Induced-hereditary graph properties, homogeneity, extensibility and universality

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Dedicated to Kieka Mynhardt on her 60th birthday

Abstract

Rado constructed a (simple) denumerable graph R with the positive integers as vertex set with the following edges: For given m and n with $m < n$, m is adjacent to n if n has a 1 in the m'th position of its binary expansion. It is well known that R is a universal graph in the set $\mathcal{I}$ of all countable graphs (since every graph in $\mathcal{I}$ is isomorphic to an induced subgraph of R) and that R can be characterized using this notion and that of being homogeneous and having the extension property.

In this paper we extend these notions to arbitrary induced-hereditary properties (of graphs), relate them to the construction of a universal graph for any such property, and obtain results which remind one of some characterizations of R.

1 Introduction

For general graph theoretic notions, the notation and terminology of [8] will be used. In particular, for any two graphs G and $H = (V', E')$, we say that G is a **subgraph** of H, denoted by $G \subseteq H$, if there is a subset $V \subseteq V'$ and a subset $E \subseteq E'$ (with every edge $e \in E$ an adjacency between two vertices in V) such that (V, E) is a graph which is isomorphic to G. G is an **induced subgraph** of H, denoted by $G \leq H$, if G is isomorphic to such a graph (V, E) of which E contains all and only the edges $xy \in E'$ for which $x, y \in V$. We shall also write $G \subset H$ ($G < H$) to denote the fact that G is a subgraph (an induced subgraph respectively) of H which is not isomorphic to H.

There is (up to isomorphism) clearly only one subgraph induced by a given subset W of the vertex set V of a graph $G = (V, E)$; this subgraph is denoted by $G[W]$ and called the **subgraph of G generated (or spanned) by W**.

All graphs considered here for investigation are simple, undirected, unlabelled and have countable vertex sets. The vertex set of a graph is typically taken to be (or to be indexed by) the set, or some subset, of the positive integers $\mathbf{N} = \{1, 2, \ldots\}$.

For notions related to hereditary graph properties the notation and terminology of [1] will be used. For ease of reference we formulate some of the basic definitions in this paper too. A **(graph) property** is an isomorphism-closed subclass of the class of all countable graphs. Since we have for many purposes, in a property, no reason to distinguish between isomorphic copies of a graph, we consider the class of all (simple) graphs to be a set and we use the notation $\mathcal{I}$ to denote this set of (countable) graphs. In this paper we will often have occasion to deal with two graphs that are isomorphic and, if they are, we shall refer to any one of them as a **clone** of the other.

A property $\mathcal{P}$ is **induced-hereditary** if, whenever $G \in \mathcal{P}$ and $H \leq G$, then $H \in \mathcal{P}$ too. Let $\mathcal{P}$ be a set of countable graphs. Following [8], we define a graph U to be a **universal graph for $\mathcal{P}$** if every graph in $\mathcal{P}$ is an induced subgraph of U; it is a **universal graph in $\mathcal{P}$** if $U \in \mathcal{P}$ too. Since a universal graph U for $\mathcal{P}$ is allowed to be outside $\mathcal{P}$ and hence, presumably, to be uncountable, the existence of at least one such U becomes trivial: take U to be the disjoint union of one clone from each isomorphism class in $\mathcal{P}$ (i.e., of a "skeleton" of $\mathcal{P}$). The fact that this U is in general uncountable follows from Lemma 1 of [4]; a countable universal graph for any induced-hereditary property is constructed in that paper too – see Theorem 3 of [4].

Rado [9] constructed the following (simple) denumerable graph on $\mathbf{N}$: For given m and n with $m < n$, m is adjacent to n if n has a 1 in the m'th position of its binary expansion; i.e., in $n = \sum_{i=0}^{\infty} n_i 2^i, n_i \in \{0,1\}$ we have $n_{m-1} = 1$. We shall denote this graph by R. It is well known that R is a universal graph in the induced-hereditary property $\mathcal{I}$ of countable graphs. One of the most useful and characteristic properties of R is that it has the **extension property**: For every two finite disjoint sets U and V of vertices of R there is a vertex not in $U \cup V$ which is adjacent to every vertex of U and to no vertex of V. R is also **homogeneous**, that is, every isomorphism between two finite induced subgraphs of R extends to an automorphism of R.

These properties can be used as follows in characterisations of R.

Theorem 1 *(Rado [9], Cameron [5], [6] and [7])*
Let G be any denumerable graph. Then the following conditions on G are equivalent:
1. G is isomorphic to R.
2. G is universal in $\mathcal{I}$ and G is homogeneous.
3. G has the extension property. □

In this paper we extend these ideas to arbitrary induced-hereditary properties. In particular, we define and study a $\mathcal{P}$-extension property for an arbitrary induced-hereditary property $\mathcal{P}$ in Section 2. We construct a graph with the $\mathcal{P}$-extension property which is consequently universal for $\mathcal{P}$. Then we introduce the concept of $\mathcal{P}$-homogeneity in Section 3 and show (in Theorem 5) how all these concepts are related. These concepts and their relationships correspond to their classical counterparts (as described above) when $\mathcal{P} = \mathcal{I}$. The paper is concluded by showing in Section 4 that there is, up to isomorphism, not more than one graph in an induced-hereditary property $\mathcal{P}$ which has the $\mathcal{P}$-extension property, a result reminding one again of Theorem 1 above.

2 $\mathcal{P}$-extensibility and universality for $\mathcal{P}$

Let $\mathcal{P}$ be any induced-hereditary property. Now we generalize the classical extension property (as explained in the Introduction) by invoking $\mathcal{P}$. We say that a countable graph G **has the $\mathcal{P}$-extension property** when the following holds: whenever U and V are two disjoint finite subsets of $V(G)$ such that $G[U \cup V] \in \mathcal{P}$, while the graph obtained by enlarging $G[U \cup V]$ with one new vertex adjacent to all and only the vertices of U is also in $\mathcal{P}$, then this fact is reflected inside G, i.e., there exists a vertex z of G, outside of $U \cup V$, which is adjacent to every vertex of U and to no vertex of V, (so that then also $G[U \cup V \cup \{z\}] \in \mathcal{P}$). If $\mathcal{P}$ and $\mathcal{Q}$ are both induced-hereditary properties with $\mathcal{P} \subseteq \mathcal{Q}$, then any countable graph having the $\mathcal{Q}$-extension property also has the $\mathcal{P}$-extension property. In particular: any countable graph having the $\mathcal{I}$-extension property has the $\mathcal{P}$-extension property for any induced-hereditary property $\mathcal{P}$ of countable graphs. But the $\mathcal{I}$-extension property is just the classical extension property, and any graph having it is isomorphic to the Rado graph R, which is universal in $\mathcal{I}$. (Note that, while any countable graph with the classical extension property is denumerably infinite, the same is not always true for the $\mathcal{P}$-extension property – depending on $\mathcal{P}$; think for example of $\mathcal{P} = \{G \in \mathcal{I} \mid |V(G)| \leq 71, E(G) = \emptyset\}$.) The next theorem tells us how the $\mathcal{P}$-extension property yields universality for $\mathcal{P}$ too.

Theorem 2 *Let $\mathcal{P}$ be any induced-hereditary property. If the countable graph G has the $\mathcal{P}$-extension property, then it is universal for $\mathcal{P}$.*

Proof: Suppose the countable graph G has the $\mathcal{P}$-extension property and let $H \in \mathcal{P}$ have the (finite or denumerable) vertex set $\{w_1, w_2, \ldots\}$. By recursion on the indices $1, 2, \ldots$ of these vertices we shall build an isomorphic embedding $\alpha : H \to G$. To start with, let $\alpha(w_1)$ be any vertex of G. Since $\mathcal{P}$ is induced-hereditary, we clearly have $G[\{\alpha(w_1)\}] \in \mathcal{P}$.

Now summon w_2. If in H it is adjacent to w_1, define $U = \{\alpha(w_1)\}$ and $V = \emptyset$; otherwise $U = \emptyset$ and $V = \{\alpha(w_1)\}$. In either case $G[\{\alpha(w_1)\}] = G[U \cup V] \in \mathcal{P}$ and $H[\{w_1, w_2\}] \in \mathcal{P}$. By the $\mathcal{P}$-extension property of G, there exists a vertex z of G, $z \neq \alpha(w_1)$, such that z is adjacent to $\alpha(w_1)$ in G if and only if w_2 is adjacent to w_1 in H. We define $\alpha(w_2)$ to be this vertex z, yielding $\alpha : H[\{w_1, w_2\}] \cong G[\{\alpha(w_1), \alpha(w_2)\}]$.

Assume now that $\alpha(w_1), \alpha(w_2), \ldots, \alpha(w_k)$ have been defined in such a way that $\alpha : H[\{w_1, w_2, \ldots, w_k\}] \cong G[\{\alpha(w_1), \alpha(w_2), \ldots, \alpha(w_k)\}]$, and we want to define $\alpha(w_{k+1})$. Define $U = \{\alpha(w_i) \mid 1 \leq i \leq k \text{ and } w_i w_{k+1} \in E(H)\}$ and $V = \{\alpha(w_1), \alpha(w_2), \ldots, \alpha(w_k)\} \backslash U$. Then $G[\{\alpha(w_1), \alpha(w_2), \ldots, \alpha(w_k)\}] = G[U \cup V] \in \mathcal{P}$, while also $H[\{w_1, w_2, \ldots, w_k, w_{k+1}\}] \in \mathcal{P}$.

By the $\mathcal{P}$-extension property of G there exists a vertex z of G, $z \notin \{\alpha(w_1), \alpha(w_2), \ldots, \alpha(w_k)\}$, such that, for $1 \leq i \leq k$, z is adjacent to $\alpha(w_i)$ in G if and only if w_{k+1} is adjacent to w_i in H. By defining $\alpha(w_{k+1}) = z$ we obtain

$$\alpha : H[\{w_1, w_2, \ldots, w_k, w_{k+1}\}] \cong G[\{\alpha(w_1), \alpha(w_2), \ldots, \alpha(w_k), \alpha(w_{k+1})\}].$$

It is clear that this recursive step can be repeated throughout $V(H)$ to obtain the required isomorphic embedding $\alpha : H \to G$. □

For any induced-hereditary property $\mathcal{P}$ we now want to build (by a recursive process, called in [4] "of the B-type") a countable graph having the $\mathcal{P}$-extension property. By Theorem 2 this graph, which we shall call $X(\mathcal{P})$, is universal for $\mathcal{P}$.

We start with the graph $X(\mathcal{P})^1$ which has a single vertex, say v. As a first recursive step we describe $X(\mathcal{P})^2$. Let $V_1^1 = \emptyset$ and $V_2^1 = \{v\}$ be the two elements of $Pow(V(X(\mathcal{P})^1))$, the power set of $V(X(\mathcal{P})^1)$. Add two new elements, v_1^1 and v_2^1 (one for each V_i^1) to $V(X(\mathcal{P})^1)$ to obtain $V(X(\mathcal{P})^2) = \{v, v_1^1, v_2^1\}$. For the non-empty V_i^1, i.e., V_2^1, look at the graph $X(\mathcal{P})^1[V_2^1] + v_2^1$, in this case the graph K_2. If this graph is in $\mathcal{P}$, then join each vertex in V_2^1 (here only v) to v_2^1 by an edge in $E(X(\mathcal{P})^2)$. If not, then not. This describes $E(X(\mathcal{P})^2)$.

Now suppose that the graphs $X(\mathcal{P})^1, X(\mathcal{P})^2, \ldots, X(\mathcal{P})^k$ have already been constructed, and consider all the subsets $V_1^k, V_2^k, \ldots, V_n^k$ of $V(X(\mathcal{P})^k)$ one by one, where $V_1^k = \emptyset$. (If $|V(X(\mathcal{P})^k)| = m$, then $n = 2^m$.) For each V_i^k, add one new vertex v_i^k to $V(X(\mathcal{P})^k)$ to obtain $V(X(\mathcal{P})^{k+1}) = V(X(\mathcal{P})^k) \cup \{v_1^k, v_2^k, \ldots, v_n^k\}$. For each i, $1 \le i \le n$, look whether the graph $X(\mathcal{P})^k[V_i^k] + v_i^k \in \mathcal{P}$ or not. If so, add to $E(X(\mathcal{P})^k)$ all the edges wv_i^k for $w \in V_i^k$ into $E(X(\mathcal{P})^{k+1})$; if not, add no edges between v_i^k and elements of V_i^k. This gives $E(X(\mathcal{P})^{k+1})$.

We note that the induced subgraph relations $X(\mathcal{P})^1 < X(\mathcal{P})^2 < \cdots$ hold and define $X(\mathcal{P})$ as the limit of this ascending sequence:

$$X(\mathcal{P}) = \bigcup \{X(\mathcal{P})^k \mid k \text{ is a positive integer}\}.$$

Note that in forming this limit, we simply take the (denumerable) union of the ascending chain of vertex sets as vertex set and we simply take the union of the ascending chain of edge sets as edge set.

We are now ready for

Theorem 3 *Let $\mathcal{P}$ be any induced-hereditary property. Then $X(\mathcal{P})$ has the $\mathcal{P}$-extension property.*

Proof: Suppose that U and V are two finite disjoint subsets of $V(X(\mathcal{P})) = V(X(\mathcal{P})^1) \cup V(X(\mathcal{P})^2) \cup \ldots$ such that $X(\mathcal{P})[U \cup V] \in \mathcal{P}$. Then there is some positive integer k such that $U \cup V \subseteq V(X(\mathcal{P})^k)$. Assume also that if $X(\mathcal{P})[U \cup V]$ is enlarged by a single new vertex adjacent to precisely the vertices of U, the new graph is in $\mathcal{P}$. By the construction of $X(\mathcal{P})^{k+1}$, there exists a vertex z of $X(\mathcal{P})^{k+1}$ – and hence of $X(\mathcal{P})$ – outside of $U \cup V$ which in $X(\mathcal{P})^{k+1}[U \cup V \cup \{z\}]$ – and hence in $X(\mathcal{P})^{k+1}$ and in $X(\mathcal{P})$ – is adjacent to every element of U and to no element of V. Hence $X(\mathcal{P})$ has the $\mathcal{P}$-extension property. □

Corollary 3.1 *Let $\mathcal{P}$ be any induced-hereditary property. Then $X(\mathcal{P})$ is universal for $\mathcal{P}$.*

Proof: By Theorems 2 and 3. □

Corollary 3.2 *$X(\mathcal{I})$ is a clone of R, the Rado graph.*

Proof: $X(\mathcal{I})$ has the $\mathcal{I}$-extension property and this is just the classical extension property, which (by Theorem 1) characterizes R. □

Note that $X(\mathcal{I})$ adds another construction of R – this time of the "B-type" [4] – to the many of the "A-type" in [2]. It seems like a challenging problem to find an explicit isomorphism between $X(\mathcal{I})$ and R.

3 $\mathcal{P}$-homogeneity

If a graph G has the $\mathcal{P}$-extension property then G has to have some degree of symmetry. One can therefore expect that the requirement that G is universal for some property $\mathcal{P}$ is by itself not a sufficient condition for G to have the $\mathcal{P}$-extension property. We shall take this remark further by defining a condition which, together with universality for $\mathcal{P}$, will be shown to be sufficient for the $\mathcal{P}$-extension property. But first we give an alternative description of classical homogeneity (as defined in the Introduction).

We say that a graph G is **locally homogeneous** when the following holds: whenever X and Y are two finite subsets of $V(G)$ inducing isomorphic subgraphs of G and μ is an isomorphism from $G[X]$ onto $G[Y]$, and u is a vertex of G not in X, then there exists a vertex v of G not in Y such that $\mu \cup \{(u,v)\}$ is an isomorphism from $G[X \cup \{u\}]$ onto $G[Y \cup \{v\}]$. At first sight local homogeneity may seem to be a weaker property than homogeneity, but in fact they are equivalent:

Lemma 4 *A countable graph is locally homogeneous if and only if it is homogeneous.*

Proof: The truth of the "if" part of the statement is easy to see. To demonstrate the "only if" part, we start by assuming that the countable graph G is locally homogeneous and that X and Y are two finite subsets of $V(G)$, while we have an isomorphism $\alpha : G[X] \cong G[Y]$. We need to extend α to an automorphism α^+ of G.

We label the elements of $V(G)$ as $v_1, v_2, \dots$ and construct α^+ by recursion on their indices $1, 2, \dots$. The recursion interleaves two alternating recursion steps, shifting back and forth between the two cases for the next available vertex of G outlined below. Assume now that
(i) I is a finite subset of the index set $\{1, 2, \dots\}$ of $V(G)$;
(ii) all the indices of the vertices in X are in I, so $X \subseteq V_I := \{v_i \in V(G) \mid$

$i \in I\}$;
(iii) for each $i \in I$, $\alpha^+(v_i) \in V(G)$ has already been defined, with $\alpha^+(v_i) = \alpha(v_i)$ for every $v_i \in X$; and
(iv) $G[V_I] \cong G[\alpha^+(V_I)]$ under $\alpha^+|V_I$.
In the beginning situation, when we start with just α, I is the set of indices of elements of X.
Case 1: Let $v_t \in V(G)$ be the vertex with the lowest index of all those outside V_I. We want to define $\alpha^+(v_t)$. Consider $G[V_I \cup \{v_t\}]$. By the local homogeneity of G there exists some $v_s \in V(G) \setminus \alpha(V_I)$ such that $\alpha^+ \cup \{(v_t, v_s)\}$ is an isomorphism from $G[V_I \cup \{v_t\}]$ onto $G[\alpha(V_I) \cup \{v_s\}]$. We define $\alpha^+(v_t) := v_s$, establishing the extended α^+ and concomitant extended I (now including t).
Case 2: Let $v_r \in V(G)$ be the vertex with the lowest index of all those outside $\alpha^+(V_I)$. We want to find some $v_s \in V(G) \setminus V_I$ suitable for defining $\alpha^+(v_s) = v_r$. Consider $G[\alpha^+(V_I) \cup \{v_r\}]$. By the local homogeneity of G (applied to $(\alpha^+)^{-1}$) there exists some $v_s \in V(G) \setminus V_I$ such that $\alpha^+ \cup \{(v_s, v_r)\}$ is an isomorphism from $G[V_I \cup \{v_s\}]$ onto $G[\alpha^+(V_I) \cup \{v_r\}]$. By defining $\alpha^+(v_s) := v_r$ we establish the extended α^+ and concomitant extended I (now including s). (Note that the Case 2 steps ensure that the α^+ under construction will be surjective onto $V(G)$, by forcing every element of $V(G)$ eventually to be an α^+-value.)
By alternating the steps of the two cases we extend α to the automorphism α^+ of G in a (finite or denumerable) recursive construction. □

When we now invoke the induced-hereditary property $\mathcal{P}$ in order to generalize the concept of homogeneity to "$\mathcal{P}$-homogeneity", it is convenient to let *local homogeneity* be the spring-board. We say that a graph G is **$\mathcal{P}$-homogeneous** when the following holds: whenever X and Y are two finite subsets of $V(G)$ inducing (isomorphic) subgraphs $G[X], G[Y] \in \mathcal{P}$, $\mu : G[X] \cong G[Y]$, and $u \in V(G) \setminus X$ with $G[X \cup \{u\}] \in \mathcal{P}$, then there exists a $v \in V(G) \setminus Y$ such that $\mu \cup \{(u, v)\} : G[X \cup \{u\}] \cong G[Y \cup \{v\}]$, (and then $G[Y \cup \{v\}] \in \mathcal{P}$).

We note that since local homogeneity is (by Lemma 4) precisely classical homogeneity, $\mathcal{P}$-homogeneity properly generalizes homogeneity. The next theorem is a relativization to $\mathcal{P}$ of 2. $\Leftrightarrow$ 3. in Theorem 1.

Theorem 5 *Let G be any countable graph and $\mathcal{P}$ be any induced-hereditary property. Then the following are equivalent:*
(i) G is universal for $\mathcal{P}$ and G is $\mathcal{P}$-homogeneous.
(ii) G has the $\mathcal{P}$-extension property.

Proof: (i) implies (ii): Let G be a relevant graph and let U and V be two disjoint finite subsets of $V(G)$ such that $G[U \cup V] \in \mathcal{P}$ and such that the graph G' obtained by enlarging $G[U \cup V]$ with one new vertex x adjacent to all and only the vertices of U is also in $\mathcal{P}$. Then we need to find a vertex in $V(G) \setminus (U \cup V)$ (which will be called v) which is adjacent in G to every vertex of U and to no vertex of V.

In order to find v we prepare to apply the $\mathcal{P}$-homogeneity of G: Since $G' \in \mathcal{P}$ and G is universal for $\mathcal{P}$, there is a (finite) set $X \subset V(G)$ and a vertex $u \in V(G)\setminus X$ such that $G[X \cup \{u\}] \cong G'$; suppose μ is an isomorphism between them and suppose $\mu(u) = x$.

Now let $Y = U \cup V$. Then $|X| = |Y|$ and, by the $\mathcal{P}$-homogeneity of G, there is a vertex v of G which is not in Y such that $\mu \cup \{(u, v)\}$ is an isomorphism from $G[X \cup \{u\}]$ onto $G[Y \cup \{v\}]$, proving that v is adjacent in G to every element of U and to no element of V.
(ii) implies (i): Suppose G has the $\mathcal{P}$-extension property. Then G is universal for $\mathcal{P}$ by Theorem 2. In order to prove that G is $\mathcal{P}$-homogeneous, let X and Y be any two finite subsets of $V(G)$ inducing isomorphic subgraphs of G, both in $\mathcal{P}$, and let μ be an isomorphism from $G[X]$ onto $G[Y]$. If u is any vertex of G not in X with $G[X \cup \{u\}] \in \mathcal{P}$, we define a partition of X: U is the set of vertices from X adjacent to u in G and V is the set of vertices from X not adjacent to u in G. Then we have that $G[U \cup V \cup \{u\}] \in \mathcal{P}$. Hence the graph obtained by enlarging $G[\mu(U) \cup \mu(V)]$ with one new vertex adjacent to all and only the vertices of $\mu(U)$ is isomorphic to $G[U \cup V \cup \{u\}]$. To reflect this inside G, there exists a vertex v of G not in $\mu(U) \cup \mu(V)$ which is adjacent to every vertex of $\mu(U)$ and to no vertex of $\mu(V)$. But then $\mu \cup \{(u, v)\}$ is an isomorphism from $G[X \cup \{u\}]$ onto $G[Y \cup \{v\}]$. □

Corollary 5.1 *If the countable graph G is universal for (and hence in) $\mathcal{I}$ and ($\mathcal{I}$-)homogeneous, then $G \cong R$, the Rado graph.*

Proof: By Theorem 5 the relevant G has the $\mathcal{I}$-extension property, i.e., the classical extension property characterizing R by Theorem 1. □

Corollary 5.2 *Let $\mathcal{P}$ be induced-hereditary. Then $X(\mathcal{P})$ is $\mathcal{P}$-homogeneous.*

Proof: This follows from Theorems 3 and 5. □

Corollary 5.3 *If $\mathcal{P}$ is induced-hereditary, $G \in \mathcal{P}$, and G is $\mathcal{P}$-homogeneous, then G is homogeneous.*

Proof: In the relevant G *all* induced subgraphs are in $\mathcal{P}$, making the $\mathcal{P}$-homogeneity of G equivalent to its local homogeneity, which by Lemma 4 is equivalent to its homogeneity. □

4 At most one graph with $\mathcal{P}$-extension in $\mathcal{P}$

The classical extension property characterizes R: R is (up to isomorphism) the unique graph in $\mathcal{I}$ with this property (by Theorem 1). For an induced-hereditary property $\mathcal{P}$ we have, in general, no guarantee that there is a graph *in* $\mathcal{P}$ which has the $\mathcal{P}$-extension property. But there cannot be non-isomorphic ones.

Theorem 6 *Let $\mathcal{P}$ be an induced-hereditary property and let G and H be two graphs in $\mathcal{P}$ which both have the $\mathcal{P}$-extension property. Then $G \cong H$.*

Proof: By Theorem 5 we know that both G and H are universal in $\mathcal{P}$ and both are $\mathcal{P}$-homogeneous. Suppose that $V(G) = \{v_1, v_2, \ldots\}$ and $V(H) = \{u_1, u_2, \ldots\}$. By interleaved recursion, stepping back and forth between the index of the next available vertex of G, and then of H, we construct an isomorphism $\alpha : G \cong H$, established by a bijection $\alpha : V(G) \to V(H)$. Define $\alpha(v_1) = u_1$. Suppose that I is a finite subset of the index set $\{1, 2, \ldots\}$ of $V(G)$, and $V_I = \{v_i \in V(G) \mid i \in I\}$. Assume that, for each $i \in I$, we have now already defined $\alpha(v_i) \in V(H)$, while $G[V_I] \cong H[\alpha(V_I)]$ under $\alpha|V_I$. To continue the construction of α we alternate the steps described in the following two cases:
Case 1: Let $v_t \in V(G)$ be the vertex with the lowest index of all those outside V_I. We need to define $\alpha(v_t) \in V(H)$. We define two disjoint finite subsets of $V(H)$ as follows: $X := \{\alpha(v_i) \mid i \in I \text{ and } v_iv_t \in E(G)\}$ and $Y := \{\alpha(v_i) \mid i \in I \text{ and } v_iv_t \notin E(G)\}$. Since $H[X \cup Y] \leq H \in \mathcal{P}$, $H[X \cup Y] \in \mathcal{P}$. Also $G[V_I \cup \{v_t\}] \leq G \in \mathcal{P}$, hence $G[V_I \cup \{v_t\}] \in \mathcal{P}$, demonstrating thereby that the graph obtained by enlarging $H[X \cup Y]$ with one new vertex adjacent to all and only the vertices of X is also in $\mathcal{P}$. By the $\mathcal{P}$-extension property of H there exists a vertex, say $u_s \in V(H) \setminus \alpha(V_I)$, which is adjacent in H to every $\alpha(v_i)$, $i \in I$, for which $v_iv_t \in E(G)$, but to no $\alpha(v_i)$ for which $v_iv_t \notin E(G)$. By defining $\alpha(v_t) = u_s$, we now have extended α to establish $G[V_I \cup \{v_t\}] \cong H[\alpha(V_I) \cup \{u_s\}]$.
Case 2: Let u_r be the vertex of H with the least index r among those not already in $\alpha(V_I)$. We need to find a $v_s \in V(G)$ suitable for defining $\alpha(v_s) = u_r$. Define two disjoint finite subsets of $V(G)$: $X := \{v_i \mid i \in I \text{ and } \alpha(v_i)u_r \in E(H)\}$ and $Y := \{v_i \mid i \in I \text{ and } \alpha(v_i)u_r \notin E(H)\}$. Since $G[X \cup Y] \leq G \in \mathcal{P}$, $G[X \cup Y] \in \mathcal{P}$. Since $H[\alpha(V_I) \cup \{u_r\}] \in \mathcal{P}$, it follows that the graph obtained by enlarging $G[X \cup Y]$ with one new vertex adjacent to all and only the vertices in X is also in $\mathcal{P}$. By the $\mathcal{P}$-extension property of G there exists a vertex, say $v_s \in V(G) \setminus V_I$, which is adjacent in G to every v_i, $i \in I$, for which $\alpha(v_i)u_r \in E(H)$, but to no v_i for which $\alpha(v_i)u_r \notin E(H)$. By defining $\alpha(v_s) = u_r$, we now have extended α to establish $G[V_I \cup \{v_s\}] \cong H[\alpha(V_I) \cup \{u_r\}]$.

By alternating the two cases through countably many recursive steps we construct the surjective isomorphism $\alpha : G \cong H$. □

Corollary 6.1 *Let $\mathcal{P}$ be induced-hereditary and $X(\mathcal{P}) \in \mathcal{P}$. Then every countable graph in $\mathcal{P}$ with the $\mathcal{P}$-extension property is a clone of $X(\mathcal{P})$.*

Proof: This follows from Theorem 6, since $X(\mathcal{P})$ has the $\mathcal{P}$-extension property too by Theorem 3. □

One may now reasonably ask whether Theorem 6 (and Corollary 6.1) are perhaps trivial in the sense that the *only* situation (maybe with both G and H denumerably infinite) in which the assumptions are realized is the case $\mathcal{P} = \mathcal{I}$ and $G \cong H \cong R$. The answer is "no". Pick any fixed countable graph H (of which there are uncountably many pairwise non-isomorphic exemplars) and consider the induced-hereditary hom-property $\mathcal{P} = \rightarrow H$ of countable "H-colourable" graphs, i.e., graphs which have a homomorphism into H. In [3] a graph $U(H)$, universal in $\rightarrow H$, is constructed with the $\rightarrow H$-extension property (in fact a somewhat stronger property there called "H-extension"). Hence $U(H)$ is unique with this property, up to isomorphism; a fact which is also shown in Theorem 2 of [3]. By Theorem 4 of [3] $U(H)$ is not isomorphic to R – unless H has the "weak extension property".

The following question remains open for now: For which induced-hereditary properties $\mathcal{P}$ do we have that $X(\mathcal{P}) \in \mathcal{P}$, so that $X(\mathcal{P})$ is the unique (homogeneous, by Corollary 5.3) universal graph in $\mathcal{P}$ with the $\mathcal{P}$-extension property, playing in $\mathcal{P}$ the role that (by Theorem 1) R plays in $\mathcal{I}$?

References

[1] M. Borowiecki, I. Broere, M. Frick, G. Semanišin and P. Mihók, *A survey of hereditary properties of graphs.* Discussiones Mathematicae Graph Theory 17(1997), 5-50.

[2] I. Broere and J. Heidema, *Constructing an abundance of Rado graphs.* Utilitas Mathematica 84(2011), 139-152.

[3] I. Broere and J. Heidema, *Universal H-colourable graphs.* Accepted for publication in Graphs and Combinatorics.

[4] I. Broere, J. Heidema and P. Mihók, *Constructing universal graphs for induced-hereditary graph properties.* Accepted for publication in Mathematica Slovaca.

[5] P. J. Cameron, *Oligomorphic permutation groups* (London Mathematical Society Lecture Note Series, 152, Cambridge University Press, Cambridge, 1990).

[6] P. J. Cameron, *The Rado graph and the Urysohn space.* Reading Combinatorics Conference, 18 May 2006.
http://www.maths.qmw.ac.uk/pjc/preprints/stpbg.pdf

[7] P. J. Cameron, *The random graph revisited.*
http://www.math.uni-bielefeld.de/rehmann/ECM/cdrom/3ecm/pdfs/pant3/camer.pdf

[8] R. Diestel, *Graph theory,* Fourth Edition (Graduate Texts in Mathematics, 173, Springer, Heidelberg, 2010).

[9] R. Rado, *Universal graphs and universal functions.* Acta Arith. 9(1964), 331-340.