex in P, so $|R|$ is at least as large as S. But since two vertices in N may be replaced by a common grandparent, and since two vertices in M may be replaced by a common parent, it is possible that there may be more vertices in $(N \cup M)$ that are deleted from R than are replaced by vertices in $(G \cup P)$. Thus, $|S| \leq |R|$.

Next, we must show that $A \subset N[S]$, that is, the γ-set A is dominated by S. Suppose this is not true, that $A - N[S] \neq \emptyset$. Let v be a vertex in $A - N[S]$ at greatest distance from the root r of T. Since we are assuming that the γ-set A does not contain a leaf, we know that v is not a leaf and there must exist a subtree of T as in Figure 1, where u is the parent of v and $w_1, w_2, \ldots, w_k$, for some $k \geq 1$, are the children of v.

Since we are assuming that $v \in A - N[S]$, we know that $v \notin N[S]$. It follows that u, v, and all w_j are not in S, since if u or any w_j is in S, then v would be in $N[S]$, and if v is in S then it is also in $N[S]$. We wish to show that $u \notin R$, $v \notin R$ and $w_1, w_2, \ldots, w_k \notin R$.

Figure 1: A subtree of T.



(i) $w_j \notin R$. If some $w_j \in R$, then either $w_j \in N$, $w_j \in M$, or $w_j \notin (N \cup M)$. But if $w_j \in N$ then its grandparent $u \in G$, which means that $u \in S$, a contradiction. Similarly, if $w_j \in M$, then its parent $v \in P$, which means that $v \in S$ and again $v \in N[S]$, a contradiction. Finally, if $w_j \notin (N \cup M)$ but $w_j \in R$, then $w_j \in S$ and $v \in N[S]$, a contradiction.

(ii) $v \notin R$. Assume that $v \in R$. If $v \in M$ then its parent u is in P, which means that $u \in S$, and therefore $v \in N[S]$, a contradiction. Similarly, if $v \in N$, then as it is not it's own private neighbor, it must have a child, say $w_j \in R$, a contradiction to (i) above. Finally, if $v \in R - (N \cup M)$ then $v \in S$ and hence $v \in N[S]$, a contradiction.

(iii) $u \notin R$. Assume that $u \in R$. If $u \in N$, and u only has a parent as a private neighbor, then it must be the case that its child $v \in R$. But we have just shown in (ii) that $v \notin R$, a contradiction. If $u \in M$ and therefore v is not a private neighbor of u, it follows that either $v \in R$, a contradiction to (ii), or some $w_j \in R$, a contradiction to (i). Therefore, if $u \in R$ but is not in $(N \cup M)$, it must be the case that $u \in S$, and therefore $v \in N[S]$, a contradiction.

Thus, we have shown that no vertex in Figure 1, u, v or any w_j is a member of the ir-set R. Since R is a maximal irredundant set, consider what must happen if we add a vertex w_j to R. The resulting set would no longer be irredundant. But in the set $R \cup \{w_j\}$ the vertex w_j has a private neighbor, namely the vertex v, since no neighbor of v is in R. This means that in adding w_j to R, some vertex, say $y \in R$, must no longer have a private neighbor. This vertex y is either a child or a grandchild of w_j, and further vertex y cannot have a child of y as a private neighbor. But this means that vertex w_j is either a grandparent of a vertex $y \in R$ or a parent of a vertex $y \in R$. In either case, this implies that $w_j \in G$ or $w_j \in P$, which means that $w_j \in S$, which means that $v \in N[S]$, which contradicts our assumption that $v \notin N[S]$.

Thus, we have shown that the assumption that there exists a vertex $v \in A - N[S]$ leads to a contradiction. Therefore, we can conclude that $A - N[S] = \emptyset$, i.e. $A \subseteq N[S]$, which means that the set S, of cardinality at most $|R| = ir(T)$, is a dominating set of a γ-set A, and therefore, $\gamma_\gamma(T) \leq |S| \leq ir(T)$. □

We next show that the inequality $\gamma_\gamma(G) \leq ir(G)$ does not hold for all graphs, that is, there exist graphs for which $\gamma_\gamma(G) > ir(G)$. For example, let graph G consists of the sets of vertices A, B, C, D, A^* and D^*, where

- $A = \{a_1, a_2, a_3, a_4\}$
- $B = \{b_1, b_2, ..., b_6\}$
- $C = \{c_1, c_2, ..., c_6\}$
- $D = \{d_1, d_2, d_3\}$
- $A^* = \{a^i_{jk} | 1 \leq j < k \leq 6, 1 \leq i \leq 8\}$
- $D^* = \{d^i_{jk} | 1 \leq j < k \leq 6, 1 \leq i \leq 6\}$.

C is complete (K_6). For $i \in \{1,2,3\}$, a_i is adjacent to b_i. a_4 is adjacent to each of b_4, b_5 and b_6. For all $i \in \{1, ..., 6\}$, b_i is adjacent to c_i. For all $i \in \{1,2,3\}$, d_i is adjacent to both c_{2i-1} and c_{2i}. For each pair j,k with $1 \leq j < k \leq 6$, the vertices b_j and b_k are both adjacent to the eight vertices a^i_{jk} for all $1 \leq i \leq 8$. For all $i \in \{1, ..., 4\}$ the vertex a_i is adjacent to the fifteen vertices a^{2i-1}_{jk} for all $1 \leq j < k \leq 6$. Similarly, for each pair j,k with $1 \leq j < k \leq 6$, c_j and c_k are both adjacent to the six vertices d^i_{jk} for $1 \leq i \leq 6$. For all $i \in \{1,2,3\}$ the vertex d_i is adjacent to the fifteen vertices of the form d^{2i-1}_{jk} for all $1 \leq j < k \leq 6$ and the fifteen vertices d^{2i-1}_{jk} for all $1 \leq j < k \leq 6$. Excluding the sets A^* and D^* the graph appears as in Figure 2.

Then for each pair of vertices b_j, b_k with $1 \leq j < k \leq 6$ there is an induced subgraph of the form shown in Figure 3.

Similarly, for each pair of vertices c_j, c_k with $1 \leq j < k \leq 6$ there is a subgraph of the form shown in Figure 4.

We now show the set C is maximal irredundant. Each b_i is the only private neighbor of c_i. Any vertex from D or D^* would have no private neighbor, and throwing in a vertex from A or A^* would destroy the private neighbor for a vertex in C. So we have $ir(G) \leq 6$.

The set $S = A \cup D$ dominates G and is in fact a γ-set. To see this, note that the closed neighborhoods $N[a_i]$ and $N[d_i]$ of the vertices in $A \cup D$ are

Figure 2: G with A^* and D^* excluded.

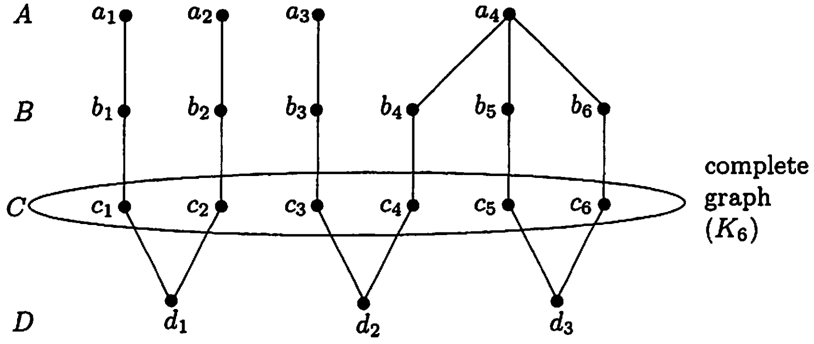
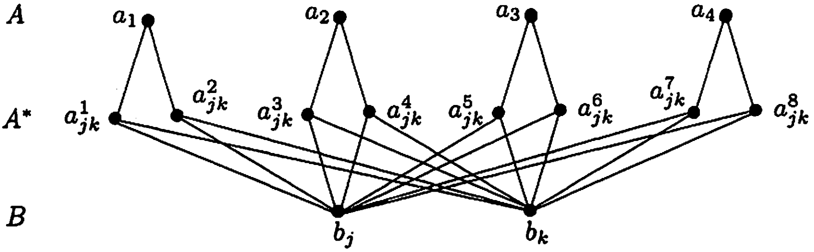


Figure 3: A subgraph of G with the vertex pair b_j, b_k.



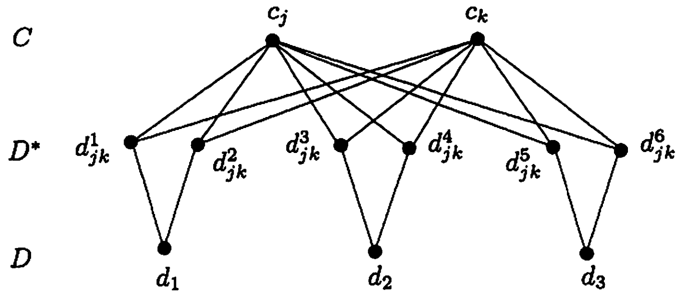
all disjoint, so at least 7 vertices will be required to dominate $A \cup D$. So, any dominating set must have at least 7 elements, and $|S| = 7$, so it is a γ-set.

In fact, S is the unique γ-set. To dominate A and be minimal, we must have 4 vertices from $A \cup A^* \cup B$. Since B has six vertices, there must exist b_j and b_k that are not in the γ-set. The 8 vertices a_{jk}^t can only be dominated by themselves or by A. Since any particular a_{jk}^t in this collection does not dominate any of the others, the only way to dominate all 8 of them using only 4 vertices is using all of A. Similarly, in order to dominate the lower portion of the graph with only 3 vertices, any γ-set must contain D.

Now, we saw above that to dominate $S = A \cup D$ requires at least 7 vertices, so $7 \leq \gamma_\gamma$. Combining this with the above result for $ir(G)$ and the standard inequality gives

$$ir \leq 6 < 7 \leq \gamma_\gamma \leq i_\gamma.$$

Figure 4: A subgraph of G with the vertex pair c_j, c_k.



We close this section by raising the question: does the inequality $\gamma_\gamma(G) \leq ir(G)$ hold for bipartite graphs?

3 Other possible inequalities involving $\gamma_\gamma(G)$

Several parameters are known which provide either lower bounds for $ir(G)$ for all graphs G, or lower bounds for $ir(G)$ when restricted to trees. For each of these we can ask how do these parameters compare with $\gamma_\gamma(G)$. In the interests of brevity we will not define each of these parameters, as it would take several pages to do this. Instead we will only mention them by name and refer the reader to an appropriate paper.

In [6] Cockayne, Hattingh, Hedetniemi, Hedetniemi and McRae introduced a parameter called the *external redundance number*, denoted $er(G)$ and they observed that for any graph G, $er(G) \leq ir(G)$. Thus, we ask: how does $\gamma_\gamma(G)$ compare with $er(G)$?

In [4] Cockayne, Favaron, Puech and Mynhardt proved the following inequality chain for trees.

Theorem 2 *For any tree T, $\gamma_{\leq 2}(T) \leq \theta(T) \leq \theta_i(T) \leq \rho(T) \leq ra(T) \leq \{rai(T), er(T)\} \leq ir(T)$.*

In this inequality chain,

1. $\theta(T)$ equals the *perfect neighborhood number*, the minimum cardinality of a perfect neighborhood set.

2. $\theta_i(T)$ equals the *independent perfect neighborhood number*, the minimum cardinality of an independent perfect neighborhood set.
3. $\rho(T)$ equals the *lower 2-packing number*, the minimum cardinality of a maximal 2-packing.
4. $ra(T)$ equals the *R-annihilation number*, the minimum cardinality of an R-annihilating set.
5. $rai(T)$ equals the *irredundant R-annihilation number*, the minimum cardinality of an R-annihilating set that is also irredundant.

Thus, we ask: how do the parameters $\theta(G) \leq \theta_i(G) \leq \rho(G) \leq ra(G) \leq rai(G)$ compare with $\gamma_\gamma(G)$?

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