Nordhaus-Gaddum type results for the induced path number with relative complements in $K_{m,n}$

Johannes H. Hattingh[1,2], Ossama A. Saleh[3], Lucas C. Van der Merwe[3], and Terry J. Walters[3]

[1]Department of Mathematics, East Carolina University, Greenville, NC 27858 USA
[2]Department of Mathematics, University of Johannesburg, Auckland Park, 2006, SOUTH AFRICA
[3]Department of Mathematics, University of Tennessee at Chattanooga, Chattanooga, TN 37403 USA

Abstract

The *induced path number* $\rho(G)$ of a graph G is defined as the minimum number of subsets into which the vertex set of G can be partitioned so that each subset induces a path. A *Nordhaus-Gaddum type result* is a (tight) lower or upper bound on the sum (or product) of a parameter of a graph and its complement. If G is a subgraph of H, then the graph $H - E(G)$ is the *complement of* G *relative to* H. In this paper, we consider Nordhaus-Gaddum type results for the parameter ρ when the relative complement is taken with respect to the complete bipartite graph $K_{m,n}$.

1 Introduction

In this paper, we follow the notation of [3]. Nordhaus and Gaddum present best possible bounds on the sum and the product of the chromatic number of a graph and its complement in [11]: for a graph G of order n, $2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1$, while $n \leq \chi(G)\chi(\overline{G}) \leq (\frac{n+1}{2})^2$.

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Since then such results have been given for several parameters, see, for example, [4]. They include the following best possible bounds on the domination number, $\gamma(G)$, due to Jaeger and Payan [9] and Payan and Xuong [12]: if G is a graph of order n, then $\gamma(G)+\gamma(\overline{G}) \leq n+1$ and $\gamma(G)\gamma(\overline{G}) \leq n$. The upper bound on the sum of the domination numbers of a graph and its complement given before can be improved if we restrict our attention to those graphs G for which G and $\overline{G}$ both have no isolated vertices, see [10]: if G is a graph of order $n \geq 2$ such that G and $\overline{G}$ have no isolated vertices, then $\gamma(G) + \gamma(\overline{G}) \leq (n+4)/2$.

For a graph G, the *induced path number* $\rho(G)$, is defined by Chartrand, Hashmi, Hossain, McCanna and Sherwani [2] as the minimum number of subsets into which the vertex set $V(G)$ of G can be partitioned such that each subset induces a path. In [1], Broere, Domke, Jonck and Markus, investigate best possible bounds on the sum of the induced path number of a graph and its complement. Specifically, the authors prove the following

Theorem 1 *If G is a graph of order n, then* $\sqrt{n} \leq \rho(G) + \rho(\overline{G}) \leq \lceil \frac{3n}{2} \rceil$.

As noted in [1], the upper bound in Theorem 1 is achieved when either G or $\overline{G}$ is the complete graph K_n. In [8], Hattingh, Saleh, Van der Merwe and Walters characterized graphs G attaining the upper bound, improved the lower bound by one when n is the square of an odd integer, and showed that the improved lower bound on $\rho(G) + \rho(\overline{G})$ is best possible for every positive integer $n \geq 4$. Moreover, $\rho(G)+\rho(\overline{G})$ is bounded from above when neither G nor $\overline{G}$ has isolated vertices. We will use $\psi(G)$ for $\rho(G) + \rho(\overline{G})$.

Another direction was pursued by Plesník [13] who extended Nordhaus and Gaddum's results to the case when the complete graph is factored into several factors.

If G is a subgraph of H, then the graph $H - E(G)$ is the *complement of G relative to H*. If $H = K_n$, and G is a subgraph of H, then the graph $K_n - E(G)$, the complement of G relative to K_n, is the classical complement $\overline{G}$ of the graph G. The concept of relative complement is due to Cockayne [5]. In [6], Goddard, Henning and Swart examine sums and products of $\pi(G_1)$ and $\pi(G_2)$ where $G_1 \oplus G_2 = K(n,n)$ with π being the independence, domination or independent domination number, respectively. Thus, the authors derive Nordhaus-Gaddum results for independence, domination and independent domination, but now considering the relative complement with respect to the complete bipartite graph $K_{n,n}$. In [7], Hattingh, Saleh, Van der Merwe and Walters derive Nordhaus Gaddum results for the induced path number by considering the relative complement with respect to the complete bipartite graph $K_{n,n}$.

A bipartite graph G with bipartition (T, B) such that $|T| = m \geq 1$ and $|B| = n \geq 1$ will be called an *(m, n)-bipartite graph.* Throughout, $\overline{G}$ will denote the relative complement of an (m, n)-bipartite graph with respect to $K_{m,n}$. In this paper, we determine best possible lower and upper bounds for $\psi(G)$ where G is an (m, n)-bipartite graph, thus extending the results of [7].

2 Lower bounds on $\psi(G)$

In this section, we determine best possible lower bounds for $\psi(G)$ where G is an (m, n)-bipartite graph. Throughout, let $U_1, \ldots, U_s$, where $s = \rho(G)$, be a partition of $V(G)$ such that each U_i induces a path in G. Let $V_1, \ldots, V_t$, where $t = \rho(\overline{G})$, be a partition of $V(\overline{G})$ such that each V_j induces a path in $\overline{G}$.

We begin with the following lemmas. Throughout G will be an (m, n)-graph where $n \geq m$.

Lemma 2 *If $U_i \cap V_j \cap T \neq \emptyset$, then $|U_i \cap V_j \cap B| \leq 4$. Moreover, if $U_i \cap V_j \cap B \neq \emptyset$, then $|U_i \cap V_j \cap T| \leq 4$.*

Proof. Let $v \in U_i \cap V_j \cap T$, and suppose, to the contrary, that $\{w_1, \ldots, w_5\} \subseteq U_i \cap V_j \cap B$. We may assume, without loss of generality, that three of the five edges $vw_1, \ldots, vw_5$ of $K_{m,n}$ are edges of G - suppose $\{vw_1, vw_2, vw_3\} \subseteq E(G)$. But then U_i does not induce a path in G, which is a contradiction. Thus, $|U_i \cap V_j \cap B| \leq 4$. □

Corollary 3 *Let $m \geq 3$. If $\psi(G) = 2$, then $n \leq 4$.*

Proof. Note that $\rho(G) = \rho(\overline{G}) = 1$. Then $U_1 \cap V_1 \cap T \neq \emptyset$, and, by Lemma 2, $|U_1 \cap V_1 \cap B| \leq 4$. Thus $n = |V(G) \cap B| = |V(G) \cap V(\overline{G}) \cap B| = |U_1 \cap V_1 \cap B| \leq 4$. □

Corollary 4 *Let $m \geq 2$. If $\psi(G) = 3$, then $n \leq 8$.*

Proof. Suppose, without loss of generality, that $\rho(G) = 1$ and $\rho(\overline{G}) = 2$. Then $U_1 \cap V_i \cap T \neq \emptyset$ for $i = 1, 2$, and, by Lemma 2, $|U_1 \cap V_i \cap B| \leq 4$ for $i = 1, 2$. Thus $n = |V(G) \cap B| = |V(G) \cap V(\overline{G}) \cap B| = |U_1 \cap (V_1 \cup V_2) \cap B| = |U_1 \cap V_1 \cap B| + |U_1 \cap V_2 \cap B| \leq 4 + 4 = 8$. □

Corollary 5 *If* $|U_i \cap V_j \cap T| \geq 5$, *then* $U_i \cap V_j \cap B = \emptyset$. *Moreover, if* $|U_i \cap V_j \cap B| \geq 5$, *then* $U_i \cap V_j \cap T = \emptyset$.

Lemma 6 *If* $|U_i \cap V_j \cap T| = 4$, *then* $|U_i \cap V_j \cap B| \leq 3$. *Moreover, if* $|U_i \cap V_j \cap B| = 4$, *then* $|U_i \cap V_j \cap T| \leq 3$.

Proof. Suppose $|U_i \cap V_j \cap T| = 4$. Then, by Lemma 2, $|U_i \cap V_j \cap B| \leq 4$. Suppose $|U_i \cap V_j \cap B| = 4$, and consider the $(4,4)$ bipartite graph H induced by $U_i \cap V_j$ in G. Then the number of edges in the union of H and $\overline{H}$ is 16. On the other hand, since the 8 vertices of H are vertices in the induced path $\langle U_i \rangle$, the total number of edges in H is at most 7. Similarly, the total number of edges in $\overline{H}$ is at most 7. Therefore the number of edges in the union of H and $\overline{H}$ is at most 14, which is a contradiction. Thus, $|U_i \cap V_j \cap B| \leq 3$. □

Lemma 7 $\rho(G) \geq n - m$.

Proof. Let $t_i = |U_i \cap T|$ and $b_i = |U_i \cap B|$ for $i = 1, \ldots, \rho(G)$. As G is bipartite, $|b_i - t_i| \leq 1$, for $i = 1, \ldots, \rho(G)$. Therefore $n - m = \sum_{i=1}^{\rho(G)} b_i - \sum_{i=1}^{\rho(G)} t_i \leq |\sum_{i=1}^{\rho(G)} b_i - \sum_{i=1}^{\rho(G)} t_i| \leq \sum_{i=1}^{\rho(G)} |b_i - t_i| \leq \rho(G)$. □

As an immediate consequence we have:

Corollary 8 $\psi(G) \geq 2(n - m)$.

<u>Case 1: $n = m$</u>

Theorem 9 (Hattingh, Saleh, Van der Merwe and Walters [8]) *Let G be an (n,n)-bipartite graph. Then*

$$\psi(G) \geq \begin{cases} 3 & \text{if } n \in \{1,2,3,4,5,6,7\} \\ 4 & \text{if } n \geq 8 \end{cases}$$

Moreover, these lower bounds are best possible.

<u>Case 2: $n = m + 1$</u>

For $m \geq 1$, construct $G_{m,m+1}$ as follows: Let T and B be disjoint sets of cardinality m and $m+1$ respectively. Partition T into two sets T_1 and T_2 where $|T_1| = \lfloor \frac{m}{2} \rfloor$ and $|T_2| = \lceil \frac{m}{2} \rceil$. Partition B into two sets B_1 and B_2

where $|B_1| = \lfloor \frac{m+1}{2} \rfloor$ and $|B_2| = \lceil \frac{m+1}{2} \rceil$. Add edges between T_1 and B_1 (T_2 and B_2, respectively) such that $\langle T_1 \cup B_1 \rangle$ ($\langle T_2 \cup B_2 \rangle$, respectively) is a path. Add edges between T_1 and B_2 (T_2 and B_1, respectively) such that $\overline{\langle T_1 \cup B_2 \rangle}$ ($\overline{\langle T_2 \cup B_1 \rangle}$, respectively) is a path.

Theorem 10 *If $m \geq 6$, then $\psi(G) \geq 4$. The graph $G_{m,m+1}$ shows that the lower bound is best possible.*

Proof. For $m \geq 8$, we have, by Corollaries 3 and 4, that $\psi(G) \geq 4$.

Suppose G is a $(7,8)$-graph. By Corollary 3, $\psi(G) \geq 3$. We show that $\psi(G) \geq 4$. Assume, to the contrary, that $\rho(G) = 1$ and that $\rho(\overline{G}) = 2$. Then $U_1 \cap V_i \cap T \neq \emptyset$ for $i = 1, 2$, and, by Lemma 2, we have $|U_1 \cap V_i \cap B| \leq 4$ for $i = 1, 2$. But then $8 = |B| = |V(G) \cap V(\overline{G}) \cap B| = |U_1 \cap (V_1 \cup V_2) \cap B| \leq |U_1 \cap V_1 \cap B| + |U_2 \cap V_2 \cap B| \leq 4 + 4 = 8$, which implies that $|U_1 \cap V_1 \cap B| = |U_1 \cap V_2 \cap B| = 4$. By Lemma 6, $|U_1 \cap V_1 \cap T| \leq 3$ and $|U_1 \cap V_2 \cap T| \leq 3$, whence $7 = |T| = |V(G) \cap V(\overline{G}) \cap T| = |U_1 \cap (V_1 \cup V_2) \cap T| \leq |U_1 \cap V_1 \cap T| + |U_2 \cap V_2 \cap T| \leq 3 + 3 = 6$, which is a contradiction.

Now, by the construction of $G_{m,m+1}$, one concludes that $\psi(G_{m,m+1}) = 4$. □

For $m \in \{4, 5, 6\}$, by Corollary 3, we have $\psi(G) \geq 3$. Clearly $\psi(G) \geq 2$ when $m = 3$. Suppose $m \in \{1, 2\}$. Assume $\psi(G) \leq 3$, and, without loss of generality, suppose $\langle U_1 \rangle \cong G$. Then $\overline{G}$ is either $\overline{K}_3$ or $2K_2 \cup K_1$, whence $\rho(\overline{G}) = 3$, and so $\psi(G) = 4$, which is a contradiction. Thus, $\psi(G) \geq 4$ when $m \in \{1, 2\}$.

We now show that these lower bounds are best possible for $1 \leq m \leq 6$. Let $T = \{t_1, \ldots, t_m\}$, let $B = \{b_1, \ldots, b_{m+1}\}$, and let G be the path with consecutive vertices $b_1, t_1, \ldots, t_m, b_{m+1}$. In all cases, $\rho(G) = 1$. For $1 \leq m \leq 2$, it follows that $\rho(\overline{G}) = 3$, whence $\psi(G) = 4$. For $m = 3$, $\overline{G}$ is the path with consecutive vertices $b_2, t_3, b_1, t_2, b_4, t_1, b_3$, so $\rho(\overline{G}) = 2$, whence $\psi(G) = 2$. For $m = 4$, the sets $\{t_2, b_1, t_4, b_3\}$ and $\{b_2, t_3, b_5, t_1, b_4\}$ each induces a path in $\overline{G}$, whence $\rho(\overline{G}) = 2$, and so $\psi(G) = 3$. For $m = 5$, the sets $\{b_6, t_1, b_3, t_5, b_1\}$ and $\{t_4, b_2, t_3, b_5, t_2, b_4\}$ each induces a path in $\overline{G}$, whence $\rho(\overline{G}) = 2$, and so $\psi(G) = 3$. For $m = 6$, the sets $\{b_2, t_6, b_1, t_2, b_7, t_1\}$ and $\{b_4, t_5, b_3, t_4, b_6, t_3, b_5\}$ each induces a path in $\overline{G}$, whence $\rho(\overline{G}) = 2$, and so $\psi(G) = 3$. These best possible lower bounds are summarized in Table 1.

Case 3.1: $n \geq m + 2$ and m is even

Construct the graph $G_{m,m+2}$ by using the same steps in constructing $G_{m,m+1}$ with the adjustment that the partition B_1 and B_2 of B satisfy $|B_1| = \frac{m}{2} + 1$

m	1	2	3	4	5	6
Best possible lower bound on $\psi(G)$	4	4	2	3	3	3

Table 1: G is an $(m, m+1)$-graph

and $|B_2| = \frac{m}{2} + 1$. Then $\psi(G_{m,m+2}) \leq 4$. By Corollary 8, it follows that $\psi(G_{m,m+2}) = 4$.

Theorem 11 *$\psi(G) \geq 2(n-m)$, and the bound is best possible.*

Proof. Corollary 8 gives the lower bound. Construct an (m,n)-graph H in the following way. Let $V(H) = T \cup B \cup B_2$ where $|T| = m$, $|B| = m+2$, and $|B_2| = n-m-2$. Add edges between the sets T and $B \cup B_2$ such that $T \cup B$ induces a graph isomorphic to $G_{m,m+2}$. Then $\rho(H) \leq \rho(G)+n-m-2$, while $\rho(\overline{H}) \leq \rho(\overline{G_{m,m+2}})+n-m-2$, whence $\psi(H) \leq \psi(G_{m,m+2})+2(n-m-2) = 4+2(n-m-2) = 2(n-m)$. By Corollary 8, $\psi(H) \geq 2(n-m)$, and so $\psi(H) = 2(n-m)$, showing that the lower bound is best possible. □

Case 3.2: $n \geq m+2$ and m is odd

Lemma 12 *If $\psi(G) = 4$, then $n = m+2$ and $m \leq 11$.*

Proof. Since $n \geq m+2$, by Lemma 7, $\rho(G) \geq 2$ and $\rho(\overline{G}) \geq 2$. As $\psi(G) = 4$, $\rho(G) = \rho(\overline{G}) = 2$. As $n-m \leq \rho(G) = 2$, we have $m+2 \leq n \leq m+2$, and so $n = m+2$. Moreover, each of the two induced paths $\langle U_1 \rangle$ and $\langle U_2 \rangle$ in G have their endpoints in B.

As m is odd, we assume, without loss of generality, that $|U_1 \cap T| > |U_2 \cap T|$, and that $|V_1 \cap T| > |V_2 \cap T|$. As $|U_1 \cap T| + |U_2 \cap T| = m$, it follows that $|U_1 \cap T| \geq \frac{m+1}{2}$, while $|U_2 \cap T| \leq \frac{m-1}{2}$. Similarly, $|V_1 \cap T| \geq \frac{m+1}{2}$, while $|V_2 \cap T| \leq \frac{m-1}{2}$.

Suppose $U_1 \cap V_1 \cap T = \emptyset$. Then $(U_1 \cap T) \subseteq (V_2 \cap T)$, whence $\frac{m+1}{2} \leq |U_1 \cap T| \leq |V_2 \cap T| \leq \frac{m-1}{2}$, which is a contradiction.

We may therefore assume that $U_1 \cap V_1 \cap T \neq \emptyset$. By Lemma 2, we have $|U_1 \cap V_1 \cap B| \leq 4$.

Suppose $U_1 \cap V_2 \cap T \neq \emptyset$. By Lemma 2, $|U_1 \cap V_2 \cap B| \leq 4$. Thus, $|U_1 \cap B| \leq |U_1 \cap (V_1 \cup V_2) \cap B| = |U_1 \cap V_1 \cap B| + |U_1 \cap V_2 \cap B| \leq 4+4 = 8$.

Suppose $|U_1 \cap B| = 8$. Then $|U_1 \cap V_1 \cap B| = |U_1 \cap V_2 \cap B| = 4$, and, by Lemma 6, $|U_1 \cap V_1 \cap T| \leq 3$ and $|U_1 \cap V_2 \cap T| \leq 3$, whence $|U_1 \cap T| \leq$

$|U_1 \cap V_1 \cap T| + |U_1 \cap V_2 \cap T| \leq 6$. As $|U_1 \cap B| = 8$ and $\langle U_1 \rangle$ is a path in G with both endvertices in B, we must have that $|U_1 \cap T| = 7$, which contradicts the fact that $|U_1 \cap T| \leq 6$.

Therefore $|U_1 \cap B| \leq 7$, and so $|U_1 \cap T| \leq 6$. As $\frac{m+1}{2} \leq |U_1 \cap T| \leq 6$, it follows that $m \leq 11$.

If $V_1 \cap U_2 \cap T \neq \emptyset$, an analogous argument to the previous case again establishes that $m \leq 11$.

We therefore assume that $U_1 \cap V_2 \cap T = \emptyset$ and $V_1 \cap U_2 \cap T = \emptyset$, which is equivalent to $U_1 \cap T = V_1 \cap T$ and $U_2 \cap T = V_2 \cap T$. Since $\langle U_1 \rangle$ induces a path in G with both endpoints in B, we have $|U_1 \cap B| = |U_1 \cap T| + 1$. Similarly, $|V_2 \cap B| = |V_2 \cap T| + 1$. But $|U_1 \cap T| > |U_2 \cap T| = |V_2 \cap T|$, and so $|U_1 \cap B| = |U_1 \cap T| + 1 > |V_2 \cap T| + 1 = |V_2 \cap B|$, whence $|U_1 \cap B| > |V_2 \cap B|$. If $U_1 \cap V_1 \cap B = \emptyset$, then $U_1 \cap B \subseteq V_2 \cap B$, which is a contradiction.

Thus, $U_1 \cap V_1 \cap B = \emptyset$, and by Lemma 2, we have $|U_1 \cap V_1 \cap T| \leq 4$. As $U_1 \cap T = V_1 \cap T$, we have $U_1 \cap V_1 \cap T = U_1 \cap T$, and so $\frac{m+1}{2} \leq |U_1 \cap T| = |U_1 \cap V_1 \cap T| \leq 4$, whence $m \leq 7$. □

The following lemma provides the lower bound when m is odd and $n = m + 2$.

Lemma 13 *Let $n = m + 2$. Then*

$$\psi(G) = \begin{cases} 4 & \text{if } 3 \leq m \leq 11 \\ 5 & \text{if } m = 1 \text{ or } m \geq 13. \end{cases}$$

These lower bounds are best possible.

Proof. Suppose $m = 1$. Clearly $\psi(G) \geq 5$. The $(1,3)$-graph $K_{1,2} \cup K_1$ (where it is understood that the isolated vertex belongs to the partitite set of cardinality 3) shows that 5 is a best possible lower bound for $\psi(G)$.

Corollary 8 gives $\psi(G) \geq 4$.

Let $m \geq 13$. By Lemma 12, $\psi(G) \geq 5$. We now construct an $(m, m+2)$-graph H such that $\psi(H) = 5$. Let T and B be disjoint sets of cardinality m and $m + 2$ respectively. Partition T into two sets T_1 and T_2 where $|T_1| = \frac{m-1}{2}$ and $|T_2| = \frac{m+1}{2}$. Let $x \in B$, and partition $B - \{x\}$ into two sets B_1 and B_2 where $|B_1| = |B_2| = \frac{m+1}{2}$. Add edges between T_1 and B_1 (T_2 and $B_2 \cup \{x\}$, respectively) such that $\langle T_1 \cup B_1 \rangle$ ($\langle T_2 \cup (B_2 \cup \{x\}) \rangle$, respectively) is a path. Add edges between T_1 and B_2 (T_2 and B_1, respectively) such that $\overline{\langle T_1 \cup B_2 \rangle}$ ($\overline{\langle T_2 \cup B_1 \rangle}$, respectively) is a path. Thus the paths of H induced by $T_1 \cup B_1$ and $T_2 \cup (B_2 \cup \{x\})$ of H and the paths of $\overline{H}$ induced by $T_1 \cup B_2$, $T_2 \cup B_1$ and $\{x\}$ show that $\psi(H) = 5$, and so the lower bound is best possible.

We now show that these lower bounds are best possible for $m \in \{3, 5, 7, 9, 11\}$. Let $T = \{t_1, \ldots, t_m\}$, let $B = \{b_1, \ldots, b_{m+2}\}$.

For $m = 3$, let P_1 (P_2, respectively) be the induced path with consecutive vertices b_1, t_1, b_2 (b_3, t_2, b_4, t_3, b_5, respectively). Construct the $(3,5)$-graph H from $P_1 \cup P_2$ by adding the edge b_2t_2. Then each of the sets $\{b_3, t_1, b_4\}$ and $\{b_5, t_2, b_1, t_3, b_2\}$ induces a path in $\overline{H}$, whence $\rho(\overline{H}) \leq 2$. We conclude that $\psi(H) = 4$, and so the lower bound is best possible.

For $m = 5$, let P_1 (P_2, respectively) be the induced path with consecutive vertices b_1, t_1, b_2, t_2, b_3, t_3, b_4 (b_5, t_4, b_6, t_5, b_7, respectively). Construct the $(5,7)$-graph H from $P_1 \cup P_2$ by adding all edges between $V(P_1) \cap T$ and $V(P_2) \cap B$ and all edges between $V(P_2) \cap T$ and $V(P_1) \cap B$ except for the edges b_1t_4, b_1t_5, b_2t_5, b_3t_4, b_5t_2, t_2b_6, t_3b_6 and t_3b_7. Then each of the sets $\{b_4, t_1, b_3, t_4, b_1, t_5, b_2\}$ and $\{b_5, t_2, b_6, t_3, b_7\}$ induces a path in $\overline{H}$, whence $\rho(\overline{H}) \leq 2$. We conclude that $\psi(H) = 4$, and so the lower bound is best possible.

For $m = 7$, let P_1 (P_2, respectively) be the induced path with consecutive vertices b_1, t_1, b_2, $t_2, b_3, t_3, b_4, t_4, b_5$ ($b_6, t_5, b_7, t_6, b_8, t_7, b_9$, respectively). Construct the $(7,9)$-graph H from $P_1 \cup P_2$ by adding all edges between $V(P_1) \cap T$ and $V(P_2) \cap B$ and all edges between $V(P_2) \cap T$ and $V(P_1) \cap B$ except for the edge t_4b_7. Then each of the sets $\{b_1, t_2, b_4, t_1, b_3\}$ and $\{b_5, t_3, b_2, t_4, b_7, t_7, b_6, t_6, b_9, t_5, b_8\}$ induces a path in $\overline{H}$, whence $\rho(\overline{H}) \leq 2$. We conclude that $\psi(H) = 4$, and so the lower bound is best possible.

For $m = 9$, let P_1 (P_2, respectively) be the induced path with consecutive vertices b_1, t_1, b_2, $t_2, b_3, t_3, b_4, t_4, b_5, t_5, b_6, t_6, b_7$ ($b_8, t_7, b_9, t_8, b_{10}, t_9, b_{11}$, respectively). Construct the $(9,11)$-graph H from $P_1 \cup P_2$ by adding all edges between $V(P_1) \cap T$ and $V(P_2) \cap B$ and all edges between $V(P_2) \cap T$ and $V(P_1) \cap B$ except for the edges b_4t_7, t_5b_{10} and $b_{10}t_6$. Then each of the sets $\{b_2, t_3, b_1, t_2, b_4, t_7, b_{11}, t_8, b_8, t_9, b_9\}$ and $\{b_6, t_4, b_7, t_5, b_{10}, t_6, b_5\}$ induces a path in $\overline{H}$, whence $\rho(\overline{H}) \leq 2$. We conclude that $\psi(H) = 4$, and so the lower bound is best possible.

For $m = 11$, let P_1 (P_2, respectively) be the induced path with consecutive vertices b_1, t_1, b_2, $t_2, b_3, t_3, b_4, t_4, b_5, t_5, b_6, t_6, b_7$ ($b_8, t_7, b_9, t_8, b_{10}, t_9, b_{11}, t_{10}$, b_{12}, t_{11}, b_{13}, respectively). Construct the $(11,13)$-graph H from $P_1 \cup P_2$ by adding all edges between $V(P_1) \cap T$ and $V(P_2) \cap B$ and all edges between $V(P_2) \cap T$ and $V(P_1) \cap B$ except for the edges b_3t_7, b_5t_9, t_6b_{11}

and t_5b_{10}. Then each of the sets $\{b_2, t_3, b_1, t_2, b_4, t_1, b_3, t_7, b_{12}, t_8, b_8, t_{10}, b_9\}$ and $\{b_6, t_4, b_7, t_5, b_{10}, t_{11}, b_{11}, t_6, b_5, t_9, b_{13}\}$ induces a path in $\overline{H}$, whence $\rho(\overline{H}) \leq 2$. We conclude that $\psi(H) = 4$, and so the lower bound is best possible.

□

Lemma 14 *Let $n \geq m+3$. Then $\psi(G) \geq 2(n-m)$, and this bound is best possible.*

Proof. Corollary 8 gives $\psi(G) \geq 4$.

When $m = 1$, the $(1, n)$-graph $G := K_{1,2} \cup \overline{K}_{n-2}$ has $\psi(G) = 2(n-1) = 2(n-m)$, and so the bound is best possible.

So suppose $m \geq 2$. We now construct an (m, n)-graph I such that $\psi(I) = 4$. Let T and B be disjoint sets of cardinality m and n respectively. Partition T into two sets T_1 and T_2 where $|T_1| = \frac{m-1}{2}$ and $|T_2| = \frac{m+1}{2}$. Let x and y be distinct vertices B, and let B_1 and B_2 be two disjoint subsets of $B - \{x, y\}$ with $|B_1| = |B_2| = \frac{m+1}{2}$. Add edges between T_1 and B_1 (T_2 and $B_2 \cup \{x\}$, respectively) such that $\langle T_1 \cup B_1 \rangle$ ($\langle T_2 \cup (B_2 \cup \{x\}) \rangle$, respectively) is a path. Add edges between T_1 and B_2 (T_2 and $B_1 \cup \{y\}$, respectively) such that $\overline{\langle T_1 \cup B_2 \rangle}$ ($\overline{\langle T_2 \cup (B_1 \cup \{y\}) \rangle}$, respectively) is a path. The paths induced by $T_1 \cup B_1$, $T_2 \cup (B_2 \cup \{x\})$ and $\{z\}$ (for $z \in B - (B_1 \cup B_2 \cup \{x\})$) of I and the paths induced by $T_1 \cup B_2$, $T_2 \cup (B_1 \cup \{y\})$ and $\{z\}$ (for $z \in B - (B_1 \cup B_2 \cup \{y\})$) of $\overline{I}$ show that $\psi(I) \leq 2(n-m)$, and so the lower bound is best possible. □

In summary then we have the following result:

Theorem 3. Suppose $n \geq m+2$. Then we have the following best possible lower bounds:

$$\psi(G) \geq \begin{cases} 5 & \text{if } m \text{ is odd, } n = m+2, \text{ where } m = 1 \text{ or } m \geq 13 \\ 2(n-m) & \text{otherwise.} \end{cases}$$

3 Upper bounds on $\psi(G)$

We next establish an upper bound on $\psi(G)$. We first prove a few useful lemmas. Throughout $n \geq m \geq 1$, while $G_{m,n}$ (or just G when the context is clear) will denote an (m, n)-bipartite graph with bipartition (T, B).

Lemma 15 *$G \ncong K_{m,n}$ and $\overline{G} \ncong K_{m,n}$ if and only if there exists a vertex u in G such that* $\deg_G(u)\deg_{\overline{G}}(u) \neq 0$.

Proof. Clearly, if there exists a vertex u in G such that $\deg_G(u)\deg_{\overline{G}}(u) \neq 0$, then $G \ncong K_{m,n}$ and $\overline{G} \ncong K_{m,n}$.

For the converse, suppose $\deg_G(u) = 0$ or $\deg_{\overline{G}}(u) = 0$ for every $u \in G$.

Consider the case when there are distinct vertices u and v such that $\deg_G(u) = 0$ and $\deg_{\overline{G}}(v) = 0$. Notice that u and v must either both be in T or both in B - without loss of generality, assume $\{u, v\} \subseteq T$. Let $w \in B$. Then w is non-adjacent to u in G, and adjacent to v in G, and so $\deg_G(w)\deg_{\overline{G}}(w) \neq 0$, which is a contradiction.

Thus, either $\deg_G(u) = 0$ for every vertex u in G, in which case $\overline{G} \cong K_{m,n}$ or $\deg_{\overline{G}}(u) = 0$ for every vertex u in G, in which case $G \cong K_{m,n}$. □

The following result was established in [2].

Lemma 16

$$\rho(K_{m,n}) = \begin{cases} \lceil (m+n)/3 \rceil & \text{if } 1 \leq m \leq n \leq 2m \\ n - m & \text{otherwise} \end{cases}$$

As a direct consequence, we have:

Lemma 17 *Suppose either G or $\overline{G}$ is a complete bipartite graph. Then*

$$\psi(G) = \begin{cases} (m+n) + \lceil (m+n)/3 \rceil & \text{if } 1 \leq m \leq n \leq 2m \\ 2n & \text{otherwise} \end{cases}$$

Lemma 18 *Let $G = G_{2,3}$. Then $\psi(G) \leq 7$, while $\psi(G) = 7$ if and only if G or $\overline{G}$ is complete.*

Proof. If either G or $\overline{G}$ is complete, then (cf. Lemma 17) $\psi(G) = 7$. Assume neither G nor $\overline{G}$ is complete. If G contains an induced P_3, P_4 or P_5, then it is easy to see that $\psi(G) \leq 6$. Otherwise, G either has exactly one edge or two independent edges. In both cases, $\psi(G) \leq 6$. □

The following result [7] is due to Hattingh, Saleh, Van der Merwe and Walters.

Theorem 19 *Let* $G = G_{m,m}$. *Then* $\psi(G) \le 2m + \lceil \frac{2m}{3} \rceil$. *Moreover,* $\psi(G) = 2m + \lceil \frac{2m}{3} \rceil$ *if and only if* G *or* $\overline{G}$ *is complete.*

Theorem 20 *Let* $m \ge 1$ *and let* $G = G_{m,m+1}$. *Then* $\psi(G) \le (2m+1) + \lceil (2m+1)/3 \rceil$. *Moreover, if* $m \ge 2$, *then* $\psi(G) = (2m+1) + \lceil (2m+1)/3 \rceil$ *if and only if* G *or* $\overline{G}$ *is complete.*

Proof. We prove the assertion by induction on m. Note that $\psi(G_{1,2}) = 4$, and so the assertion follows for $m = 1$. By Lemma 18, $\psi(G_{2,3}) \le 7$, while $\psi(G_{2,3}) = 7$ if and only if $G_{2,3}$ or $\overline{G}_{2,3}$ is complete. Thus, the assertion is also true for $m = 2$. Now assume that the assertion is true for $m \ge 2$ and consider the graph $G = G_{m+1,m+2}$. If either G or $\overline{G}$ is complete, then, by Lemma 17, we have $\psi(G) \le (2m+3) + \lceil (2m+3)/3 \rceil$. So assume that neither G nor $\overline{G}$ is complete. By Lemma 15, there exists a vertex w_1 in T (say) of G and two vertices w_2 and w_3 in B such that w_1w_2 is an edge in G and w_1w_3 is an edge in $\overline{G}$. Factor G into $G_{m,m} \oplus G_{1,2}$, such that $\{w_1, w_2, w_3\} \subseteq V(G_{m,m})$. As neither $G_{m,m}$ nor $\overline{G}_{m,m}$ is complete, it follows from the inductive hypothesis that $\psi(G) \le \psi(G_{m,m}) + \psi(G_{1,2}) < 2m + \lceil \frac{2m}{3} \rceil + 4 = (2m+3) + \lceil (2m+3)/3 \rceil$. □

Lemma 21 *Let* $i \ge 1$, *and let* $G = G_{i+s,i+2s}$ *for some* $s \ge 0$. *Then* $\psi(G) \le (i+s) + (i+2s) + \lceil ((i+s) + (i+2s))/3 \rceil$. *Moreover,* $\psi(G) = (i+s) + (i+2s) + \lceil ((i+s) + (i+2s))/3 \rceil$ *if and only if* G *or* $\overline{G}$ *is complete.*

Proof. We prove the assertion by induction on s. The assertion for $s = 0$ is the statement of Theorem 19, while the assertion for $s = 1$ is the statement of Theorem 20. Now assume that the assertion is true for $s \ge 2$ and consider the graph $G = G_{i+s+1,i+2(s+1)}$. If either G or $\overline{G}$ is complete, then, by Lemma 17, we have $\psi(G) \le (i+s+1) + (i+2(s+1)) + \lceil ((i+s+1) + (i+2(s+1)))/3 \rceil$. So assume that neither G nor $\overline{G}$ is complete. By Lemma 15, there exists a vertex w_1 in T (say) of G and two vertices w_2 and w_3 in B such that w_1w_2 is an edge in G and w_1w_3 is an edge in $\overline{G}$. Factor G into $G_{i+s,i+2s} \oplus G_{1,2}$ such that $\{w_1, w_2, w_3\} \subseteq V(G_{i+s,i+2s})$. As neither $G_{i+s,i+2s}$ nor $\overline{G}_{i+s,i+2s}$ is complete, the inductive hypothesis implies that $\psi(G) \le \psi(G_{i+s,i+2s}) + \psi(G_{1,2}) < (i+s) + (i+2s) + \lceil ((i+s)+(i+2s))/3 \rceil + 4 = (i+s+1) + (i+2(s+1)) + \lceil ((i+s) + (i+2s))/3 \rceil$. □

Theorem 22 *For* $1 \le m \le n < 2m$, *let* $G = G_{m,n}$. *Then* $\psi(G) \le (m+n) + \lceil (m+n)/3 \rceil$. *Moreover,* $\psi(G) = (m+n) + \lceil (m+n)/3 \rceil$ *if and only if* G *or* $\overline{G}$ *is complete.*

Proof. Let $i = 2m - n$ and $s = n - m$. Then $m = i + s$ and $n = i + 2s$, while $(i + s) + (i + 2s) = m + n$, and so the result follows directly from Lemma 21. □

Lemma 23 *Let $m \geq 1$ and let $G = G_{m,2m}$. Then $\psi(G) \leq 4m$.*

Proof. We prove the assertion by induction on m. Note that $\psi(G_{1,2}) = 4 = 2 \times 2$, and so the assertion follows for $m = 1$. Now assume that the assertion is true for $m \geq 1$ and consider the graph $G = G_{m+1,2m+2}$. Factor G into $G_{m,2m} \oplus G_{1,2}$. Then, by the induction hypothesis, $\psi(G) \leq \psi(G_{m,2m}) + \psi(G_{1,2}) \leq 4m + 4 = 4(m + 1)$. □

Theorem 24 *For $n \geq 2m$, we have $\psi(G_{m,n}) \leq 2n$. The graph $K_{m,n}$ shows that this bound is best possible.*

Proof. We have (cf. Lemma 23) $\psi(G_{m,n}) \leq \psi(G_{m,2m}) + \psi(G_{0,n-2m}) \leq 4m + 2(n - 2m) = 2n$. □

It is not true that the upper bound of Theorem 24 is only attained when $G := G_{m,n}$ or $\overline{G}$ is complete. For example, let $m \geq 4$ be an even integer, and let $T = \{t_1, \ldots, t_m\}$ and $B = \{b_1, \ldots, b_n\}$. For $i = 1, \ldots, \frac{m}{2}$, join u_i to every b_j, $j = 1, \ldots, n$. Then $\rho(G) \leq n$ and $\rho(\overline{G}) \leq n$, and so $\psi(G) = 2n$, but neither G nor $\overline{G}$ is complete.

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