Homomorphically full reflexive graphs and digraphs

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Abstract

A (di)graph G is *homomorphically full* if every homomorphic image of G is a sub(di)graph of G. This class of (di)graphs arose in the study of whether a homomorphism from a given graph G to a fixed graph H can be factored through a fixed graph Y. Brewster and MacGillivray proved that the homomorphically full irreflexive graphs are precisely the graphs that contain neither P_4 nor $2K_2$ as an induced subgraph. In this paper, we show that the homomorphically full reflexive graphs are precisely threshold graphs, i.e. the graphs that contain none of P_4, $2K_2$ and C_4 as an induced subgraph. We also characterize the reflexive semicomplete digraphs that are homomorphically full, and discuss the relationship of these digraphs and Ferrers digraphs.

1 Introduction

The (di)graphs considered in this paper are finite. They may contain loops but not multiple edges (arcs). When each vertex of a (di)graph has a loop, the (di)graph is called *reflexive*; when no vertex has a loop, it is called *irreflexive*.

Let G and H be (di)graphs. A *homomorphism* from G to H is a function $f : V(G) \to V(H)$ such that if xy is an edge (arc) of G, then $f(x)f(y)$ is an

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edge (arc) of H. A wealth of information on homomorphisms of (di)graphs and their applications can be found in the recent book [15] and the survey articles [12, 13].

A homomorphism f from G to H is *complete* if f is surjective, and for every edge (arc) h_1h_2 of H there is an edge (arc) xy of G such that $f(x) = h_1$ and $f(y) = h_2$. If there is a complete homomorphism from G to H, then H is called a *homomorphic image* of G.

A (di)graph G is called *homomorphically full* if every homomorphic image of G is a sub(di)graph of G. Homomorphically full graphs arose in the context of the homomorphism factoring problem [3]: for a fixed pair of (di)graphs Y, H and a homomorphism h from Y to H, the question is whether a given homomorphism from a (di)graph G to H can be expressed as a composition of h and a homomorphism g from G to Y, i.e., $f = h \circ g$. It is proved in [3] that if Y is homomorphically full, then the problem is polynomial time solvable. This led naturally to the question of which (di)graphs are homomorphically full.

There is a subtle difference in the concept 'homomorphically full' depending on whether the (di)graph is irreflexive or reflexive. For irreflexive (di)graphs, attention is restricted to irreflexive homomorphic images, otherwise no irreflexive (di)graph with at least one edge (arc) is homomorphically full. For reflexive (di)graphs no such restriction is imposed. In particular, a homomorphism may assign adjacent vertices the same image.

The homomorphically full irreflexive graphs have been characterized in terms of six equivalent conditions [2]. In particular, an irreflexive graph is homomorphically full if and only if it contains neither $2K_2$ nor P_4 as an induced subgraph.

In this paper, we focus on homomorphically full reflexive (di)graphs. We show that homomorphically full graphs are precisely threshold graphs, i.e. the graphs that contain none of P_4, $2K_2$ and C_4 as an induced subgraph. Threshold graphs enjoy many beautiful applications and characterizations [4, 5, 6, 7, 8, 9, 11, 16, 17, 18, 19, 20]. For the purpose of our interest we shall list five well-known equivalent conditions which characterize the class of threshold graphs and hence the class of homomorphically full reflexive graphs. While our purpose here is not to study threshold graphs, the reader may refer to Mahdev and Peled [17] for a wealth of further information.

Three of the conditions mentioned above carry over to reflexive digraphs in a straightforward way and each of them ensures a graph to be homomorphically full. Our examples and the obvious implications of the three conditions show that they define at least two nested classes of reflexive digraphs, and leave open the possibility that two of the conditions define the same class of digraphs. We show that among reflexive semicomplete digraphs these two conditions indeed coincide. Thus the three conditions define exactly two classes of reflexive semicomplete digraphs, both are subclasses of reflexive Ferrers digraphs. Finally we give a complete structural characterization of homomorphically full reflexive semicomplete digraphs.

2 Terminology and notation

Let G be a graph and H be a labelled induced subgraph of G. A *retraction* of G to H is a homomorphism f of G to H such that $f(v) = v$ for each vertex $v \in V(H)$. The definition is the same when G and H are digraphs. When there is a retraction from G to some induced subgraph isomorphic to H, the (di)graph H is called a *retract* of G.

Let G be a graph and v a vertex of G. The *neighbourhood* of v, denoted by $N(v)$, is the set of all vertices $x \neq v$ which are adjacent to v. Note that, by definition, v is *not* in $N(v)$ even when v has a loop. The *closed neighbourhood* of v, denoted by $N[v]$, is defined to be $N(v) \cup \{v\}$. Two vertices u and v of G are *neighbourhood comparable* if either $N(u) - \{v\} \subseteq N(v) - \{u\}$ or $N(v) - \{u\} \subseteq N(u) - \{v\}$.

A digraph is called a *Ferrers digraph* if whenever u, w, x and y are vertices (not necessarily distinct), and uw and xy are arcs, then either uy or xw is also an arc. It follows that a reflexive Ferrers digraph is semicomplete. The forbidden subgraph characterization of threshold graphs implies that these are precisely the underlying graphs of irreflexive symmetric Ferrers digraphs.

It follows from the definition that if D is a Ferrers digraph then for any two vertices x and y we have $N^+(x) \subseteq N^+(y)$ or $N^+(y) \subseteq N^+(x)$, and $N^-(x) \subseteq N^-(y)$ or $N^-(y) \subseteq N^-(x)$. This resembles neighbourhood comparability, which leads one to wonder about connections between reflexive Ferrers digraphs and homomorphically full reflexive semicomplete digraphs. This topic is explored in Section 4. These two classes of digraphs are different, but (for example) the reflexive semicomplete digraphs in which any two vertices are neighbourhood comparable are a proper subset of the reflexive Ferrers digraphs.

Let D be a digraph. If xy is an arc of D, we say that x *dominates* y or that y *is dominated by* x, and write $x \to y$. For a vertex v of D, the *out-neighbourhood* of v, denoted $N^+(v)$, consists of all vertices $x \neq v$ dominated by v and the *closed out-neighbourhood* of v is $N^+[v] = N^+(v) \cup \{v\}$. The *in-neighbourhood* of v, $N^-(v)$, and the *closed in-neighbourhood* of v, $N^-[v]$, are defined similarly. Two vertices u and v of a digraph

D are *neighbourhood comparable* if either $N^+(u) - \{v\} \subseteq N^+(v) - \{u\}$ and $N^-(u) - \{v\} \subseteq N^-(v) - \{u\}$, or $N^+(v) - \{u\} \subseteq N^+(u) - \{v\}$ and $N^-(v) - \{u\} \subseteq N^-(u) - \{v\}$.

Let D be a digraph. An arc xy of D is called *symmetric* if yx is also an arc of D, otherwise it is called *non-symmetric.* We call D *semicomplete* if there is at least one arc between every pair of vertices. We shall use D^s to denote the subdigraph of D induced by symmetric arcs. The *underlying undirected graph of* D, denoted by $U(D)$, is the graph with vertex set $V(D)$ and two vertices being adjacent if they are joined in D by at least one arc. Note that if D is reflexive then D^s and $U(D)$ are both reflexive.

Let G be a reflexive (di)graph and $x, y \in V(G)$. We use G_{xy} to denote the (di)graph obtained by identifying x and y; we think of $E(G_{xy})$ as a set, so that G_{xy} has no multiple edges (arcs). Then G_{xy} is a homomorphic image of G. If G is homomorphically full then G_{xy} is a sub(di)graph of G.

Unless otherwise mentioned, we assume in the remainder of this paper that (di)graphs are reflexive.

3 Reflexive graphs

In this section we show that the homomorphically full reflexive graphs are characterized by six equivalent statements. These are similar to those that characterize the homomorphically full irreflexive graphs [2].

Lemma 3.1 *Let G be a reflexive graph with at least two vertices. Suppose that v is a vertex with $N(v) = \emptyset$ or $N[v] = V(G)$. Then G is homomorphically full if and only if $G - v$ is homomorphically full.*

Proof: We only show the statement is true when $N(v) = \emptyset$. The proof is similar when $N[v] = V(G)$.

Suppose that G is homomorphically full. To show that $G - v$ is homomorphically full, let X be a homomorphic image of $G - v$. Let X' be the graph obtained from X by adding an isolated vertex w (i.e., w is adjacent to itself only). Then X' is a homomorphic image of G. Since G is homomorphically full, X' is a subgraph of G. Therefore, X is a subgraph of $G - v$.

Conversely, suppose that $G - v$ is homomorphically full. To show that G is homomorphically full, let Y be a homomorphic image of G. Thus there is a complete homomorphism f of G to Y. If $f^{-1}(f(v)) = \{v\}$, then $f(v)$ is an isolated vertex of Y and $Y - f(v)$ is a homomorphic image of $G - v$.

Since $G-v$ is homomorphically full, $Y-f(v)$ is a subgraph of $G-v$. Since $f(v)$ is an isolated vertex of Y, Y is a subgraph of G. On the other hand, if $f^{-1}(f(v)) \supset \{v\}$, then Y is a homomorphic image of $G-v$. Thus Y is a subgraph of $G-v$ and hence of G. ◇

The following lemma is formulated from a characterization of threshold graphs, cf. [17].

Lemma 3.2 *Let G be a reflexive graph with at least two vertices. Suppose that v is a vertex with $N(v) = \emptyset$ or $N[v] = V(G)$. Then G is a threshold graph if and only if $G-v$ is a threshold graph.* ◇

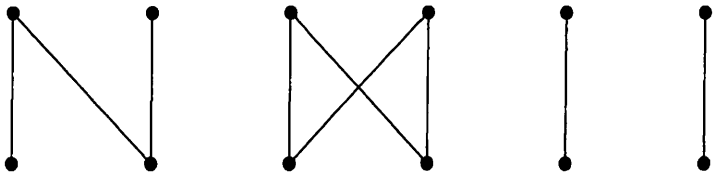

Figure 1: Forbidden suborders in threshold orders.

A poset is called a *threshold order* if it does not contain as a suborder any of the partial orders whose diagrams are shown in Fig. 1 [17]. It is known that the threshold graphs are precisely the comparability graphs of threshold orders [5].

Let G be a reflexive graph. Consider the following six statements:

(a) G is homomorphically full;

(b) Every pair of vertices of G is neighbourhood comparable;

(c) Every homomorphic image of G is a retract of G;

(d) Every homomorphic image of G is an induced subgraph of G;

(e) G is a threshold graph, i.e., G contains none of P_4, C_4, and $2K_2$ as an induced subgraph;

(f) G is the comparability graph of a threshold order.

It is clear from the definitions that $(e) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (a)$. The equivalence between (e) and (f) is proved in [5]. Thus, in order to show the equivalence of these six statements for reflexive graphs, it suffices to show that $(a) \Rightarrow (e)$.

Theorem 3.3 *Let G be a reflexive graph. If G is homomorphically full, then G is a threshold graph, i.e., G contains none of P_4, C_4, and $2K_2$ as an induced subgraph.*

Proof: The proof is by induction on $|V|$. A graph with one vertex is both homomorphically full and a threshold graph. Suppose that every homomorphically full graph with at most $k-1 \geq 1$ vertices is a threshold graph, and let G be a homomorphically full graph with k vertices. If G has an isolated vertex, then by Lemma 3.1, Lemma 3.2 and the induction hypothesis, G is a threshold graph. Suppose, then, that G has no isolated vertices. If G is not connected, then G_{xy}, where x, y belong to different components, is not a subgraph of G, contrary to the assumption that G is homomorphically full. Hence G is a connected graph.

We claim that G has a vertex u with $N[u] = V(G)$. Among all vertices of the maximum degree, $\Delta(G)$, choose a vertex u such that the number of edges contained in the subgraph induced by $N(u)$ is maximum. Suppose that $N[u] \neq V(G)$. Since G is connected, there is a vertex $a \in V(G) - N[u]$ which is adjacent to some vertex $b \in N(u)$. The vertex b has degree at most $\Delta(G)$ and is adjacent to $a \notin N(u)$, so it is not adjacent to some vertex $c \in N(u)$. Since G is homomorphically full, G_{ac} is a subgraph of G. But this is not possible, since the degree of u in G_{ac} is $\Delta(G)$ and the subgraph of G_{ac} induced by $N(u)$ has more edges than the subgraph of G induced by $N(u)$. Therefore $N[u] = V(G)$, proving the claim.

By Lemma 3.1, Lemma 3.2, and the induction hypothesis, G is a threshold graph. The result now follows by induction. ◇

Corollary 3.4 *Let G be a reflexive graph. Then the statements (a) through (f) given above are equivalent.* ◇

Corollary 3.5 *Every induced subgraph, and every homomorphic image, of a homomorphically full reflexive graph is itself homomorphically full.* ◇

4 Reflexive digraphs

In this section we turn to the question of which reflexive digraphs are homomorphically full. We will see that the situation is much more complex than for graphs.

Let D be a reflexive digraph. Consider the following four statements:

(a) Every pair of vertices of D is neighbourhood comparable;

(b) Every homomorphic image of D is a retract of D;

(c) Every homomorphic image of D is an induced subdigraph of D;

(d) D is homomorphically full.

It is clear from comparing the definitions that $(b) \Rightarrow (c) \Rightarrow (d)$. After proving that $(a) \Rightarrow (b)$ we will give examples to show that these four statements are not all equivalent. Then, the remainder of the section is devoted to characterizing the reflexive semicomplete digraphs that satisfy each condition.

Proposition 4.1 *Let D be a reflexive digraph. If every pair of vertices of D is neighbourhood comparable, then every homomorphic image of D is a retract of D.*

Proof: Let H be a homomorphic image of D, and f be a complete homomorphism of D to H. Since every pair of vertices of D is neighbourhood comparable, for each vertex v of H the vertices of D belonging to $f^{-1}(v)$ can be ordered as $x_{v,1}, x_{v,2}, \ldots, x_{v,|f^{-1}(v)|}$ so that, if $i < j$ then $N_D^+(x_{v,i}) \cap (V(D) - f^{-1}(v)) \supseteq N_D^+(x_{v,j}) \cap (V(D) - f^{-1}(v))$ and $N_D^-(x_{v,i}) \cap (V(D) - f^{-1}(v)) \supseteq N_D^-(x_{v,j}) \cap (V(D) - f^{-1}(v))$. Then, since f is a complete homomorphism, the set $\{x_{v,1} : v \in V(H)\}$ induces a subdigraph H' of D that is isomorphic to H. Further, the function g that maps each set $f^{-1}(v)$ to the vertex $x_{v,1}$ is a retraction of D to H'. Therefore, H is a retract of D. $\diamond$

Corollary 4.2 *Let D be a reflexive digraph. Then $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$.* $\diamond$

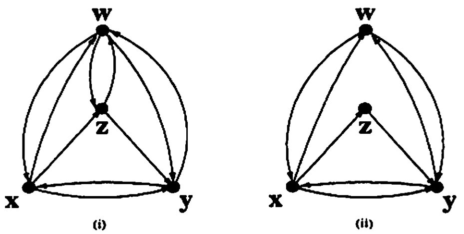

Figure 2: Reflexive digraphs (loops are not drawn) which show gaps among conditions (a), (b), (c) and (d).

The statements (a), (b), (c) and (d) above are not all equivalent. The digraph in Fig. 2(i) satisfies (b) but not (a) (since $N^+(x) - \{y\} \supset N^+(y) -$

$\{x\}$ and $N^{-}(y)-\{x\} \supset N^{-}(x)-\{y\}$), and the digraph in Fig. 2(ii) satisfies (d) but not (c) (since the homomorphic image that results from identifying x and y is not an induced subgraph). We do not know whether statements (b) and (c) are equivalent. While it is true that every retract of a reflexive digraph D is an induced subdigraph of D, the converse is not in general true. This may change under the additional hypothesis that every homomorphic image of D is an induced subdigraph of D, for example see Corollary 4.7.

Given a digraph D, recall that D^s denotes the spanning subdigraph of D induced by the symmetric arcs, and $U(D)$ denotes the underlying undirected graph of D. If D is a reflexive digraph, then both D^s and $U(D)$ are reflexive.

Lemma 4.3 *Suppose that D is a homomorphically full reflexive digraph. Then both $U(D)$ and $U(D^s)$ are homomorphically full and hence reflexive threshold graphs.*

Proof: This follows from Corollary 3.4 and the observation that, for any subdigraph H of D (of D^s), $U(H)$ $(U(H^s))$ is a subgraph of $U(D)$ (of $U(D^s)$). ◇

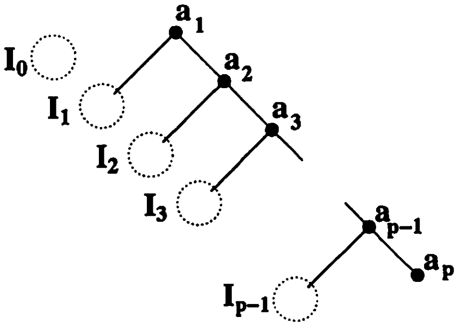

Figure 3: The threshold order of G.

We now illustrate some of the structure of threshold graphs that we will use in our work on homomorphically full reflexive semicomplete digraphs. Let G be a threshold graph. The vertex set $V(G)$ can be partitioned into a maximum clique K and an independent set I. Since any pair of vertices is neighbourhood comparable, it is possible to order the vertices of K as $a_1, a_2, \ldots, a_p$ in such a way that for each $x \in I$, $N(x) = \{a_1, a_2, \ldots, a_j\}$ for some $j = 0, 1, \ldots, p-1$. When $j = 0$, this means $N(x) = \emptyset$ and the vertex x is isolated. For each $j = 0, 1, \ldots, p-1$, let

$$I_j = \{x : \ x \in I \text{ and } N(x) = \{a_1, a_2, \ldots, a_j\}\}.$$

Thus, the (non-empty members of the collection of) sets $I_0, I_1, \ldots, I_{p-1}$ form a partition of the set I. Fig. 3 depicts the corresponding threshold order of G.

Lemma 4.4 *If D is a homomorphically full reflexive semicomplete digraph with $|V(D)| \geq 3$, then $U(D^s)$ has at most one isolated vertex.*

Proof: Suppose that x and y are isolated vertices in $U(D^s)$ and, without loss of generality, assume $x \rightarrow y$ in D. Let z be a vertex of $D - \{x, y\}$. If $y \rightarrow z$, then D_{xz} contains more symmetric arcs than D and hence is not a subdigraph of D. This contradicts the assumption that D is homomorphically full. So we must have $z \rightarrow y$ and $y \not\rightarrow z$. Similarly we must also have $x \rightarrow z$ and $z \not\rightarrow x$. Now, for the same reason, D_{xy} can not be a subdigraph of D, a contradiction. ◇

Let D be a digraph. The *converse* of D, denoted D^r, is the digraph obtained from D by reversing the orientation of all arcs of D. Observe that if H is a homomorphic image of D then H^r is a homomorphic image of D^r. Thus, D is homomorphically full if and only if D^r is homomorphically full.

Theorem 4.5 *Let D be a reflexive semicomplete digraph. Then D is homomorphically full if and only if D or D^r satisfies one of the following three conditions:*

(i) all arcs are symmetric,

(ii) there is a vertex w such that all non-symmetric arcs of D are incident with w, or

(iii) there are two vertices u and v, and a partition W, X, Y, Z of $V(D) - \{u, v\}$, some of which may be empty, such that

(a) $Y \neq \emptyset$,
(b) $(X \cup Z) \neq \emptyset$,
(c) if $Z = \emptyset$, then $|Y| = 1$,

and the non-symmetric arcs of D are precisely the arcs

(e) from all vertices in $X \cup \{u\}$ to v,
(f) from both u and v to every vertex of Y,
(g) from v to every vertex in Z.

Proof: To prove necessity, suppose that D is homomorphically full. By Lemma 4.3, $U(D^s)$ is a threshold graph. Let $K = \{a_1, a_2, \ldots, a_p\}$ induce a maximum clique in $U(D^s)$ and let $I_0, I_1, \ldots, I_{p-1}$ be defined as above. Set $I = I_0 \cup I_1 \cup \ldots \cup I_{p-1}$.

Suppose I has at least three vertices. Then the subdigraph of D induced by I must contain a directed path xyz of length two. The digraph D_{xz} contains all vertices in K and a symmetric arc not incident with a vertex in K, so it can not be a subdigraph of D, contrary to the hypothesis that D is homomorphically full. Hence, I has at most two vertices.

If I is empty, then D is symmetric and satisfies condition (i). If I contains exactly one vertex, say w, then all non-symmetric arcs of D are incident with w and D satisfies condition (ii). Thus, it remains to consider the case where I contains exactly two vertices.

Let $I = \{v, u\}$ where $v \in I_k$, $u \in I_j$ and $0 \leq k \leq j \leq p-1$. By taking the converse if necessary, we can assume $u \rightarrow v$.

We claim that $u \rightarrow a_i$ and $v \rightarrow a_i$ for each $i = j+1, j+2, \ldots, p$. Indeed, if $a_i \rightarrow u$ for some $i = j+1, j+2, \ldots, p$, then in D_{va_i} the vertex u is joined by symmetric arcs to more vertices of K than either u or v in D, so D_{va_i} can not be a subdigraph of D, a contradiction. Similarly, if $a_i \rightarrow v$ for some $i = j+1, j+2, \ldots, p$, then in D_{uv} the vertex u is joined to more vertices of K than either u or v in D, again a contradiction. This proves the claim.

We must have $k \neq j$, for otherwise D_{ua_p} would not be a subdigraph of D. Therefore, $0 \leq k < j \leq p-1$.

Let $W = \{a_1, a_2 \ldots, a_k\}$, $Y = \{a_{j+1}, a_{j+2} \ldots, a_p\}$, $X = N^-(v) \cap \{a_{k+1}, a_{k+2} \ldots, a_j\}$, and $Z = N^+(v) \cap \{a_{k+1}, a_{k+2} \ldots, a_j\}$. Then $Y \neq \emptyset$ and $(X \cup Z) \neq \emptyset$.

Suppose $Z = \emptyset$ and $|Y| \geq 2$, and consider $D_{ua_{j+1}}$. In this digraph, the set K still induces a subdigraph in which all arcs are symmetric, and there are $p - |Y| + 1$ arcs from K to v. But D has only $p - |Y|$ arcs from K to u or to v. Thus, $D_{ua_{j+1}}$ is not a subdigraph of D, a contradiction. Hence, when $Z = \emptyset$, we must have $|Y| = 1$, and D satisfies condition (iii).

For sufficiency, it is enough to show that if a digraph D satisfies one of conditions (i) – (iii) then D is homomorphically full.

The result is clear if D satisfies condition (i) or condition (ii), since in each case D has a subdigraph D' on $|V(D)| - 1$ vertices in which all arcs are symmetric. Any homomorphic image of D with $|V(D)| - 1$ or fewer vertices is therefore a subdigraph of D'. The only homomorphic image of D with $|V(D)|$ vertices is D itself.

Suppose that D satisfies condition (iii). Since $D - \{u, v\}$ is a semicomplete digraph containing only symmetric arcs, every homomorphic image with at most $|V(D)| - 2$ vertices is a subdigraph of D. Thus, we only need to verify that for any $a, b \in V(D)$, D_{ab} is a subdigraph of D.

Observe that the only non-symmetric arcs in $D - v$ are from u to the vertices in Y. Thus, if D_{ab} has a vertex that is adjacent by non-symmetric arcs to a set of at least $p = |Y|$ vertices, then $D - v$ contains D_{ab} as a subdigraph. We show that this is indeed the case.

If neither a nor b is in $Y \cup \{u\}$, then the non-symmetric arcs from u to Y are still present in D_{ab}. Similarly, if neither a nor b is in $Y \cup \{v\}$, then the non-symmetric arcs from v to Y are still present in D_{ab}. Hence, assume that $(Y \cup \{u\}) \cap \{a, b\} \neq \emptyset$ and $(Y \cup \{v\}) \cap \{a, b\} \neq \emptyset$.

If $a = u$ and $b = v$ then, in D_{ab}, the vertex resulting from identifying a and b has the required property.

Suppose that $a = u$ and $b \in Y$. If $Z \neq \emptyset$ then in D_{ab} the vertex v is adjacent by non-symmetric arcs to p vertices belonging to $Y \cup Z$. If $Z = \emptyset$, then by condition $(iii)(c)$, $p = 1$. By condition $(iii)(b)$, $X \neq \emptyset$. Thus, in D_{ab}, a vertex in X is adjacent to v by a non-symmetric arc.

Finally, if $a \in Y$ and $b \in V(D) - \{a, u\}$, then the vertex u has the desired property in D_{ab}.

In all cases, D_{ab} is a subdigraph of $D - v$, as claimed. This completes the proof. ◇

We remark that the homomorphically full reflexive semicomplete digraphs are different from the reflexive Ferrers digraphs. The digraphs described in Theorem 4.5 are Ferrers digraphs, except in (ii) when both $N^+(w) - N^-(w) \neq \emptyset$ and $N^-(w) - N^+(w) \neq \emptyset$, and in (iii) when $X \neq \emptyset$. On the other hand, any digraph obtained from a transitive tournament on at least three vertices by adding a loop at each vertex is a Ferrers digraph but not homomorphically full.

Corollary 4.6 *Let D be a reflexive semicomplete digraph. Then every homomorphic image of D is an induced subdigraph of D if and only if D or D^r satisfies one of the following three conditions:*

(1) all arcs are symmetric;

(2) there is a vertex w such that all non-symmetric arcs are incident with w and, further, either $N^+(w) - N^-(w) = \emptyset$, or $|N^+(w) - N^-(w)| = |N^-(w) - N^+(w)| = 1$;

(3) the digraph D satisfies condition (iii) of Theorem 4.5 with $X = \emptyset$ and $|Z| = 1$.

Further, if D satisfies (1), (2), or (3), then every homomorphic image of D is a retract of D.

Proof: Suppose every homomorphic image of D is an induced subdigraph of D. Then, D is homomorphically full, so D or D^r satisfies one of the three conditions in Theorem 4.5. By taking the converse if necessary, assume D satisfies one of these three conditions. If it is condition (i) then D satisfies condition (1) above.

Suppose D satisfies condition (ii) in Theorem 4.5. If $N^-(w) - N^+(w) = \emptyset$, then D^r satisfies condition (2) above. Suppose that $a \in N^+(w) - N^-(w)$ and $b \in N^-(w) - N^+(w)$. Then, in D_{ab} the vertex w is incident with one more symmetric arc than it is in D. Since w is the only vertex of D incident with all non-symmetric arcs, if w is incident with a non-symmetric arc in D_{ab}, then D_{ab} is not an induced subdigraph of D. Thus, D satisfies condition (2) above.

Suppose D satisfies condition (iii) in Theorem 4.5. For a vertex x of D, we will say that x has an *incoming non-symmetric arc* if it is incident with a non-symmetric arc from another vertex, and x has an *outgoing non-symmetric arc* if it is incident with a non-symmetric arc to another vertex. Note that, in the digraph D, the vertex v is the only vertex that has both an incoming and outgoing non-symmetric arc.

Suppose first that $X = \emptyset$. Then, by Theorem 4.5 $(iii)(b)$, $Z \neq \emptyset$. Let $z \in Z$ and consider D_{uz}. In this digraph, the vertex v has an outgoing non-symmetric arc and is incident with more symmetric arcs than the vertex v in D. Hence, the only possibility is that D_{uz} is isomorphic to the induced subdigraph $D - v$. Since the only vertex of $D - v$ that has an outgoing non-symmetric arc is u, in any isomorphism of D_{uz} to $D - v$, the vertex v of D_{uz} must map to the vertex u of $D - v$. We show that this implies $|Z| = 1$. In the digraph D_{uz}, the vertex v is incident with one more symmetric arc than it is in D. In both of the digraphs D and $D - v$ the vertex u is incident with $|Z|$ more symmetric arcs than the vertex v in D, each corresponding to a vertex of Z. Thus, in the digraph D the vertex u is incident with one more symmetric arc than v, that is, $|Z| = 1$.

Now suppose that $X \neq \emptyset$. Assume $Z \neq \emptyset$ and let $z \in Z$. Consider D_{uz}. In this digraph, the vertex v has both an incoming and an outgoing non-symmetric arc, and is incident with more symmetric arcs than the vertex v in D. Hence, D_{uv} can not be an induced subdigraph of D. Therefore, if $X \neq \emptyset$, then $Z = \emptyset$. By Theorem 4.5 $(iii)(c)$, $|Y| = 1$.

Let $x \in X$, $Y = \{y\}$, and consider D_{xy}. In this case, the vertex v of D_{xy} has an incoming non-symmetric arc (uv) and is incident with more symmetric arcs than the vertex v in D. Hence, the only possibility is that D_{xy} is isomorphic to the induced subgraph $D - v$. Since the only vertex of $D - v$ that has an incoming non-symmetric arc is y, a similar argument to that in the previous case implies that $|X| = 1$. It is now easy to see that the digraph D^r satisfies condition (3).

Therefore, if D satisfies condition (*iii*) in Theorem 4.5 then D or D^r has $X = \emptyset$ and $|Z| = 1$.

It is enough to prove the converse when D satisfies one of the three given conditions, since D has the property that every homomorphic image is both an induced subgraph and a retract of D if and only if D^r has both of these properties.

If D satisfies condition (1) or (2) above, then it is easy to see that every homomorphic image of D is both an induced subdigraph and a retract of D.

Suppose D satisfies condition (3) with $X = \emptyset$ and $|Z| = 1$. Let $Z = \{z\}$ and $Y = \{y_1, y_2, \ldots, y_t\}, t \geq 1$. The only pairs of vertices that are not neighbourhood comparable are u and z, and u and $y_i, 1 \leq i \leq t$. Let H be a homomorphic image of D and f be a complete homomorphism of D to H. We regard H as having been constructed by taking a partition of $V(D)$ and identifying all vertices in each cell.

If each cell contains only neighbourhood comparable vertices, then H is both a homomorphic image and a retract of D, as in the proof of Proposition 4.1.

Suppose some $y_i \in f^{-1}(f(u))$. Then H is a homomorphic image of D_{uy_i}, which is isomorphic to the induced subdigraph $D - v$ and is also a retract of D (map v to y_i). Since D_{uy_i} satisfies condition (2), the digraph H is both an induced subdigraph and a retract of D_{uy_i}. Hence, it is also both an induced subdigraph and a retract of D.

Similarly, if $z \in f^{-1}(f(u))$, then H is a homomorphic image of D_{uz}, which is isomorphic to the induced subdigraph $D - v$, and is both an induced subdigraph and a retract of D.

This completes the proof. ◇

Since every retract of a digraph D is an induced subdigraph of D, Corollary 4.6 implies the following:

Corollary 4.7 *Let D be a reflexive semicomplete digraph. Then, every homomorphic image of D is an induced subdigraph of D if and only if* *every homomorphic image of D is a retract of D.* ◇

By our previous remarks, the reflexive Ferrers digraphs properly contain the reflexive digraphs such that every homomorphic image is an induced subdigraph, or a retract.

Proposition 4.8 *Let D be a reflexive semicomplete digraph. Then, any two vertices of D are neighbourhood comparable if and only if either all arcs of D are symmetric, or D^r satisfies condition (2) of Corollary 4.6 with $N^+(w) - N^-(w) = \emptyset$.*

Proof: It is clear that any two vertices of a reflexive semicomplete digraph satisfying the given conditions are neighbourhood comparable.

Suppose D is a reflexive semicomplete digraph in which any two vertices are neighbourhood comparable. Then, every homomorphic image of D is a retract of D so, without loss of generality, D satisfies one of the three conditions in Corollary 4.6. But, D can not satisfy condition (3), since the vertex u is not neighbourhood comparable with the vertex in Z. Further, D can not satisfy condition (2) with $|N^+(w)-N^-(w)| = |N^-(w)-N^+(w)| = 1$: if $x \in N^+(w) - N^-(w)$ and $y \in N^-(w) - N^+(w)$, then x and y are not neighbourhood comparable.

This completes the proof. ◇

Again, it follows from our earlier remarks that the reflexive digraphs for which any two vertices are neighbourhood comparable are a proper subset of the reflexive Ferrers digraphs.

For $x \in \{a, b, c, d\}$, let P_x be the set of reflexive semicomplete digraphs that satisfy condition (x) (at the start of this section). Then, by the above results, $P_a \subset P_b = P_c \subset P_d$. The reader is invited to compare this statement with the discussion following Proposition 4.1.

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