

# An Eternal Vertex Cover Problem

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Dedicated to Kieka Mynhardt, a wonderful colleague and friend, on this special occasion.

## Abstract

Mobile guards on the vertices of a graph are used to defend it against an infinite sequence of attacks on its vertices. The locations of the guards must induce a vertex cover at all times. We compare this new model of graph protection with other previously studied parameters such as the eternal domination number and the variation of the eternal vertex cover problem in which attacks occur at edges.

## 1 Introduction

Let  $G = (V, E)$  be a simple graph with  $n$  vertices. A *dominating set* of  $G$  is a set  $D \subseteq V$  such that each vertex in  $V - D$  is adjacent to a vertex in  $D$ . The minimum cardinality among all dominating sets of  $G$  is the *domination number*  $\gamma(G)$ . A *vertex cover* of  $G$  is a set  $C \subseteq V$  such that for each edge  $uv \in E$  at least one of  $u$  and  $v$  is in  $C$ . Let  $\tau(G)$  be the *vertex cover number* of  $G$ , the minimum cardinality of any vertex cover of  $G$ .

Mobile guards are used to protect graph  $G$  against a sequence of attacks. Many recent papers have studied these types of problems; see for instance [1, 4, 6, 7, 9], just to list a few. A classic version of this problem is the *eternal domination problem*: at most one guard is located at each vertex; a guard can protect the vertex where it is located and can move to a neighboring vertex to defend an attack there. The sequence of attacks is infinitely long and requires the configuration of guards induce a dominating set before and after each attack has been defended. Alternatively, an infinite sequences of attacks may occur on edges rather than on vertices. In this case, a guard from an incident vertex moves across the attacked edge. This is the

*eternal vertex cover problem.* We define these problems formally below. Each eternal protection problem has two models, depending on whether one guard or all guards are allowed to move in response to an attack.

A vertex (edge) is *protected* if there is a guard on the vertex or on an adjacent (incident) vertex. A vertex  $v$  is *occupied* if there is a guard on  $v$ , otherwise  $v$  is *unoccupied*. An attack is *defended* if a guard moves to the attacked vertex (across the attacked edge).

We define three models of protection that have been studied in the literature. Let  $\{D_i\}$ ,  $D_i \subseteq V$ ,  $i \geq 1$ , be a collection of sets of vertices of the same cardinality, with one guard located on each vertex of  $D_i$ . Each protection strategy can be modeled as a two-player game between a *defender* and an *attacker*: the defender chooses  $D_1$  as well as each  $D_i$ ,  $i > 1$ ; the attacker chooses the locations of the attacks  $r_1, r_2, \dots$ . Each attack is handled by the defender by choosing the next  $D_i$  subject to the constraints of the particular game. The defender wins the game if they can successfully defend any sequence of attacks, the attacker wins otherwise.

In the **eternal domination problem**, each  $D_i$ ,  $i \geq 1$ , is required to be a dominating set,  $r_i \in V$  (assume without loss of generality  $r_i \notin D_i$ ), and  $D_{i+1}$  is obtained from  $D_i$  by moving one guard to  $r_i$  from an adjacent vertex  $v \in D_i$ . If the defender can win the game with the sets  $\{D_i\}$ , then each  $D_i$  is an *eternal dominating set*. The size of a smallest eternal dominating set of  $G$  is the *eternal domination number*  $\gamma^\infty(G)$ . This problem was first studied by Burger et al. in [4] and will sometimes be referred to as the *one-guard moves* model.

In the **m-eternal dominating set problem**, each  $D_i$ ,  $i \geq 1$ , is required to be a dominating set,  $r_i \in V$  (assume without loss of generality  $r_i \notin D_i$ ), and  $D_{i+1}$  is obtained from  $D_i$  by moving guards to neighboring vertices. That is, each guard in  $D_i$  may move to an adjacent vertex, as long as one guard moves to  $r_i$ . Thus it is required that  $r_i \in D_{i+1}$ . The size of a smallest *m-eternal dominating set* (defined similar to an eternal dominating set) of  $G$  is the *m-eternal domination number*  $\gamma_m^\infty(G)$ . This “all guards move” version of the problem was introduced by Goddard, Hedetniemi and Hedetniemi [6]. It is clear that  $\gamma^\infty(G) \geq \gamma_m^\infty(G) \geq \gamma(G)$  for all graphs  $G$ .

In the **m-eternal vertex covering problem**, each  $D_i$ ,  $i \geq 1$ , is required to be a vertex cover,  $r_i \in E$ , and  $D_{i+1}$  is obtained from  $D_i$  by moving guards to neighboring vertices; all guards in  $D_i$  may move to adjacent vertices provided that one of them moves across edge  $r_i$  (we assume without loss of generality that one endvertex of  $r_i$  is not in  $D_i$ , otherwise the guards on the endvertices of  $r_i$  interchange positions). If the defender can win the game with the sets  $\{D_i\}$ , then each  $D_i$  is an *eternal vertex cover*. The size of a smallest eternal vertex cover of  $G$  is the *eternal covering number*

$\tau_m^\infty(G)$ . The  $m$ -eternal vertex covering problem (a.k.a. the eternal vertex covering problem) was introduced by Klostermeyer and Mynhardt [11] and was studied by Fomin et al.[5] and Anderson et al. [2, 3]. Of course,  $\tau_m^\infty(G) \geq \tau(G)$  for all graphs  $G$ . For any graph  $G$  without isolated vertices,  $\tau(G) \geq \gamma(G)$  and  $\tau_m^\infty(G) \geq \gamma_m^\infty(G)$ .

When edges are attacked, the only model that turns out to be interesting allows each guard to move (or not) across an incident edge when an edge is attacked: one guard moves to defend the attack and others may move to better configure themselves. As a simple example, consider an even cycle  $C_{2n}$  with vertices numbered 1 to  $2n$ . Initially have guards on the odd numbered vertices. When an edge is attacked, the guards rotate to the even numbered vertices.

Two new models are introduced, which are the primary subject of this paper. They were inspired by a referee's interpretation of the problems posed in the initial submission of [13], which led to an interesting dialogue with Professor Mynhardt.

In the **eternal vc problem**, each  $D_i$ ,  $i \geq 1$ , is required to be a vertex cover,  $r_i \in V$  (assume without loss of generality  $r_i \notin D_i$ ), and  $D_{i+1}$  is obtained from  $D_i$  by moving one guard to  $r_i$  from an adjacent vertex  $v \in D_i$ . If the defender can win the game with the sets  $\{D_i\}$ , then each  $D_i$  is an *eternal vc set*. The size of a smallest such set of  $G$  is the *eternal vc number*  $\nu^\infty(G)$ .

In the  **$m$ -eternal vc problem**, each  $D_i$ ,  $i \geq 1$ , is required to be a vertex cover,  $r_i \in V$ , and  $D_{i+1}$  is obtained from  $D_i$  by moving guards to neighboring vertices such that  $r_i \in D_{i+1}$ ; all guards in  $D_i$  may move to adjacent vertices. If the defender can win the game with the sets  $\{D_i\}$ , then each  $D_i$  is an  *$m$ -eternal vc set*. The size of a smallest such set of  $G$  is the  *$m$ -eternal vc number*  $\nu_m^\infty(G)$ .

## 2 Terminology and Background

An *independent set* of  $G$  is a set  $I \subseteq V$  such that no two vertices in  $I$  are adjacent. The maximum cardinality among all independent sets is the *independence number*  $\alpha(G)$ . The clique number of  $G$ ,  $\omega(G)$ , is the number of vertices in a largest complete subgraph of  $G$ . The *clique covering number*  $\theta(G)$  is the minimum number,  $k$ , of sets in a partition  $V = V_1 \cup \dots \cup V_k$  such that each induced subgraph  $G[V_i]$  is complete. Hence  $\theta(G)$  equals the chromatic number  $\chi(\bar{G})$  of the complement,  $\bar{G}$ , of  $G$ .

Some fundamental bounds on the graph protection parameters are the following.

**Theorem 1** [6] *For any graph  $G$ ,  $\gamma(G) \leq \gamma_m^\infty(G) \leq \alpha(G) \leq \gamma^\infty(G) \leq \theta(G)$ .*

**Theorem 2** [11] (i) *For any nontrivial, connected graph  $G$ ,  $\tau(G) \leq \tau_m^\infty(G) \leq 2\tau(G)$ .*

(ii) *For any nontrivial tree  $T$ ,  $\tau_m^\infty(T) = |V - L| + 1$ .*

From the definitions above, we state this chain:

$$\gamma(G) \leq \gamma_m^\infty(G) \leq \nu_m^\infty(G) \leq \tau_m^\infty(G)$$

### 3 Results: one guard moves

It is obvious that  $\gamma^\infty(G) \leq \nu^\infty(G)$  and that  $\tau(G) \leq \nu^\infty(G)$  for all graphs  $G$ .  $K_2$ ,  $P_3$ , and  $P_5$  are examples for which  $\gamma^\infty(G) = \nu^\infty(G)$ .  $K_n$  and  $C_5$  are graphs with  $\tau(G) = \nu^\infty(G)$ .  $P_4$  and  $K_3$  are small graphs for which  $\gamma^\infty(G) < \nu^\infty(G)$ .  $P_5$  is an example having  $\tau(G) < \nu^\infty(G)$  and  $K_4 - e$  is a graph with  $\gamma^\infty(G) = \tau(G) < \nu^\infty(G)$ .  $K_{3,4}$  is an example with  $\tau(G) < \gamma^\infty(G) < \nu^\infty(G)$ . There exist graphs  $G$  having  $\tau(G) < c\nu^\infty(G)$ , for any constant  $c$ , consider  $K_{1,n}$ .

We state some elementary results:  $\nu^\infty(K_n) = n - 1$  and  $\nu^\infty(K_{n,m}) = n + m - 1$ . In fact,  $\nu^\infty(G) = n - 1$  for any complete  $k$ -partite graph  $G$ .

**Problem 1** *Characterize the graphs  $G$  with  $n$  vertices for which  $\nu^\infty(G) = n - 1$ .*

**Proposition 3** (i) *For all  $n \geq 3$ ,  $\nu^\infty(C_n) = \lceil \frac{2n}{3} \rceil$ , except that  $\nu^\infty(C_5) = 3$ .*  
(ii)  *$\nu^\infty(P_n) = \frac{2n}{3}$  if  $n \equiv 0 \pmod{3}$ ;  $\nu^\infty(P_n) = \lfloor \frac{2n}{3} \rfloor + 1$  if  $n \not\equiv 0 \pmod{3}$ .*

*Proof:* Consider  $C_n$ . Number the vertices around the cycle from 0 to  $n - 1$ . Partition the vertices of  $C_n$  into parts of three vertices each (possibly with one group of one or two vertices) so that each part induces a  $P_2$  with possibly one part that is  $K_1$  or  $P_1$ . Place a guard on the two highest numbered vertices of each part having three vertices, one guard in a part with one vertex, and two guards in a part with two vertices. The guard on the highest numbered vertex in a part with three vertices will never move and the other guard can move back and forth between the other two vertices in the part. If the part has two vertices, the guards will remain fixed on those vertices; if the part has one vertex, the guard will remain fixed on that vertex. Hence the number of guards specified is sufficient.

For the other direction, suppose  $\nu^\infty(C_n)$  is less than the number specified in (i). First, let  $n \equiv 0 \pmod{3}$ . It is easy to verify the claim when  $n \leq 6$ , so assume  $n \geq 9$ . Partition the vertices into parts,  $Q_1, Q_2, \dots, Q_k$ ,  $k = n/3$ , with three vertices each. Number the parts clockwise around the cycle in the obvious way. Some part, say  $Q_1$ , must contain exactly one guard, and that guard is on the middle vertex of the part, else there exist adjacent vertices such that neither vertex has a guard. Let us first suppose all parts except  $Q_1$  contain at most two guards. Then part  $Q_2$  ( $Q_n$ ) has guards on its lowest numbered (highest numbered) two vertices (else one attack in  $Q_1$  forces an edge to be unprotected), which in turn implies the same for  $Q_3$  ( $Q_{n-1}$ ), and so on until there is either a pair of adjacent vertices with no guard or, if  $n \equiv 3 \pmod{6}$ , part  $Q_{k/2}$  has guards on both its leftmost and rightmost vertex. Attacking the unoccupied vertex in  $Q_{k/2}$  forces there to be an unprotected edge.

On the other hand, if some part has three guards, let us say part  $Q_i$ ,  $i > 1$ , is the closest such part, in the clockwise direction. Then there can be no other part with only one guard between  $Q_1$  and  $Q_i$ , else there would be a pair of adjacent vertices with no guard. So there must exist a part  $Q_j$ ,  $k \geq j > i$ , with exactly one guard (since there exists a part with three guards and there are fewer than  $2n/3$  guards). But then traversing the cycle in a counterclockwise direction, there must be a part  $Q_m$  with three guards between  $Q_1$  and  $Q_j$ . Thus either there is another part closer to  $Q_1$  with only one guard (and we repeat the same argument as before) or there are  $2n/3$  guards. Hence, fewer than  $2n$  guards do not suffice.

The argument when  $n \equiv 1 \pmod{3}$  is similar. Partition into  $k+1$  parts, with one part having one vertex, the others with three vertices. Suppose  $\lfloor \frac{2n}{3} \rfloor$  guards suffice. We may suppose without loss of generality that there is no guard in the part  $Q_1$  that contains one vertex  $v_0$  (this can be done by picking  $Q_1$  to be any unoccupied vertex and labeling that vertex  $v_0$ ). Also assume without loss of generality that the lowest two indexed vertices of  $Q_2$  each have guards (alternatively, the highest two indexed vertices of  $Q_{k+1}$  have guards and the argument is symmetric). That is, if both  $Q_{k+1}$  and  $Q_2$  have their middle vertex unoccupied, an attack at  $v_0 \in Q_1$  leaves an edge unprotected. If there exists a part of three vertices with only one guard, the argument proceeds as above. Otherwise, all parts with three vertices have two guards. Attacking the middle vertex of each of these parts starting first with  $Q_3$ , then  $Q_4$ , up to  $Q_k$ , we can assume that each of these parts has guards on its two lowest numbered vertices. But then edge  $v_0v_{n-1}$  is unprotected.

Next suppose  $n \equiv 2 \pmod{3}$ . The case when  $n = 5$  is easy to check

(and this case is special as  $\nu^\infty(C_5) = 3$ ), so assume  $n \geq 8$ . Partition into  $k+1$  parts, with one part having two vertices. Suppose to the contrary that  $\lfloor \frac{2n}{3} \rfloor + 1$  guards suffice. Then there must exist eight consecutive vertices such that there are at most five guards on those eight vertices. Number these vertices  $v_1, v_2, \dots, v_8$  and assume they contain exactly five guards (the case when they have fewer than five guards is very easy to argue). Assume  $v_5$  is unoccupied. Thus there are guards on  $v_4$  and  $v_6$ . There are a few cases to consider depending on the locations of the other guards.

(a) Suppose  $v_1, v_2, v_4, v_6,$  and  $v_7$  are occupied. Attack  $v_5$ . This must be defended by the guard at  $v_6$ . Now attack  $v_3$ . If this is defended by the guard at  $v_4$ , there are five consecutive vertices ( $v_4$  through  $v_8$ ) containing only two guards and it is easy for the attacker to win the game with one more attack (at  $v_6$  in this case). Otherwise, the attack is defended by the guard at  $v_2$ . Then we have guards on three consecutive vertices ( $v_3, v_4, v_5$ ) adjacent to a part with one guard ( $v_6, v_7, v_8$ ) and we can consider those six vertices to be two parts,  $Q_1, Q_2$ , and argue as follows. If  $n = 8$ , attacking  $v_8$  causes either  $v_1v_2$  or  $v_6v_7$  to be unprotected. So suppose  $n > 8$ . Let  $Q_{k+1}$  consist of  $v_1, v_2$  and it contains one guard at  $v_1$ , per above.

Suppose first that all parts other than  $Q_{k+1}$  have exactly two guards. Then each of these parts must initially have a guard on the lowest numbered vertex (since  $v_8 \in Q_2$  is unoccupied). Attacking the middle vertex of each of these parts starting from part  $Q_3$ , then  $Q_4$ , up to  $Q_k$ , we can assume that each of these parts has guards on its two lowest numbered vertices. Now attack  $v_8$ , which must be defended by the guard at  $v_9$ , then attack  $v_{11}$  (defended by guard at  $v_{12}$ ), then  $v_{14}$ , and so on around the cycle until eventually  $v_0$  is attacked and that must be defended by the guard at  $v_1$ . Then edge  $v_1v_2$  is unprotected.

If any other part besides  $Q_2$  and  $Q_{k+1}$  has only one guard, we can argue in a similar manner as above.

(b) Suppose  $v_1, v_3, v_4, v_6,$  and  $v_7$  are occupied. An attack at  $v_5$ , if defended by the guard at  $v_6$ , causes there to be guards on three consecutive vertices  $v_3, v_4, v_5$  which are adjacent to three vertices with only one guard ( $v_6, v_7, v_8$ ) and we may proceed as above, noting that if  $n = 8$ , an attack at  $v_8$  will force an edge to be unprotected. Otherwise, the attack at  $v_5$  is defended by the guard at  $v_4$  and we have  $v_5, v_6, v_7$  occupied. An attack at  $v_2$  forces the guard at  $v_1$  to move to  $v_2$ . If  $n = 8$ , we are done (since now  $v_1v_8$  is unprotected). Otherwise,  $n > 8$  and there are guards on three consecutive vertices  $v_5, v_6, v_7$  which are adjacent to three vertices with only one guard ( $v_2, v_3, v_4$ ) and we can again proceed as before.

(c) The other possible locations of the guards all either have guards on

three consecutive vertices, or five consecutive vertices containing only two guards and can be argued as above.

The proof for (ii) is similar, except that when  $n \not\equiv 0 \pmod{3}$ , let the part consisting of one or two vertices be the rightmost one or two vertices in  $P_n$ . Only one guard is needed in a part with two vertices. Hence the necessity argument is simpler than above, particularly in the  $n \equiv 2 \pmod{3}$  case.  $\square$

**Theorem 4** *Let  $G$  be a graph with  $\omega(G) > 2$ . Then  $\gamma^\infty(G) < \nu^\infty(G)$ .*

*Proof:* Suppose  $\omega(G) > 2$ . Observe that for each clique  $C_i$  with  $k_i$  vertices, there must be at least  $k_i - 1$  guards in  $C_i$  at all times, in order to maintain a vertex cover. Let  $C$  be a clique covering containing a clique  $C_j$  with at least three vertices. Suppose we could protect  $G$  with  $|C| \geq \theta(G)$  guards and configure the guards thusly, so that each clique of  $C$  with  $k_i > 1$  vertices contains  $k_i - 1$  guards. Since  $C_j$  contains at least three vertices (and thus at least two guards), it follows that there must exist a vertex  $v$  that is a  $K_1$  in  $C$  such that  $v$  has no guard. If the first attack in the sequence of attacks is an attack at  $v$ , this forces a guard to move to from a vertex  $u$  in a neighboring clique. It must be that that neighboring clique has exactly two vertices  $u, w$ , contains one guard (since the attack at  $v$  was the first attack), and there is an alternate clique covering in which  $v$  is in a  $K_2$  with  $u$ . But now edge  $uw$  is unprotected. As  $\gamma^\infty(G) \leq \theta(G)$ , we have that if  $\omega(G) > 2$ , then  $\gamma^\infty(G) < \nu^\infty(G)$ .  $\square$

We say that two disjoint cliques are adjacent if there is at least one edge between them.

**Theorem 5** *Let  $G$  be a graph with  $\omega(G) = 2$ . Then  $\theta(G) = \nu^\infty(G)$  if and only if there exists a minimum clique covering having no two  $K_2$ 's adjacent.*

*Proof:* Suppose every minimum clique covering has two adjacent  $K_2$ 's. Then assume, without loss of generality, a guard configuration in which each clique in clique covering  $C$  has exactly one guard. Then we can force the guards onto particular vertices of each of the two adjacent  $K_2$ 's, leaving an edge unprotected.

If there exists a minimum clique covering having no two  $K_2$ 's adjacent, it is easy to see that  $\nu^\infty(G) = \theta(G)$ , since  $\omega(G) = 2$ .  $\square$

It is a simple corollary to the above that if  $n > 1$ ,  $\nu^\infty(G) \geq \theta(G)$ . Take any number of  $K_2$ 's and join one end of each to a common vertex. This graph has  $\gamma^\infty(G) = \nu^\infty(G)$ . It is not clear to us whether or not  $\theta(G) = \gamma^\infty(G)$  for all graphs with  $\omega(G) = 2$  and a minimum clique covering having no two  $K_2$ 's adjacent.

**Theorem 6** *Let  $G$  be a graph with  $n > 2$  vertices. Then  $\gamma(G) < \nu^\infty(G)$ .*

*Proof:* If  $\theta(G) = 1$ , then  $G = K_n, n > 2$ , and the theorem is true. If  $\omega(G) = 2$  and  $\theta(G) = 2$ , then  $G$  has either three or four vertices and it is easy to verify the cases. Otherwise,  $\omega(G) \geq 2$  and  $\theta(G) > 2$ . If  $\omega(G) > 2$ , then the result follows from Theorem 4. Finally, suppose  $\omega(G) = 2$  and  $\theta(G) > 2$ . Fix a minimum clique covering  $C$  and note that  $|C| = \theta(G) > 1$ . If each clique in  $C$  has two vertices, then  $\nu^\infty(G) > \theta(G)$  and we are done by the same reasoning as in Theorem 5 and because  $\theta(G) \geq \gamma(G)$ . Otherwise, there is a clique in  $C$  with one vertex  $v$ . Then there exists a dominating set  $D$  of  $G$  in which  $D$  does not contain  $v$ , i.e.,  $\gamma(G) < \theta(G)$ .  $\square$

Let  $T$  be a tree with more than two vertices. Designate one vertex as the root. The parent of vertex  $v$  is its neighbor  $u$  such that  $\text{dist}(\text{root}, u) = \text{dist}(\text{root}, v) - 1$ . A *stem* of  $T$  is a vertex adjacent to a leaf and a *p-stem* is a vertex that is a parent of a stem. A vertex can be both a stem and a p-stem. We outline a simple linear-time algorithm to determine  $\nu^\infty(T)$ , though we do not know a structural way to characterize  $\nu^\infty(T)$ .

A vertex is *locked* if it has a guard and that guard never moves. Begin with a stem  $v$  furthest from the root. Place guards on all leaves adjacent to  $v$ . Do not place a guard on  $v$  but place a guard on  $v$ 's parent (the p-stem  $u$ ) and lock it, unless all of  $u$ 's children in  $T$  are (or will be) locked vertices. Delete  $v$  and its leaves and repeat the process recursively on the resulting graph  $T'$ . Note that if a locked vertex has children that are leaves in some  $T'$ , its children will have guards placed on them.

In the same manner as in Proposition 3, one can show that if a tree  $T$  has  $n > 2$  vertices, then  $\nu^\infty(T) \geq \lfloor \frac{2n}{3} \rfloor$ .

**Question 1** *For which graphs is  $\tau(G) = \nu^\infty(G)$ ?*

Grid graphs, i.e.,  $P_m \times P_n$ , are a well-studied class of graphs in domination theory. We used  $P_m \times P_n$  to denote the  $m \times n$  grid graph.

**Theorem 7** *Let  $n \geq 2$ . Then  $\nu^\infty(P_2 \times P_n) = \lfloor \frac{3n}{2} \rfloor$ .*

*Proof:* Let us first show that this many guards suffice. Start with  $n$  guards on the top row and  $\lfloor \frac{n}{2} \rfloor$  guards on the bottom row, arranged every other vertex, so they form a vertex cover of the  $P_n$  that is the bottom row. An attack on an unoccupied vertex on the bottom row is defended by sending the guard from the neighboring vertex on the top row.

To see this many guard are necessary, suppose to the contrary that  $\nu^\infty(P_2 \times P_n) < \lfloor \frac{3n}{2} \rfloor$ . Number the vertices on each row from 1 to  $n$ , from left to right.

First suppose that  $n \equiv 0 \pmod{2}$ . Partition the vertex set of  $P_2 \times P_n$  into  $C_4$ 's. Observe that each  $C_4$  must have at least two guards in it at all times and some  $C_4$  must contain exactly two guards, as  $\nu^\infty(P_2 \times P_n) < \lfloor \frac{3n}{2} \rfloor$ . Let  $C$  be the leftmost  $C_4$  that contains exactly two guards at any point in time. It is clear that  $C$  cannot be the leftmost  $C_4$  in the grid, else either an edge in that  $C_4$  is unprotected or one attack can force an edge to be unprotected. We claim  $C$  must be adjacent to a  $C_4$  which lies to the left of  $C$ , call it  $C'$ , with four guards, otherwise  $C'$  has either two guards (in which case an attack can force an edge to be unprotected) or three guards (in which case an attack in  $C$  can force  $C'$  to have two guards, contradicting that  $C$  is the leftmost such  $C_4$ ). If the  $C_4$  to the right of  $C$  has exactly two guards, we can again force an unprotected edge. If the  $C_4$  to the right of  $C$ , call it  $C''$ , has exactly three guards, then none of those guard can ever move to a cycle to the right of  $C''$  (else  $C''$  would have two guards), and we can proceed by induction. Otherwise,  $C''$  has four guards. In the latter case, there must be  $C_4$  to the right of  $C$  containing exactly two guards, and we can remove  $C$  and  $C'$  and again proceed by induction (since we can be sure none of the size guards in  $C$  or  $C'$  ever move to the right, which can be done if we never have any attacks in  $C$  or  $C'$ ).

The case when  $n \equiv 1 \pmod{2}$  can be argued in a similar fashion: partition the graph into  $\lfloor \frac{n}{2} \rfloor$   $C_4$ 's and a  $P_2$ . Since we must always keep a guard in the  $P_2$ , the argument can proceed as above.  $\square$

We next consider  $P_3 \times P_n$ . Assume the grid is laid out in three rows, each with  $n$  columns. We will refer to the vertex in row  $r$  and column  $c$  as  $(r, c)$ .

**Lemma 8** (i)  $\nu^\infty(P_2 \times P_3) = 4$ .

(ii)  $\nu^\infty(P_3 \times P_3) = 7$ .

(iii)  $\nu^\infty(P_3 \times P_4) = 9$ .

*Proof:* (i) We showed above that  $\nu^\infty(P_2 \times P_3) = 4$ .

(ii) That  $\nu^\infty(P_3 \times P_3) \leq 7$  is shown by keeping three guards fixed in any one row and keeping two guards in each other row (and the two guards in each of these rows may move within the row). We show that this many are necessary.

Suppose to the contrary we could defend with six guards. Initially suppose there are exactly two guards in each row at all times. Assume without loss of generality there are guards on  $(1, 2)$ , and  $(1, 3)$ . Then there must be a guard on  $(2, 1)$  and we assume there is a guard on  $(2, 2)$  (otherwise we can skip the next attack). Attack  $(2, 3)$ , which must be defended by the guard at  $(2, 2)$  since there are two guards in each row at

all times. Now attack  $(1, 1)$  and the guard from  $(1, 2)$  moves there, so now the guards do not induce a vertex cover.

On the other hand, suppose there are three guards in some row at some time. First suppose three guards are in row one. Then there cannot be three guards on row two (else the edges on row three are not covered), so first suppose there are two guards on row two. If there are guards on  $(2, 2)$  and  $(2, 3)$ , then there must be a guard on  $(3, 1)$  and an attack at  $(3, 2)$  forces the guards into a configuration that is not a vertex cover. If there are guards on  $(2, 1)$  and  $(2, 3)$ , then there must be a guard on  $(3, 2)$  and an attack at  $(2, 2)$  forces the guards into a configuration that is not a vertex cover. Likewise if there are guards on  $(2, 1)$  and  $(2, 2)$ . If there are two guards on row three, we argue in a similar manner.

To conclude part (ii), suppose there are three guards on row two. Assume, without loss of generality, a guard is on  $(1, 2)$  and two guards are on row three. An attack at  $(1, 1)$  would have to be defended by a guard from  $(2, 1)$ , which implies there is a guard at  $(3, 1)$ . So rather than attack  $(1, 1)$ , attack  $(3, 2)$ . This either forces three guards onto row three (which we dealt with above), or the guard from  $(3, 1)$  moves to  $(3, 2)$ . Now an attack at  $(1, 1)$  forces the guards into a configuration that is not a vertex cover.

(iii) We show that  $\nu^\infty(P_3 \times P_4) = 9$ . This can be achieved by starting with two guards in column one (and these two guards move within this column, we denote such a move as  $\updownarrow$ ), three guards in column two that never move, and two guards in each of columns three and four (two guards in the middle row of these columns) and the guard on the top and the guard on the bottom move left and right between two vertices (which we denote as  $\leftrightarrow$ ). We show that this many is necessary.

Suppose to the contrary we could defend with eight guards. Similar to part (ii) above, if there are exactly two guards in each column at all times, it is not difficult to force the guards into a configuration that is not a vertex cover. So first suppose there are three guards in column four. We can assume without loss of generality that there are guards on  $(1, 1)$ ,  $(2, 2)$ , and  $(3, 1)$ , since these are independent vertices, none of which are adjacent to column four. Now we must have guards on  $(1, 3)$  and  $(3, 3)$  and then an attack at  $(1, 2)$  forces the guards into a configuration that is not a vertex cover.

Next suppose there are three guards in column three (the argument is the same if the three guards are in column two). Suppose there is one guard in column four, in which case it must be on  $(2, 4)$ . Suppose there are two guards in column two. If  $(1, 2)$  is unoccupied, then an attack at  $(1, 4)$  forces the guards into a configuration that is not a vertex cover. Likewise if

$(3, 2)$  is unoccupied. If  $(2, 2)$  is unoccupied, let us assume without loss of generality that  $(3, 1)$  is unoccupied, since column one has two guards and  $(2, 1)$  must be occupied. We attack  $(3, 4)$ . This is defended with the guard from  $(3, 3)$ . Now attack  $(3, 1)$ . Either possible defense, from  $(2, 1)$  or  $(3, 2)$ , puts the guards into a configuration that is not a vertex cover.

The case when there are three guards in column two can be easily reduced to the case when there are two guards in column two, by placing an attack at the appropriate vertex in column one.

If there is one guard in column two, then there are three guards in column one, which we can handle as in the case when three guards were in column four.

Now suppose there are three guards in column three and two guards in column four. We can assume vertices  $(1, 1)$  and  $(3, 1)$  have guards, as they are independent and not adjacent to column three. This implies  $(2, 2)$  has a guard. An attack at  $(2, 1)$  forces the guards into a configuration that is not a vertex cover.  $\square$

**Theorem 9** *Let  $n \equiv 1 \pmod{2}$ ,  $n \geq 3$ . Then  $\nu^\infty(P_3 \times P_n) = 2n + 1$ .*

*Proof:* We showed nine guards suffice for  $n = 3$  above, so assume  $n \geq 5$ . We first show this many guards suffice. Place  $n$  guards on row two; these guards will never move. Place  $\lceil \frac{n}{2} \rceil$  in each of the first and third rows on every other vertex. The guards in the first and third rows in columns whose index is  $3 \pmod{4}$  will remain fixed in that location, each other guard will move  $\leftrightarrow$ , swapping positions between two horizontally adjacent vertices.

We claim this many guards are necessary. Suppose to the contrary that we could defend with  $2n$  guards. Each column must have at least one guard at all times, in order for the guards to protect all edges. From Lemma 8, we can force seven guards to be in the rightmost three columns, after some number of attacks. To see this, imagine a sequence of attacks occurred only in the rightmost three columns; thus to defend these, seven guards would be needed in these three columns. With seven guards being in those three columns, there must exist a pair of adjacent columns  $i, i + 1$ ,  $i < n - 2$  having at most three guards between them. Since each column must have at least one guard, let us first suppose there are guards at  $(2, i)$ ,  $(1, i + 1)$ , and  $(3, i + 1)$ . An attack at  $(3, i)$  must be defended by the guard in column  $i - 1$  else an edge would be unprotected. If  $i - 1 = 0$  the attacker wins the game, otherwise there now exists a pair of adjacent columns  $i, i + 1$ ,  $i < n - 3$  having at most three guards between them. Repeating this, the attacker either wins the game or forces there to be just three guards in columns 1 and 2, combined. An attack at an unoccupied vertex of column 1 causes the attacker to win.

On the other hand, suppose there are guards at  $(1, i)$ ,  $(3, i)$ , and  $(2, i+1)$ . An attack at  $(2, i)$  must be defended by the guard in column  $i-1$  else an edge would be unprotected. The argument is now the same as in the previous case.  $\square$

**Theorem 10** *Let  $n \equiv 0 \pmod{2}$ ,  $n \geq 4$ . Then  $\nu^\infty(P_3 \times P_n) = 2n + 1$ .*

*Proof:* We showed nine guards suffice for  $n = 4$  above, so assume  $n \geq 6$ . We first show this many guards suffice. Place the guards in the first  $n - 3$  columns as in Proposition 9 (so column  $n - 3$  has three guards). Place two guards in columns  $n - 2$ ,  $n - 1$ , and  $n$  so that there is no guard in row two of column  $n - 2$  and so the guards induce a vertex cover. Note that the rightmost three columns have six guards (which is an insufficient number to defend these three columns, without some assistance). However, we defend attacks at row two of columns  $n - 3$  and  $n - 2$  with the “special” guard initially fixed at row two, column  $n - 3$  (so this guard moves  $\leftrightarrow$ ). The remaining eight vertices in columns  $n - 2$  through  $n$  can be defended with those six guards so that the edges between them and row two column  $n - 2$  are always protected. Likewise, we defend attacks in columns  $n - 5$  through  $n - 3$  with the seven guards initially located (though the special guard sometimes leaves these columns to move to column  $n - 2$ ) and these guards will always protect all edges incident to the vertex in row two, column  $n - 3$ .

We claim this many guards are necessary. Suppose we could defend with  $2n$  guards. Since  $\nu^\infty(P_3 \times P_4) = 9$ , we must at some point have nine guards in the rightmost four columns, thus some column has at most one guard. The argument then follows as in Proposition 9.  $\square$

We leave open the problem of determining  $\nu^\infty(P_m \times P_n)$  exactly for  $m > 3$ . We believe that  $\nu^\infty(P_4 \times P_n) > 3n$ , for  $n > 3$ . In particular, it may be that  $\nu^\infty(P_4 \times P_4) = 13$ .

## 4 Results: all guards move

There exist triangle-free graphs with  $\gamma_m^\infty(G) < \nu_m^\infty(G)$ . One such example is formed by taking a  $C_5$  with vertices  $v_1, v_2, v_3, v_4, v_5$  and adding two vertices  $u, v$  such that  $uv$  is an edge and  $u$  is adjacent to  $v_1$  and  $v$  is adjacent to  $v_3$ . This graph has  $\gamma_m^\infty(G) = 3$ ,  $\nu_m^\infty(G) = 4$ . As another example,  $\nu_m^\infty(K_{m,n}) = \min\{\min\{m, n\} + 1, \max\{m, n\}\}$ , whereas  $\gamma_m^\infty(K_{m,n}) = 2$ .

There exist graphs with  $\nu_m^\infty(G) < \tau_m^\infty(G)$  such as  $K_4 - e$ . We next show a larger class of graphs with this property.

Define a *neo-colonization* to be a partition  $\{V_1, V_2, \dots, V_t\}$  of  $G$  such that each  $V_i$  induces a connected graph. A part  $V_i$  is assigned weight one if it induces a clique and  $1 + \gamma_c(G[V_i])$ , otherwise, where  $\gamma_c(G[V_i])$  is the size of the smallest connected dominating set in the subgraph induced by  $V_i$ . Recall that a connected dominating set of  $G$  is a set  $D$  such that  $D$  is a dominating set and the induced subgraph  $G[D]$  is connected. Then  $\theta_c(G)$  is the minimum weight of any neo-colonization of  $G$ . Goddard et al. [6] proved that  $\gamma_m(G) \leq \theta_c(G) \leq \gamma_c(G) + 1$ . Klostermeyer and MacGillivray proved the following [10].

**Theorem 11** [10] *Let  $T$  be a tree. Then  $\theta_c(T) = \gamma_m^\infty(T)$ .*

The next result implies that  $\nu_m^\infty(G)$  is less than  $\tau_m^\infty(G)$  for infinitely many trees.

**Theorem 12** *Let  $T$  be a tree. Then  $\nu_m^\infty(G) = \theta_c(T) = \gamma_m^\infty(T)$ .*

*Proof:* It suffices to prove that  $\nu_m^\infty(G) \leq \theta_c(T)$ . Let  $C = \{V_1, V_2, \dots, V_t\}$  be a minimum weight neo-colonization of  $T$ . Each  $V_i$  induces a  $K_1$ , a  $K_2$ , or another trees that is not a clique. For each  $K_1$  in  $C$ , place a guard on that vertex, that guard will never move. For each  $K_2$ , place one guard in it that will move back and forth between the two vertices. For each other tree in  $C$  that is part  $V_i$ , place  $\gamma_c(G[V_i]) + 1$  guards that will reside on the internal vertices of the tree  $T_i$  induced by  $V_i$  as well as one leaf of  $T_i$  (which leaf will change depending on the current attack). Consider an attack at vertex  $v \in V_j$ . Move the guards in  $V_j$  (if necessary) to defend this attack. This may leave unprotected one or more edges that connect  $V_j$  to other parts. However, each of those edges can be protected by moving guards in the neighboring parts. This may leave additional edges unprotected between those parts and the parts that are adjacent to them. Those edges can be protected in a similar manner, and the process will terminate since  $T$  is a tree.  $\square$

Likewise, if  $G$  a graph with  $\omega(G) = 2$ , then  $\theta(G) \geq \nu_m^\infty(G)$ .  $C_4$  and  $C_5$  are examples that are not trees where this bound is sharp.

For  $P_2 \times P_n, n \geq 2$ ,  $\gamma_m^\infty(P_2 \times P_n) = \lceil \frac{2n}{3} \rceil$ . For  $P_3 \times P_n, 2 \leq n \leq 8$ ,  $\gamma_m^\infty(P_3 \times P_n) = n$  and for  $n \geq 9$ ,  $\gamma_m^\infty(P_3 \times P_n) \leq \lceil \frac{8n}{9} \rceil$ , see [8].

**Theorem 13** [11]

- (i)  $\tau_m^\infty(P_1 \times P_n) = n - 1$ .
- (ii) If  $n$  is even, then  $\tau_m^\infty(P_n \times P_m) = \frac{nm}{2} = \tau(P_n \times P_m)$ .

(iii) If  $n, m > 1$  are odd,  $n \geq m$ , then  $\tau_m^\infty(P_n \times P_m) = \lceil \frac{nm}{2} \rceil = \tau(P_n \times P_m) + 1$ .

**Theorem 14** (i)  $\nu_m^\infty(P_1 \times P_n) = \lceil \frac{n}{2} \rceil$ .

(ii) If  $n$  is even, then  $\tau_m^\infty(P_n \times P_m) = \frac{nm}{2} = \tau(P_n \times P_m)$ .

(iii) If  $n, m > 1$  are odd,  $n \geq m$ , then  $\tau_m^\infty(P_n \times P_m) = \lceil \frac{nm}{2} \rceil = \tau(P_n \times P_m) + 1$ .

*Proof:* Part (i) follows from Theorem 12. Part (ii) follows from Theorem 13 (ii) and the fact that for all  $G$ ,  $\tau(G) \leq \nu_m^\infty(G) \leq \tau_m^\infty(G)$ . Part (iii) is similar, except we must show that  $\tau_m^\infty(P_n \times P_m) > \tau(P_n \times P_m)$ . To see this, observe that  $P_n \times P_m$  has a unique minimum vertex cover of cardinality  $\lceil \frac{nm}{2} \rceil$  consisting of the vertices at odd distance from the vertices of degree two, when both  $n$  and  $m$  are odd. Thus if guards are located on the vertices of this unique minimum vertex cover, an attack at any unoccupied vertex would force the guards into a configuration that is not a vertex cover.  $\square$

**Theorem 15** Let  $G$  be a connected graph with  $\theta(G) \leq 2$ . Then  $\nu_m^\infty(G) = n - 1$  if and only if (a)  $G \cong K_n, n > 1$ , or (b)  $\theta(G) = 2$  and either (i) the two distinct cliques  $C_1, C_2$  in each minimum clique covering of  $G$  are such that there do not exist  $u, v \in C_1, x, y \in C_2$  ( $u \neq v, x \neq y$ ) such that  $ux, vy \notin E(G)$  or (ii) one clique in some minimum clique covering of  $G$  is a  $K_1$ .

*Proof:* Clearly,  $\nu_m^\infty(K_n) = n - 1$  when  $n > 1$  and also  $\nu_m^\infty(G) \leq n - 1$  for every connected graph  $G$ . Now consider a graph  $G$  with  $\theta(G) = 2$ . Consider a clique covering  $C = \{C_1, C_2\}$  of  $G$ . Suppose to the contrary that  $\nu_m^\infty(G) < n - 1$ . If  $C_1$  is a  $K_1$  and  $C_2$  is not a  $K_1$ , then we cannot have two unoccupied vertices in  $C_2$  at any time. Then we can force a guard into  $C_1$  by attacking that vertex, causing there to be two unoccupied vertices in  $C_2$ .

Now suppose neither  $C_1$  nor  $C_2$  is a  $K_1$ . Then there must be an unoccupied vertex in both  $C_1$  and  $C_2$ , since each can have at most one unoccupied vertex at any time. Let  $u$  be a vertex in  $C_1$  such that there exists a  $y$  in  $C_2$  with  $uy \notin E$ . Vertex  $u$  exists since  $\theta(G) = 2$ . If we force a guard onto  $u$ , then since every other pair of vertices in  $G$  besides  $u, y$  are adjacent, by condition (i), there will be an unprotected edge between  $C_1$  and  $C_2$ .

For the other direction, we show that if  $G$  is not of the form specified in the statement of the theorem, then  $\nu_m^\infty(G) < n - 1$ . Let  $C = \{C_1, C_2\}$  be a minimum clique covering of  $G$ . Then neither  $C_1$  nor  $C_2$  is a  $K_1$ , otherwise condition (i) is not satisfied. Let  $G$  have  $n$  vertices and place  $n - 2$  guards

in  $G$  so that they form a vertex cover. Then exactly one vertex in  $C_1$  is unoccupied and exactly one vertex in  $C_2$  is unoccupied. Suppose  $v \in C_1$  is attacked. Move a guard from another vertex in  $C_1$  to  $v$ . In particular, move a guard from vertex  $u$  where  $u \neq v$  in  $C_1$  and  $x \in C_2$  satisfy condition (i). Move guards in  $C_2$  so that  $x$  is unoccupied. All vertices in  $G$  except  $u$  and  $x$  are occupied. The guards now form a vertex cover. Likewise if a vertex in  $C_2$  is attacked.  $\square$

Characterizing the graphs with  $\nu_m^\infty(G) = n-1$  in the case  $\theta(G) > 2$  takes a bit more work. Let  $C = \{C_1, C_2, \dots, C_t\}$  be a minimum clique covering of  $G$ ,  $t > 2$ . If there exists a pair of cliques in  $C$  (or in any clique covering for that matter) satisfying condition (i) in the statement of Theorem 15, then it is easy to see that  $\nu_m^\infty(G) < n-1$ . This is because we can maintain one unoccupied vertex in each clique from this pair at all times, with all other vertices in  $G$  being occupied. If  $G$  is structured so that the only edges between cliques in  $C$  are between  $K_1$ 's and non- $K_1$ 's, then we claim  $\nu_m^\infty(G) < n-1$ . In this case, the cliques from  $C$  are arranged in a graph-like structure with  $K_1$ 's adjacent only to non- $K_1$ 's and vice versa. Then some  $K_p$ ,  $p > 1$ , is adjacent to two or more  $K_1$ 's and the subgraph induced by these  $p + c \geq p + 2$  vertices can be defended by  $p$  guards (since any two distinct  $K_1$  cliques are not adjacent, by the minimality of  $C$ ). Placing and keeping a guard on each other vertex of the graph suffices to prove the claim. In fact, if  $G$  has more than one  $K_1$  clique in any minimum clique cover, the same argument applies to show that  $\nu_m^\infty(G) < n-1$ .

Hence, the one remaining case to resolve is when there exist pairs of adjacent cliques in a clique covering such that neither of them are  $K_1$ 's, but for each such pair,  $C_i, C_j$ , there do not exist  $u, v \in C_i$ ,  $x, y \in C_j$  ( $u \neq v$ ,  $x \neq y$ ) such that  $ux, vy \notin E(G)$ . Initially place guards on all vertices except  $u, v$  where  $u \in C_1$ ,  $v \in C_2$ ,  $C_1, C_2$  are adjacent, and  $uv \notin E$ . Vertices  $u, v$  exist by the minimality of  $C$ . Suppose  $u$  is attacked. The only difficult situation is when  $v$  is adjacent to each other vertex in  $C_1$  (otherwise send the guard at  $w \in C_1$  to  $u$ , where  $vw \notin E$ ). There are two cases.

**Case 1.** Suppose there exists a clique  $C_3$  adjacent to  $C_1$ . Send a guard from  $C_3$  to  $C_1$  so all vertices in  $C_1$  are occupied (this may also require us to move a guard from within  $C_1$  to  $u$ , depending on whether or not  $u$  has a neighbor in  $C_3$ ) and move guards within  $C_2$  and  $C_3$ , if necessary, so that the unoccupied vertices in  $C_3$  and  $C_2$  are not adjacent. Since there must exist vertices  $a \in C_3$ ,  $b \in C_2$  such that  $ab \notin E$ , this is possible. Note that this applies even if  $C_3$  is a  $K_1$ .

**Case 2.** Suppose there does not exist any clique adjacent to  $C_1$  except  $C_2$ . Then since  $\theta(G) > 2$  and  $G$  is connected, there must exist a clique  $C_3$  that is adjacent to  $C_2$ . But then  $C_1$  and  $C_3$  are a pair of cliques such that

there exist  $u, v \in C_1, x, y \in C_3$  ( $u \neq v, x \neq y$ ) such that  $ux, vy \notin E(G)$ .  $\square$

Therefore, the graphs  $G$  described in Theorem 15 are the only ones with  $\nu_m^\infty(G) = n - 1$ , as we state now.

**Theorem 16** *Let  $G$  be a connected graph. Then  $\nu_m^\infty(G) = n - 1$  if and only if (a)  $G \cong K_n, n > 1$  or (b)  $\theta(G) = 2$  and either (i) the two distinct cliques  $C_1, C_2$  in each minimum clique covering of  $G$  are such that there do not exist  $u, v \in C_1, x, y \in C_2$  ( $u \neq v, x \neq y$ ) such that  $ux, vy \notin E(G)$  or (ii) one clique in some minimum clique covering of  $G$  is a  $K_1$ .*

We conclude with a conjecture.

**Conjecture 2** *Let  $G$  be a connected graph other than  $C_4$  with  $\delta(G) \geq 2$ . Then  $\nu_m^\infty(G) > \gamma_m^\infty(G)$ .*

The  $\delta \geq 2$  condition is needed because of graphs such as  $K_3$  with a pendant vertex attached to each vertex of the  $K_3$ . It is known that if  $G$  is a connected graph with  $\delta(G) \geq 2$  and girth seven or girth at least nine, then  $\gamma_m^\infty(G) < \tau(G)$ , see [13].

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